Salahaddin Univ.-Erbil **College of Science** **First Semester**
Dept. of Math. **(2023-2024)** **PhD Exam.**
First Trail **On Group Theory with Applications (Prof. Dr. Abdullah M.)** **Time: 2 hours**

Q1/ (i) Define "Center of a group". If $|G|$ is a power of a prime p, then show that G has nontrivial center. **(8+8) marks**

(ii) Let G be a group with subgroups H and K. Assume that

(1) H and K are both normal in G.

(2) $H \cap K = \{1\}$.

(3) G = HK. Then, show that $G \cong H \times K$.

Q2/ (i) What is mean by:

(1) The fiber of a function and the union of a fibers. **(2)** Exact sequences and short exact sequences. **(3)** Conjugate. **(4)** Abelianization of a group. **(5)** The second Isomorphism Theorem (Correspondence Theorem). **(10+8) marks**

(ii) State and prove "Cauchy Theorem".

Q3/ (i) Prove or disprove **(10+6+10) marks**

Let (H, .) be an infinite cyclic group. Then $H \cong \mathbf{Z}$, where **Z** is the set of integer numbers.

(ii) Show that there is a 1-1 and onto mapping between any two right cosets of H in G.

(iii) If G is a group in which $(ab)^i = a^i b^i$, for three consecutive integers i and any a, b in G, then show that G is commutative.

Q4/ (i) State and prove "Factorization of Homomorphism". **(8+6+6) marks**

(ii) Let H be a subgroup of a group G. Let $g \in G$. Show that

(1) $gHg^{-1} = \{ghg^{-1}: h \in H\}$ is a subgroup of G. **(2)** $|gHg^{-1}| = |H|$.

(iii) Write "**True**" or "**False**" of the following:

(1) Suppose $a^2 \equiv b^2$ mod n, where n is a prime number, then either $a \equiv b$ mod n or $a \equiv -b$ mod n.

(2) Let (H, *) be s subgroup of the group (G, *). Two elements a and b of G are congruent modulo H, written $a \equiv b$ (mod H) **iff** $a*b^{-1} \in H$.

(3) The positive integer n such that $g^n = 1$, for all $g \in G$ is called the exponent of G.

(4) Let G be a group and let H be a nonempty subset of G. Then $H \leqslant G$ **iff** $ab \in H$, for all $a, b \in H$.

(5) Given that $(H_1, *)$ and $(H_2, *)$ are both normal subgroups of the group (G, *), the subgroup $(H_1 \cap H_2, *)$ is also normal.

(6) Given (H, *) is a normal subgroup of the group (G, *), then the quotient group $(G / H, \otimes)$ is commutative whenever (G, *) is commutative.

Good Luck