8. Separation Axioms:

Def 8.1: (T_0-space) (Kolomogorov Axiom)

A space (X, τ) is called a T_0-space if it satisfies the following axiom of Kolomogorov.

For every two distinct points of X, there exists an open set which contains one of them, but not the other.

Ex 8.2: The discrete topology (X, D) is a T_0-space since for each point $x \in X$, $\exists$ an open set $\{x\}$, which contains no point of X different than x.

Ex 8.3: Indiscrete topological space (X, I) is not T_0-space since for every points x, y of X, $x \neq y$, there does not exist an open set containing one of them since the only open set containing one of them is X, which contains the other.

Th 8.4: A topological space (X, τ) is T_0 **iff** $\forall$ $x, y \in X$ such that $x \neq y$, then $Cl(\{x\}) \neq Cl(\{y\})$.

(**Q:** State and prove an equivalent statement of T_0-space).

Proof: (H.W.)

Def 8.5: (Hereditary Property)

A property of a space (X, τ) is said to be hereditary if every subspace of the space has the same property.

Th 8.6: Every subspace of a T_0-space is T_0.

(That is, T_0-space being hereditary property).

Proof: Let (X, τ) be a T_0-space and let (Y, T_Y) be a subspace of (X, τ).

To show that (Y, T_Y) is a T_0-space.

Let y_1, y_2 be any two distinct points of Y.

Since $Y \subset X$, then y_1, y_2 also are distinct points in X.

Since (X, τ) is a T_0-space, $\exists$ an open set G containing y_1, but does not containing y_2, then $G \cap Y = G^*$ is a T_Y-open set containing y_1, but does not containing y_2.

It follows that (Y, T_Y) is a T_0-space.

Th 8.7: The property of a space being T_0-space is preserved under 1-1, onto, open function and hence it is a topological property.

Proof: (H.W.)

Def 8.8: (T_1-space) (Frechet Space)

A space (X, τ) is said to be a T_1-space if for any two distinct points x and y of X, there exist two open sets G and H such that $x \in G$, $y \notin G$ and $y \in H$, but $x \notin H$. **Or,**

(A space (X, τ) is said to be a T_1-space if for any two distinct points x and y in X, there exist two open sets, one of them containing x, but not y and the other containing y, but not x).

Ex 8.9: Let X = {a, b} and τ = {ϕ, X, {a}, {b}}. Then, (X, τ) is a T_1-space because for a, b $\in$ X, $\exists$ two open sets {a}, {b} such that a $\in$ {a}, b $\notin$ {a} and b $\in$ {b}, but a $\notin$ {b}.

Q 8.10: Every T_1-space is a T_0-space, but the converse is not true in general. Give an example **(H.W.).**

Th 8.11: A space (X, τ) is T_1 **iff** every single subset {x} of X is closed.

(**Q:** State and prove an equivalent statement of T_1-space).

Proof: ($\Leftarrow$).

Suppose {x} is closed, for every x $\in$ X.

Let a, b $\in$ X, then {a}, {b} are closed subsets of X.

Then, $b \in \{a\}^c$ is open, but $a \notin \{a\}^c$.

Also, $a \in \{b\}^c$ is open, but $b \notin \{b\}^c$.

Whence, (X, τ) is T_1-space (by Def. of T_1-space).

(⇒).

Suppose X is T_1-space and let $x \in X$.

For each $y \in X$, there exists an open set G(y), which contains y, but not x (Since X is T_1 and every T_1 is T_0).

Since G(y) is open, then $\bigcup_{y \neq x} G(y)$ is open (The union of any family of open sets is open).

Then, $\{x\}^c = \bigcup_{y \neq x} G(y)$.

Then, $\{x\}^c$ is open.

Whence, $\{x\}$ is closed.

Q*: Explain Th 8.11 by an example **(H.W.).**

Th 8.12: The property of a space being a T_1-space is preserved under 1-1, onto, open function, and hence it is a topological property.

(Without proof).

(**Q:** State a property of a T_1-space)

Def 8.13: (T_2-space) or (Hausdorff space)

A space (X, τ) is said to be a T_2-space or Hausdorff space or separated space if for every pair of distinct points x and y of X, there exist disjoint open sets containing x and y, respectively.

Or,

(A space (X, τ) is said to be a T_2-space if for every pair of distinct points x and y of X, there exist two open sets N and M such that $x \in N$, $y \in M$ and $M \cap N = \phi$).

Ex 8.14: Every discrete space is a T_2-space.

Solution: Let (X, D) be a discrete space, let $x, y \in X$, then $\{x\}$ and $\{y\}$ are two disjoint open sets such that $x \in \{x\}$ and $y \in \{y\}$.

Whence, (X, D) is a T_2-space.

Lemma 8.15: Every T_2-space is a T_1-space.

Proof: Let (X, τ) be a T_2-space, and let x, y be any two distinct points of X.

Since the space is T_2, there exist disjoint open set N of x and M of y such that $N \cap M = \phi$. This implies that the space is T_1 because directly follows from the Definition of T_1-space.

The converse of Lemma 8.15 is not true in general as it is shown by the following example.

Ex 8.16: (H.W.)

Th 8.17: Every subspace of a T_2-space is T_2.

(**Q:** T_2-space being hereditary property).

Proof:

Let x_1, x_2 be any two distinct points of Y.

Since $Y \subset X$, then $x_1 \neq x_2$ in X.

Since X is a T_2-space, $\exists$ two open sets G and H such that $x_1 \in G$, $x_2 \in H$ and $G \cap H = \phi$.

By Def. of subspace, $G \cap Y$ and $H \cap Y$ are τ_Y-open sets.

Also, $x_1 \in G$, $x_1 \in Y \Rightarrow x_1 \in G \cap Y$.

And $x_2 \in H$, $x_2 \in Y \Rightarrow x_2 \in H \cap Y$.

Since $G \cap H = \phi$, then we have,

$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \phi \cap Y = \phi$.

Thus, $G_1 = G \cap Y$ and $H_1 = H \cap Y$ are two disjoint τ_Y-open sets such that $x_1 \in G_1$, $x_2 \in H_1$ and $G_1 \cap H_1 = \phi$.

Hence (Y, τ_Y) is T_2-space.

Some Questions about Chapter 8

Q1/ Show that a finite subset of a T_1-space X has no accumulation points.

Q2/ Show that every finite T_1-space X is a discrete space.

Q3/ Let τ be the topology on the real line R generated by the open-closed interval (a, b]. Show that (R, τ) is T_2-space.