Salahaddin Univ.-Erbil | **College of Science** | **First Semester**
Dept. of Math. | **(2023-2024)** | **MSc Exam.**
Final Exam. | **Non-Commutative Algebra** | **Time: 3 hours**

Q1/ (i) Define "Imbedded ring". State and prove "Dorroh Extension Theorem". (6+4) marks

(ii) Show that any homomorphism from any arbitrary ring R onto the ring **Z** of integers is uniquely determined by its kernel.

Q2/ (i) Define "nil ideal" and "nilpotent ideal". What is the relation between them ? Prove it. Define "nil (nilpotent) ring". If R contains an ideal I such that I and R / I are both nil ring, then show that R is a nil ring. (6+4) marks

(ii) Show that the set $I_i = \{(0, \ldots, 0, a_i, 0, \ldots, 0): ai \in Ri\}$ forms an ideal of R isomorphic to Ri under the mapping which sends $(0, \ldots, 0, a_i, 0, \ldots, 0)$ to the element a_i.

Q3/ (i) State or what is mean by: (4+5+5) marks

(1) Factorization of Homomorphism **(2)** External direct sum

(ii) Let R be a commutative ring with identity and I is an ideal of R. If J is an ideal of R such that $I \subseteq J$, then show that J / I is an ideal of R / I.

(iii) Let R be a commutative ring with identity and I is an ideal of R. If P is a prime ideal of R such that $I \subseteq P$, then show that P / I is a prime ideal of R / I.

[Hint: Use part (i)].

Q4/ (i) Assume that R is a ring and $a \in R$. If C(a) denotes the set of all elements with commute with a, $C(a) = \{r \in R: ar = ra\}$. Show that C(a) is a subring of R. Also verify the equality Cent R =

Q1/ (i) State and prove "Correspondence Theorem". **(12+8) marks**

(ii) Let I and J be two ideals of the ring R. Define "The right quotient of I by J, $I:_rJ$". Show that $I:_rJ$ is an ideal of R.

Q2/ **(8+12) marks**

(i) Show that the set of elements of the form ar+na is a right ideal of the ring R.

(ii) Let $I_1, I_2, \ldots, I_n$ be n ideals of a ring R. Show that $I_1+I_2+ \ldots +I_n = \langle I_1 \cup I_2 \cup \ldots \cup I_n \rangle$.

Q3/ (i) What is mean by **(12+8) marks**

(1) Internal direct sum (2) Let $\{I_i\}$ be an arbitrary indexed collection of ideals of a ring R, what is $\sum I_i$? (3) The evaluation homomorphism at a point. (4) n-fold sum of 1.

(ii) Let R be a commutative ring with unity. If every ideal of R is prime, then show that R is a field.

Q4/ (i) Suppose that R is an integral domain. Determine all the idempotent elements of R. **(6+6+8) marks**

(ii) Consider the map f : **C**→**C** such that f (a+ib) = a-ib. Is f a ring homomorphism or not ? Explain, where **C** = complex numbers.

(iii) Write "**True**" or "**False**" of the following:

(1) In a ring (R, +, .) with identity. No divisor of zero can posses a multiplicative inverse.

(2) Let (R, +, .) be a ring which has the property that $a^2 = a$, for every $a \in R$. Then, (R,+, .) is commutative ring.

(3) In an integral domain the zero element is the only idempotent.

(4) The ring of real numbers (**R**, +, .) is a simple ring.

(5) Given that (I, +, .) is an ideal of the ring (R,+, .). If (R,+, .) is a principal ideal ring, then so is the quotient ring (R / I,+, .).

(6) If R is any ring with identity 1, then R has characteristic n > 0 **iff** n is the positive integer for which n1 = 0.

(7) $I:_r (JK) = (I:_r K):_r J$, where I, J, K are ideals in R.

(8) If I is an ideal of the ring R and J is an ideal of I, then J is an ideal of R.

Q1/ (i) Let G be a group with subgroups H and K. Assume that **(1)** H and K are both normal in G. **(2)** $H \cap K = \{1\}$. **(3)** G = HK. Then, show that $G \cong H \times K$. (4+4) marks

(ii) What is mean by:

(1) The general linear group, GL_n **(2)** Algebraically closed field with some examples

(3) Exact sequence and short exact sequence **(4)** Descending decomposition

Q2/ (i) Show that two elements $\sigma, \sigma' \in S_X$ are conjugate **iff** $\lambda(\sigma) = \lambda(\sigma')$, where $\lambda(\sigma)$ and $\lambda(\sigma')$ are cycle type of σ and σ', respectively. (6+6) marks

(ii) Define "Center of a group". If $|G|$ is a power of a prime p, then show that G has nontrivial center.

Q3/ (i) Let H be a subgroup of a group G. Let $g \in G$. Show that (5+5) marks

(1) $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G. **(2)** $|gHg^{-1}| = |H|$.

(ii) Let G be a group, $a \in G$. Define $\sigma_a(x)$. Show that σ_a is an isomorphism.

Q4/ (i) Prove or disprove. (5+5) marks

Let (G, .) be a finite cyclic group of order n. Then, (G, .) ≅ ($\mathbf{Z_n}$, $+_n$), where $\mathbf{Z_n}$ is the set of integers modulo n.

(ii) Show that every quotient group of a cyclic group is cyclic.

Q5/ (i) What is the conditions on a group (G, *) to become an abelian group ? State (5) of them.

(ii) Write "**True**" or "**False**" of the following: (5+5) marks

(1) Given a and b are elements of a group (G, *), with a*b = b*a, then $(a*b)^k = a^k * b^k$, for some integer $k \in \mathbf{Z}$, where **Z** is the set of integers.

(2) Given G = {1, -1, i, -i}with $i^2 = -1$, then (G, .) is a cyclic group.

(3) Let (G, *) be a group and a, b ∈G. If a is of order n, then $a^i = a^j$ **iff** $i \equiv j \pmod{n}$.

(4) If the quotient group (G / Cent G, ⊗) is cyclic, then (G, *) is cyclic.

(5) If the equation $x^2 \equiv a \bmod (n)$ has a solution x_1, then $x_2 = n - x_1$ is also a solution.

Good Luck

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