| | | |
|---|---|---|
| **Salahaddin Univ.-Erbil** | **College of Science** | **First Semester** |
| **Dept. of Math.** | **(2023-2024)** | **PhD Exam.** |
| **Final Exam.** | **On Group Theory with Applications** | **Time: 4 hours** |

---

**Q1/ (i)** Let G be a group with subgroups H and K. Assume that **(1)** H and K are both normal in G. **(2)** $H \cap K = \{1\}$. **(3)** $G = HK$. Then, show that $G \cong H \times K$. (4+4) marks

**(ii)** What is mean by:

**(1)** The general linear group, $GL_n$ **(2)** Algebraically closed field with some examples

**(3)** Exact sequence and short exact sequence **(4)** Descending decomposition

**Q2/ (i)** Show that two elements $\sigma, \sigma' \in S_X$ are conjugate **iff** $\lambda(\sigma) = \lambda(\sigma')$, where $\lambda(\sigma)$ and $\lambda(\sigma')$ are cycle type of $\sigma$ and $\sigma'$, respectively. (6+6) marks

**(ii)** Define "Center of a group". If $|G|$ is a power of a prime p, then show that G has nontrivial center.

**Q3/ (i)** Let H be a subgroup of a group G. Let $g \in G$. Show that (5+5) marks

**(1)** $gHg^{-1} = \{ghg^{-1}: h \in H\}$ is a subgroup of G. **(2)** $|gHg^{-1}| = |H|$.

**(ii)** Let G be a group, $a \in G$. Define $\sigma_a(x)$. Show that $\sigma_a$ is an isomorphism.

**Q4/ (i)** Prove or disprove. (5+5) marks

Let (G, .) be a finite cyclic group of order n. Then, $(G, .) \cong (\mathbf{Z}_n, +_n)$, where $\mathbf{Z}_n$ is the set of integers modulo n.

**(ii)** Show that every quotient group of a cyclic group is cyclic.

**Q5/ (i)** What is the conditions on a group (G, *) to become an abelian group ? State (5) of them.

**(ii)** Write "**True**" or "**False**" of the following: (5+5) marks

**(1)** Given a and b are elements of a group (G, *), with a*b = b*a, then $(a*b)^k = a^k * b^k$, for some integer $k \in \mathbf{Z}$, where **Z** is the set of integers.

**(2)** Given $G = \{1, -1, i, -i\}$ with $i^2 = -1$, then (G, .) is a cyclic group.

**(3)** Let (G, *) be a group and $a, b \in G$. If a is of order n, then $a^i = a^j$ **iff** $i \equiv j \pmod n$.

**(4)** If the quotient group $(G / \text{Cent } G, \otimes)$ is cyclic, then (G, *) is cyclic.

**(5)** If the equation $x^2 \equiv a \bmod (n)$ has a solution $x_1$, then $x_2 = n - x_1$ is also a solution.

**Good Luck**

**Prof. Dr. Abdullah M. Abdul-Jabbar**