Theorem 4.25: (Fundamental Theorem of Isomorphism) <u>or</u>

(First Isomorphism Group Theorem)

Let (G, *) and (G′, .) be two groups. Let f: (G, *)$\rightarrow$(G′, .) be an onto homomorphism, then (G / Ker (f), $\otimes$) $\cong$ (G′, .).

Proof: Let H = Ker (f).

Define a mapping φ: G / H $\rightarrow$ G′ by

$\varphi(a*H) = f(a), \forall\, a \in G$.

First we show that φ is well-defined, i.e.,

let a*H, b*H $\in$ G / H.

we show that $a*H = b*H \Rightarrow \varphi(a*H) = \varphi(b*H)$.

Let $a*H = b*H \Rightarrow a^{-1}*b \in H = \text{Ker}(f)$

$\Rightarrow f(a^{-1}*b) = e'$, where e′ is the identity element of G′.

$\Rightarrow f(a^{-1}) \,.\, f(b) = e'$ (since f is homomorphism)

Since $a\in G$, then $f(a)\in G'$. Therefore, $\exists\, (f(a))^{-1} \in G'$ such that

$f(a) \,.\, (f(a))^{-1} = e'$.

Now, we have

$f(a^{-1}) \,.\, f(b) = e'$

$\Rightarrow f(a) \,.\, (f(a))^{-1} \,.\, f(b) = f(a) \,.\, e'$ [since $f(a^{-1}) = (f(a))^{-1}$, where f is homomorphism]

$\Rightarrow e' .\, f(b) = f(a)$

$\Rightarrow f(b) = f(a)$

$\Rightarrow \varphi(b*H) = \varphi(a*H)$

$\therefore\ \varphi$ is well-defined.

Now, we show that φ is a homomorphism.

Let a, b $\in$G.

We show that $\varphi\ [(a*H) \otimes (b*H)] = \varphi(a*H).\varphi(b*H)$, $\forall$ a*H, b*H $\in$ G / H.

Consider,

$\varphi\ [(a*H) \otimes (b*H)] = \varphi\ [(a*b)*H] = f\ (a*b) = f\ (a).f\ (b)$ [since f is homo.]

$\varphi\ [(a*H) \otimes (b*H)] = \varphi(a*H).\varphi(b*H)$.

$\therefore$ φ is homomorphism.

To show that φ is 1-1, i.e.,

we show that $\forall$ a*H, b*H $\in$ G / H, $\varphi(a*H) = \varphi(b*H) \Rightarrow a*H = b*H$.

Now, $\varphi(a*H) = \varphi(b*H) \Rightarrow f\ (a) = f\ (b)$.

$\Rightarrow (f(a))^{-1}.\ f\ (a) = (f(a))^{-1}.f\ (b)$

$\Rightarrow$ $e' = f\ (a^{-1})\ .\ f\ (b)$ [since $f\ (a^{-1}) = (f(a))^{-1}$, where f is homomorphism]

$\Rightarrow$ $e' = f\ (a^{-1} * b)$ [since f is homomorphism]

$\Rightarrow a^{-1} * b \in H = \text{Ker}\ (f)$.

$\Rightarrow a*H = b*H$ [why ?]

$\therefore$ φ is 1-1.

Since f is onto, then φ is onto (By Corollary 4.23)

$\therefore$ φ is isomorphism.

$\therefore G / \text{Ker}\ (f) \cong G'$.

Th. 4.26: (Second Isomorphism Group Theorem)

Let (G, *) be a group, $H \leqslant G$ and $K \nabla G$, then

(1) $H \cap K \nabla H$.

(2) $H*K \leqslant G$.

(3) $H / H \cap K \cong H*K / K$.

Proof: (1) Since $H \leqslant G$ and $K \leqslant G$, then $e \in H$ and $e \in K \Rightarrow e \in H \cap K \Rightarrow$ $H \cap K \neq \phi$.

Also, let $a, b \in H \cap K$.

Now, we show that $H \cap K \leqslant G$, i.e.,

we show that $a*b^{-1} \in H \cap K$.

Since $a, b \in H \cap K \Rightarrow a, b \in H$ and $a, b \in K$.

$\Rightarrow a*b^{-1} \in H$ and $a*b^{-1} \in K$ (since $H \leqslant G$ and $K \leqslant G$)

$\Rightarrow a*b^{-1} \in H \cap K$

$\therefore H \cap K \leqslant G$.

Now, we want to show that $H \cap K \nabla H$.

Let $x \in H$ and $h \in H \cap K$.

We show that $x*h*x^{-1} \in H \cap K$, i.e.,

we show that $x*h*x^{-1} \in H$ and $x*h*x^{-1} \in K$.

Since $x \in H \subseteq G$ and $h \in H \cap K \Rightarrow h \in H$ and $h \in K$.

Then, $x \in G$, so $x*h*x^{-1} \in K$ (since $K \nabla G$)

Since $x \in H$ and $h \in H \Rightarrow x*h \in H$ (since $H \leqslant G$)

Since $x \in H$, then $x^{-1} \in H$ (since $H \leqslant G$)

So, $x*h*x^{-1} \in H$

$\therefore x*h*x^{-1} \in H \cap K$.

(2) H.W.

(3) H.W.

Q 4.26′ (H.W.): Let $(\mathbb{Z}, +)$ and $(\mathbb{Z}_2, +_2)$ be two groups, then show that $\mathbb{Z} / (2) \cong \mathbb{Z}_2$.

Q 4.26′′ (H.W.): Let $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +_n)$ be two groups, then show that $\mathbb{Z} / (n) \cong \mathbb{Z}_n$.

Th. 4.27: (Third Isomorphism Group Theorem)

Let (G, *) be a group, $H \nabla G$, $K \nabla G$ and $H \nabla K$, then

$$\frac{\frac{G}{H}}{\frac{K}{H}} \cong \frac{G}{K}$$

Proof: Let $\varphi: G / H \to G / K$ such that

$\varphi(a*H) = a*K, \forall a \in G$.

First we show that φ is well-defined.

Let $a*H, b*H \in G / H$.

We show that $a*H = b*H \Rightarrow \varphi(a*H) = \varphi(b*H)$.

Since $a*H = b*H \Rightarrow a^{-1}*b \in H \subseteq K \Rightarrow a^{-1}*b \in K \Rightarrow a*K = b*K$ [why ?]

$\therefore \varphi(a*H) = \varphi(b*H)$.

So, φ is well-defined.

Now, we show that φ is homomorphism.

Let $a*H, b*H \in G / H$.

We show that $\varphi((a*H)\otimes(b*H)) = \varphi(a*H)\otimes(b*H)$.

Now, $\varphi((a*H)\otimes(b*H)) = \varphi(a*b*H)$ [by Definition of quotient group]

$$= a*b*K = (a*K)\otimes(b*K) = \varphi(a*H)\otimes(b*H).$$

So, φ is homomorphism.

Now, we show that φ is onto.

$\forall\ x \in G / K, \exists\ a \in G$ such that $x = a*K = \varphi(a*H)$.

So, $x = \varphi(a*H)$.

$\therefore \varphi$ is onto.

It remains to show that $Ker(\varphi) = K / H$ **(H.W.)**.

Direct Product For Groups

Def. 4.28: (Direct Product For Groups)

Let $(G, *)$ and (H, o) be two distinct groups, then define a binary operation $\otimes$ on $G \times H = \{(g, h): g \in G, h \in H\}$ as follows:

For each $(g_1, h_1), (g_2, h_2) \in G \times H$,

$(g_1, h_1) \otimes (g_2, h_2) = (g_1*g_2, h_1 o h_2)$.

Q 4.29: Show that $(G \times H, \otimes)$ form a group.

Solution: (1) Let $(g_1, h_1), (g_2, h_2) \in G \times H$.

We show that $(g_1, h_1) \otimes (g_2, h_2) \in G \times H$.

Now, $(g_1, h_1) \otimes (g_2, h_2) = (g_1*g_2, h_1 o h_2) \in G \times H$.

$\therefore G \times H$ is closed under $\otimes$.

(2) Let $(g_1, h_1), (g_2, h_2), (g_3, h_3) \in G \times H$.

We show that

$((g_1, h_1) \otimes (g_2, h_2)) \otimes (g_3, h_3) = (g_1, h_1) \otimes ((g_2, h_2) \otimes (g_3, h_3))$ … (1)

L.H.S. of (1) $= ((g_1, h_1) \otimes (g_2, h_2)) \otimes (g_3, h_3)$

$= ((g_1 * g_2), (h_1 \circ h_2)) \otimes (g_3, h_3)$

$= ((g_1 * g_2) * g_3, (h_1 \circ h_2) \circ h_3)$

$= (g_1 * (g_2 * g_3), h_1 \circ (h_2 \circ h_3))$ [why ?]

$= (g_1, h_1) \otimes ((g_2, h_2) \otimes (g_3, h_3))$ [write in detail]

$\therefore \otimes$ is associative.

(3) $(g_1, h_1) \otimes (e_G, e_H) = (g_1 * e_G, h_1 \circ e_H) = (g_1, h_1)$.

Also, $(e_G, e_H) \otimes (g_1, h_1) = (e_G * g_1, e_H \circ h_1) = (g_1, h_1)$.

$\therefore (e_G, e_H)$ is the identity element.

(4) **(H.W.)**

Q.4.30 (H.W.): Give an example on Def. 4.28.

Q.4.31 (H.W.): State and prove some (three) properties of direct product of two groups.