Chapter 8

Quotient and Homomorphism Rings

Some Examples and Results

Question: (8.14+1)

Show that $I \subseteq \sqrt{I}$

Answer: Let $a \in I$, put n = 1, then $a^n = a^1 \in I \Rightarrow a \in \sqrt{I}$.

Question: (8.14+2)

If I is an ideal of R, then show that

$\sqrt{\sqrt{I}} = \sqrt{I}$

Answer: We prove that $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$ ..... (1)

and $\sqrt{\sqrt{I}} \subseteq \sqrt{I}$ ..... (2)

Since $I \subseteq \sqrt{I}$ (By **Question: (8.14+1)**) and I is an ideal of R, then $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$.

Let $x \in \sqrt{\sqrt{I}}$.

We prove that $x \in \sqrt{I}$

$x \in \sqrt{\sqrt{I}} \Rightarrow \exists\, n \in \mathbf{N}$ such that $x^n \in \sqrt{I}$

$\Rightarrow \exists\, m \in \mathbf{N}$ such that $(x^n)^m \in I$.

$\Rightarrow x^{nm} \in I$, choose $nm = k \in \mathbf{N}$

$\Rightarrow x^k \in I \Rightarrow x \in \sqrt{I}$

Therefore, $\sqrt{\sqrt{I}} \subseteq \sqrt{I}$

Whence, $\sqrt{\sqrt{I}} = \sqrt{I}$.

Remarks: (8.14+3)

Let (**Z**, +,.) be the ring of integers. Then,

$\sqrt{(p)} = (p)$, where p is a prime number.

If $n = p_1^{\alpha 1}.p_2^{\alpha 2} . \dots . p_n^{\alpha n}$, then

$\sqrt{n} = (p_1.p_2. \dots .p_n)$, where $p_1, p_2, \dots, p_n$ are distinct primes.

Example: **(8.14+4)**

Let (**Z**, +,.) be the ring of integers. Find $\sqrt{(120)}$, $\sqrt{(8)}$, $\sqrt{(11)}$.

Solution:

(1) $\sqrt{(120)} = (2.3.5) = (30)$

(2) $\sqrt{(8)}$ = (2).

(3) $\sqrt{(11)} = (11)$

Primary Ideal

<u>**Defintion :**</u> **(8.14+5) (Primary Ideal)**

Let (R, +, .) be a commutative ring with identity. An ideal I of a ring R is said to be primary if for every a, b $\in$ R such that a.b $\in$ I, and a $\notin$I, then there exista a positive integer n such that $b^n \in$ I.

Remark: (8.14+6)

$a \in I \Rightarrow a^2 \in I \Rightarrow a^3 \in I \Rightarrow ... \Rightarrow a^n \in I$.

Question (H.W.): (8.14+7)

What is relation between prime ideal and primary ideal ? Prove it. Is the converse part true? If **YES,** prove it but if **NO,** give an example.

Example: (8.14+8)

If we take (**Z**, +, .), the ring of integers.

((9), +, .) is an ideal of **Z**.

(9) = {..., -9, 0, 9, 18, 27, ...}

$9 = 3^2.1$

Whence, ((9), +, .) is primary ideal.

Question (H.W.): (8.14+9)

Give some examples and properties on primary ideal.

Theorem: (8.14+10)

Let I be an ideal of the ring R. Then, I is a primary ideal iff every zero divisor of R / I is nilpotent.

(**Q:** State and prove an equivalent statement of primary ideal).

Proof: Suppose that I is a primary ideal of R and a+I is a zero divisor of R / I. Then,

$\exists$ b+I $\neq$ I such that (a+I)(b+I) = I.

$\Rightarrow$ ab+I = I $\Rightarrow$ ab $\in$ I.

Since b+I $\neq$ I $\Rightarrow$ b$\notin$I.

Since I is primary, so that $a^n \in$ I, for some positive integer n.

$\Rightarrow$ $(a+I)^n = a^n + I = I$,

Whence, a+I is nilpotent.

Conversely **(H.W.).**

Homomorphism Rings

Defintion : (8.15)

A mapping f from a ring $(R, +\,, .)$ into a ring $(R', +', .')$ is said to be a ring homomorphism if $\forall\, a, b \in R$

(1) $f(a + b) = f(a) +' f(b)$.
(2) $f(a.b) = f(a).' f(b)$.

Example: (8.16)

Let $(R, +, .)$ and $(R', +', .')$ be two rings. Define $f(a) = 0', \forall\, a \in R$, where $0'$ is the zero of R'. Show that f is a homomorphism ring.

Sol:

(1) $f(a + b) = f(a) +' f(b)$ _____(1)

$L.H.S\ of\ (1) = f(a + b) = 0'$.

$R.H.S\ of\ (1) = f(a) +' f(b) = 0' +' 0' = 0'$.

$\therefore f(a + b) = f(a) +' f(b)$.

(2) $f(a.b) = f(a) .' f(b)$______(2)

$L.H.S.of\ (2) = f(a.b) = 0'$.

$R.H.S. of\ (2) = f(a) .' f(b) = 0' .' 0' = 0'$.

f is a homomorphism ring.

Example: (8.17)

Let $f: (\mathbb{Z}, +, .) \rightarrow (\mathbb{Z}, +, .)$ be a mapping define by $f(n) = 3n$, $n \in \mathbb{Z}$.

Is f a homomorphism ring?

Solution:

(1) $f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y)$.
(2) $f(x.y) = 3(x.y) \neq 3x.3y = f(x).f(y)$.

$\therefore f$ is not a ring homomorphism.

Theorem: (8.18)

If f is a homomorphism from a ring $(R, +, .)$ onto a ring $(R', +', .')$, then

(1) $f(0) = 0'$.
(2) $f(1) = 1'$.
(3) If I is an ideal of R, then $f(I)$ is an ideal of R'.
(4) If I' is an ideal of R', then $f^{-1}(I')$ is an ideal of R.

Proof:

(1) $f(0) = f(0 + 0)$

$= f(0) +' f(0)$ (since f is homomorphism)

$f(0) +' 0' = f(0) +' f(0)$.

Since $(R', +')$ is an abelian group, then by cancellation law, $f(0) = 0'$.

(2) Let $a \in R, f(a) .' 1' = f(a) = f(a.1) = f(a) .' f(1)$ (since f is homomorphism).
$\therefore f(a) .' 1' = f(a) .' f(1)$
We cannot use cancellation law because R' is not an integral domain.
Since f is onto and $f(1)$ is the identity element of R'.

$\therefore f(1) = 1'$ (by uniqueness of the identity element)

(3) H.W.
(4) H.W.

Defintion : **(8.19)**

A ring homomorphism $f: (R, +, .) \to (R', +', .')$ is said to be.

(1) monomorphism if f is 1 – 1.

(2) epimorphism if f is onto.

(3) isomorphism if f is 1 – 1 & onto.

(4) automorphism if f is isomorphism and $R = R'$.

Defintion : **(8.20)**

Let (R, + , .) and $(R', +', .')$ be two rings. Then $R \simeq R'$, (i.e., R is isomorphic to R') if ∃ an isomorphism mapping $f: R' \to R$.

Defintion : **(8.21) (Kernel of homomorphism ring)**

Let $(R, +, .)$ and $(R', +', .')$ be two rings and $f: (R, +, .) \to (R', +', .')$ be a ring homomorphism. Then $Ker\, f = \{x \in R: f(x) = 0'\}$

Theorem: **(8.22)**

Let $f: (R, +, .) \to (R', +', .')$ be a homomorphism.

Then

(1) $Ker\, f = \{0\}$ if and only if f is 1 – 1

(2) $Ker\, f$ is an ideal of R.

proof:

(1) suppose f is 1 – 1.

We prove that $Ker\, f = \{0\}$.

Let $x \in Ker\, f$.

We show that $x = 0$?

Now, $f(x) = 0' = f(0)$ $[by\ Th\ 8.18(2)]$

$So, f(x) = f(0)$.

$\Rightarrow x = 0\ (since\ f\ is\ 1 - 1)$

Conversely, suppose $Ker\, f = \{0\}$.

We show that f is 1 – 1.

Let $f(r_1) = f(r_2)$.

We show that $r_1 = r_2$.

Now, $f(r_1) -' f(r_2) = 0'$; where $0'$ is the identity element of R'

$f(r_1 - r_2) = 0'$ (since f is homomorphism).

$r_1 = r_2 \in Ker\, f = \{0\}$.

$\Rightarrow r_1 - r_2 = 0 \Rightarrow r_1 = r_2$.

$\therefore f$ is 1–1.

(2) $Since\ 0 \in R\ and\ f(0) = 0' \Rightarrow 0 \in Ker\, f \Rightarrow Ker\, f \neq \emptyset$.

(i) $let\ a, b \in Ker\, f \Rightarrow f(a) = 0'\ and\ f(b) = 0'$.

Now, $f(a - b) = f(a + (-b))$

$= f(a) +' f(-b)$ $(since\ f\ is\ homomorphism)$

$= f(a) +' (-f(b))$

$= 0' +' (-0')$

$= 0'$

$\therefore a - b \in Ker\, f$.

(ii) ***H.W.***

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(Fundamental Theorem of Isomorphism) <u>or</u> (First Isomorphism Ring Theorem)

<u>Theorem:</u> (8.23)

Let $(R, +, .)$ and $(R', +', .')$ be two rings.Let f be a ring homomorphism from R $onto$ R', then $R/Ker\, f \simeq R'$.

Proof: We define a mapping $g: R/\,Ker\, f \rightarrow R'$ by

$g(a + Ker\, f) = f(a), \forall a \in R$.

Let $K = Ker\, f$.

$\therefore g(a + K) = f(a), \forall a \in R$.

(1) First we show that g is well- defined.

Let $a + K = b + K, for\ a, b \in R$

$\Rightarrow a - b \in K = Ker\, f$.

$\Rightarrow f(a - b) = 0'$.

$\Rightarrow f(a) -' f(b) = 0'$ $(since\ f\ is\ a\ homomorphism)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \; g(a + K) =$ g(b +K)

So g is well- defined.

(2) Second we show that g is a homomorphism.

Let $a, b \in R$.

Now, $g\big((a + K) + (b + K)\big) = g\big((a + b) + K\big)$.

$$= f(a + b) = f(a) +' f(b) \; (since \; f \; is \; a \; homomorphism)$$

$$= g(a + K) +' g(b + K).$$

Also, $g((a + K) . (b + K) = g(a.b. + K) = f(a.b)$

$$= f(a) .' f(b) \; (since \; f \; is \; a \; homomorphism)$$

$$= g(a + K) .' \; g(b + K).$$

$\therefore g$ is a homomorphism

(3) Let $g(a + K) = g(b + K), for \; each \; a + K, b + K \in R/K$

$\Rightarrow f(a) = f(b) \Rightarrow f(a) -' f(b) = 0'$.

$\Rightarrow f(a - b) = 0' \; (since \; f \; is \; a \; homomorphism)$

$\Rightarrow a - b \in K = Kerf$

$\Rightarrow a + K = b + K$.

$\therefore \; g \; is \; 1 - 1$.

(4) $\forall \, y \in R' , \exists \, x \in R \; such \; that \; f(x) = y \; (since \; f \; is \; onto)$.

$\therefore \; y = f(x) = g(x + K)$.

$\therefore \; g \; is \; onto$.

$\therefore \; R/Kerf \simeq R'$

<u>Question:</u> Show that

(1) $Z/(2) \simeq Z_2$.

(2) $Z/(n) \simeq Z_n$.

Theorem: **(8.24) (Second Isomorphism Ring Theorem)**

Let A and B be two ideals of a ring $(R,+,.)$ such that $B \subseteq A$ and B an ideal of A, then $\frac{R/B}{A/B} \cong R/A$

Proof:

Let $\varphi : R/B \to R/A$ such that $\varphi(r + B) = r + A, r \in R$

First we show that φ is well- defined.

Let $r_1 + B, r_2 + B \in R/B$ and

if $r_1 + B = r_2 + B$.

We show that $\varphi(r_1 + B) = \varphi(r_2 + B)$.

Since $r_1 + B = r_2 + B$

$\Rightarrow r_1 - r_2 \in B \subseteq A \Rightarrow r_1 - r_2 \in A$

$\Rightarrow r_1 + A = r_2 + A$

$$\varphi(r_1 + B) = \varphi(r_2 + B)$$

$\therefore \varphi\, is\, well - defined.$

Now, we prove that φ is homomorphism.

Let $r_1 + B, r_2 + B \in R/B$.

We show that $\varphi\big((r_1 + B) + (r_2 + B)\big) = \varphi(r_1 + B) + \varphi(r_2 + B)$.

$$Now, \varphi\big((r_1 + B) + (r_2 + B)\big) = \varphi(r_1 + r_2 + B) = r_1 + r_2 + A$$
$$= (r_1 + A) + (r_2 + A) = \varphi(r_1 + B) + \varphi(r_2 + B).$$

Now, we show that

$\varphi\big((r_1 + B).(r_2 + B)\big) = \varphi(r_1 + B).\varphi(r_2 + B)$

$Now, \varphi\big((r_1 + B).(r_2 + B)\big) = \varphi(r_1 . r_2 + B) = r_1.r_2 + A$

$= (r_1 + A).(r_2 + A) = \varphi(r_1 + B).\varphi(r_2 + B)$

So φ is a homomorphism.

To prove that φ is onto.

Let $y \in R/A$ such that $y = r + A \Rightarrow \exists\, x \in R/B,\ x = r + B$ such that

$$\varphi(r + B) = r + A = y$$
$$\Rightarrow y = \varphi(r + B).$$

So φ is onto.

It remains to show that $Ker\ \varphi = A/B$ **(H.W.)**

∴ By first isomorphism ring theorem, $\frac{R/B}{A/B} \cong R/A$.

Question (H.W.): (8.24+1)

Explain **Second Isomorphism Ring Theorem** by an example.

Theorem: (8.25) (Third Isomorphism Ring Theorem)

Let A and B be two ideals of a ring R, then $B/(A \cap B) \cong (A + B)/A$

Proof:

Define $\varphi: B \to (A + B)/A$ such that $\varphi(b) = b + A,\ \forall\, b \in B$

Let $b_1, b_2 \in B\ and\ if\ b_1 = b_2$. We show that $\varphi(b_1) = \varphi(b_2)$

First we show that φ is well- defined.

$$\varphi(b_1) = b_1 + A = b_2 + A = \varphi(b_2).$$

So $\varphi(b_1) = \varphi(b_2)$.

∴ φ is well- defined.

Now, we show that φ is a homomorphism.

Let $b_1, b_2 \in B$. We show that $\varphi(b_1 + b_2) = \varphi(b_1) + \varphi(b_2)$

$\varphi(b_1 + b_2) = (b_1 + b_2) + A$

$= (b_1 + A) + (b_2 + A)$

$= \varphi(b_1) + \varphi(b_2)$

$\varphi(b_1 . b_2) = b_1 . b_2 + A$

$= (b_1 + A).(b_2 + A)$

$= \varphi(b_1) . \varphi(b_2)$

So φ is a homomorphism.

Now, we show that φ is onto.

Let $x \in (A + B)/A, \exists\, a \in A, b \in B\ such\ that\ x = (a + b) + A$

$\therefore\ x = (b + a) + A$

$= b + (a + A)$

$= b + A = \varphi(b).$

$\therefore\ x = \varphi(b), then\ \varphi\ is\ onto.$

To show that $Ker\ \varphi = A \cap B$ ($\boldsymbol{H.W.}$)

∴ By First Isomorphism Ring Theorem.

$B/\ Ker\ \varphi \simeq (A + B)/A$

$\therefore\ B/\ A \cap B \simeq (A + B)/A$

Question (H.W.): (8.26)

Explain **Third Isomorphism Ring Theorem** by an example.