Chapter 5

5. Open mappings and Homeomorphisms

Def. 5.1: Let (X, τ) and (Y, σ) be two top. spaces and let f :X$\rightarrow$Y be a mapping. Then

(1) f is said to be an open mapping if f (G) is σ-open set in Y, for each τ-open set G of X.

(2) f is said to be a closed mapping if f (F) is σ-closed set in Y, for each τ-closed set F of X.

(3) f is said to be bicontinuous if f is open and continuous, i.e.,

(f and f^{-1} are continuous).

(4) f is said to be homeomorphism if

(i) f is bijective (one-one and onto)

(ii) f is a τ-σ continuous.

(iii) f^{-1} is a σ-τ continuous (or f is an open mapping).

Def. 5.2: (Homeomorphic Spaces)

Two top. spaces are homeomorphic if there exists a homeomorphism mapping from one of them into the other.

Then Y is said to be a homeomorphic to X and denoted by Y $\cong$ X.

Q*: Show that $\cong$ is an equivalence relation **(H.W.)**.

Def. 5.3: (Topological Property)

A property of a space X is said to be a topological property if it's preserved under homeomorphism mapping, i.e.,

if (X, τ) and (Y, σ) are two topological spaces and f :X$\rightarrow$Y is a homeomorphism mapping, then a property **P** of X has a topological property if f (X) has the same property **P**, i.e.,

if Y has the same property **P**.

Ex 5.4: Connectedness and Compactness have topological property.

Remark 5.5: A property which is preserved under continuous mapping is also preserved under homeomorphism.

Th 5.6: Let (X, τ) and (Y, σ) be two top. spaces and let $f: X \to Y$ be a bijective mapping, then the following statements are equivalent:

(1) f is homeomorphism.

(2) f is continuous and open.

(3) f is continuous and closed.

Proof: H.W.

Ex 5.7: Consider the topology

$\tau = \{\phi, \{a\}, \{b, c\}, X\}$ on $X = \{a, b, c\}$ and

$\sigma = \{\phi, \{r\}, \{p, q\}, Y\}$ on $Y = \{r, p, q\}$.

Find which of the mappings defined as follows are:

(1) cont. fun.?

(2) open fun.?

(3) closed fun.?

(4) bicontinuous fun.?

(5) homeomorphism.?

(i) $f(a) = r$, $f(b) = r$, $f(c) = r$.

(ii) $\mu(a) = p$, $\mu(b) = q$, $\mu(c) = r$.

Solution: (i)

$f(a) = r$, $f(b) = r$, $f(c) = r$.

(1) $f^{-1}(\phi) = \phi$, $f^{-1}(\{r\}) = \{a, b, c\} = X$.

$f^{-1}(\{p, q\}) = \phi$, $f^{-1}(Y) = X$.

Hence f is cont.

(2) $f(\phi) = \phi$, $f(\{a\}) = \{r\}$, $f(\{b, c\}) = \{r\}$, $f(X) = \{r\}$. Also, $\{r\}$ is open in (Y, σ).

Hence f is open fun.

(3) Now, τ-closed sets are

$X, \{b, c\}, \{a\}, \phi$.

The σ-closed sets are

$Y, \{p, q\}, \{r\}, \phi$.

$f(X) = \{r\}$, $f(\{b, c\}) = \{r\}$, $f(\{a\}) = \{r\}$, $f(\phi) = \phi$.

Hence f is closed fun.

(4) Since f is cont. and open, so it is bicontinuous.

(5) **H.W.**

(ii) **H.W.**

Th 5.8: Let (X, τ) and (Y, σ) be two top. spaces. Then a mapping $f : X \to Y$ is open **iff**

$f(A^o) \subset (f(A))^o$, $\forall A \subset X$.

(Q: State and prove an equivalent statement of open function).

Proof: ($\Rightarrow$).

Let f be an open mapping and $A \subset X$. Then A^o is a τ-open set in X.

Since f is an open mapping, then $f(A^o)$ is σ-open set in Y (by Def. of open mapping).

$\Rightarrow (f(A^o))^o = f(A^o)$ (Th. A is open $\leftrightarrow A^o = A$).

But, $A^o \subset A \Rightarrow f(A^o) \subset f(A)$ [How ?]

$\Rightarrow (f(A^o))^o \subset (f(A))^o$ [since $A \subset B \to A^o \subset B^o$]

So, $f(A^o) \subset (f(A))^o$.

Conversely, let $f(A^o) \subset (f(A))^o$, $\forall A \subset X$.

To show that f is an open mapping.

Let G be any τ-open set in X.

$\Rightarrow G^o = G$ [Th: A is open $\leftrightarrow A^o = A$]

$\Rightarrow f(G^o) = f(G)$.

i.e., $f(G) = f(G^o) \subset (f(G))^o$ [By given].

But, $(f(G))^o \subset f(G)$ [Since $A^o \subset A$]

$\Rightarrow f(G) = (f(G))^o$.

Hence $f(G)$ is an open set (by Def. of open mapping).

Whence f is an open mapping (By Def. of open fun.).

Th 5.9: Let (X, τ) and (Y, σ) be two top. spaces.

A mapping $f: X \rightarrow Y$ is closed **iff**

$Cl(f(A)) \subset f(Cl(A))$, $\forall A \subset X$.

(Without proof)

Th 5.10: Let (X, τ) and (Y, σ) be any two top. spaces.

Let the mapping $f: X \rightarrow Y$ be bijective.

Then f is a homeomorphism **iff**

$f(Cl(A)) = Cl(f(A))$, $\forall A \subset X$.

(**Q:** State and proof an equivalent statements of homeomorphism)

Proof: ($\Rightarrow$).

Suppose that f is a homeomorphism, then by Th. 5.6, we obtain

(f is bijective, f is cont. and f is closed).

If $A \subset X$, then $f(Cl(A)) \subset Cl(f(A))$ [Since f is cont.(by Th. 4.16)]

Since f is closed mapping, then $\mathrm{Cl}(f(A)) \subset f(\mathrm{Cl}(A))$ [by Th. 5.9], and so

$f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$.

Conversely, let $f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$, $\forall A \subset X$.

$\Rightarrow f(\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A)) \Rightarrow f$ is cont [by Th. 4.16].

Also, $\mathrm{Cl}(f(A)) \subset f(\mathrm{Cl}(A)) \Rightarrow f$ is closed [by Th 5.9].

Hence f is a homeomorphism.

Some Questions about Chapter 5

Q1/ Let the real function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Show that f is not open fun.

Q2/ Let **B** be a base for a topological space X. Show that if $f : X \to Y$ has the property that $f[B]$ is open for every $B \in \mathbf{B}$, then f is an open fun.

Q3/ Show that the closed interval $A = [a, b]$ is homeomorphic to the closed unit interval $I = [0, 1]$.

Q4/ Give an example of a real function $f : \mathbf{R} \to \mathbf{R}$ such that f is continuous and closed, but not open.