زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Loss Distributions

Research Project
Submitted to the department of (mathematics)in partial fulfillment of the requirements for the degree of BSc in (mathematic)

Prepared By: Ismail Ghafur Muhammed
Supervised by: Dr. Awaz Kakamam Muhammad

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Certification of the supervisors:

I certify that this work was prepared under my supervision at the department of mathematics /college of education/Salahaddin university –Erbil in partial fulfilment of the requirements for the degree of bachelor of philosophy of science in mathematics.

Signature:

Supervisor: Dr. Awaz Kakamam Muhammad

Scientific grade: Lecturer

In view of the available recommendations, forward this work for debate by the examining committee.

Signature:

Name: Dr. Rashad R. Haji

Scientific grade: Assistant Professor

Chairman of the Mathematics Department

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Abstract.

This study is an attempt to describe Loss Distribution. In this project firstly, we provide a definition for Loss Distribution. We seek to find out the risk in insurance business. The differences between both the insurer and the policyholder are highlighted aiming at a deeper understanding of both insurer and the policyholder. The second aim of this project is to show that we have positive skewness and also discuss the distributions widely used for modelling loss in insurance. In doing that, those distributions that are used to Loss Distributions that are highlighted. Then, we come across to discuss about the types of distribution such us gamma, exponential, Pareto, normal, lognormal, Weibull and Burr.

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Introduction

Insurance is a way of managing risk by transferring the risk of financial loss to another party called the insurer or insurance company. The latter is a commercial enterprise, which makes money out of risk, selling policies to its clients called the policyholders. The policy is a contract between the insurer and the policyholder. Under that contract, the policyholder pays a price for a policy called premium to the insurer, in order to cover various specified adverse situations (fire destruction, road accident etc), in the even of which the company pays back to the policyholder a certain amount of money called the benefit. The claims for benefits cause a financial loss to the insurance company, so, to manage its finance effectively, the latter needs to be able to estimate various parameters of future loss, such as the frequency or the size of the claims. One of the methods used in insurance is the statistical study of similar events in the past to make a prognosis for the future, based on appropriate mathematical models.

The quantities of interest are regarded as random variables, hence the most important problem is to determine their probability distribution. This can be done using the past observations and approximating them by a suitable density function, as in Figure 1. This function is further used in mathematical modelling, that allows the insurer to estimate possible loss and derive the premiums for various types of policyholders. A widespread method used in practice is an approximation of the empirical histogram within a given family of model distributions depending on a set of parameters. The task is then to determine the value of parameters using such techniques as the method of moments or the maximal likelihood.

From experience, the loss samples exhibit fat-tail behaviour and typically have positive skewness (asymmetric around the mean).

This project contains two chapters. In chapter one, we discuss the distributions widely used for modelling loss in insurance. We also calculate probabilities and moments of the loss distributions both with and without limits and risk-sharing arrangements. In particular, we derive moments and moment generating functions (where defined) of the gamma, exponential, Pareto, generalized Pareto, normal, lognormal, Weibull Burr loss distributions. We have explanation and example for each one.

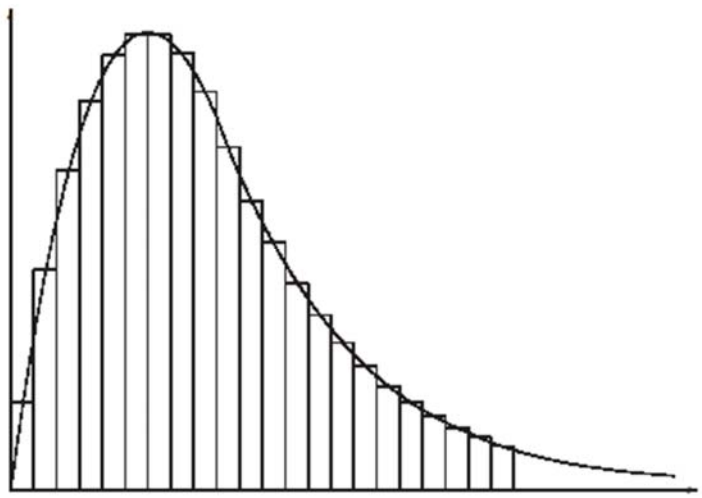

Figure 1: Approximation of empiric distribution histogram

You will be familiar with a few of these distributions already. We apply the principles of statistical inference to select suitable loss distributions for sets of claims.

In chapter two, we talk about applications to reinsurance. We explain the concepts of excesses, and retention limits. We describe the operation of simple forms of proportional and excess of loss reinsurance. We derive the distribution and corresponding moments of the claim amounts paid by the insurer and the reinsurer in the presence of excesses and reinsurance.

Chapter One:

Basic distributions

In this chapter, we present a summary of the most important probability distributions used in various areas of economics and finance, including actuarial science. The properties of interest to us are the density and distribution functions, the moments and the mgf.

Definition (Continuous random variable):

The distribution function F_X of a random variable X is called **absolutely continuous** if there is a positive function f_X, such that $\int_R \; f_X(t)\,dt = 1$, and F(x) = $\int_{-\infty}^{x} f_X(t)dt.$ We shall call such X continuous random variables, and f_X is said to be its ***probability density function***.

1.1 Normal distribution:

Definition: The normal distribution N (μ, σ 2) is completely characterized by two parameters μ ∈ R, and σ > 0.
The density function is

$$\boldsymbol{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,, x \in R}$$

The distribution $N(\mu, \sigma 2)$ can be obtained from the distribution N(0, 1) via the substitution x → $\frac{x-\mu}{\sqrt{2}\sigma}$.
The mean value of $N(\mu, \sigma 2)$ is μ, and the density function is symmetric around the mean. The cumulative distribution function Erf(x) is the integral of the density and it cannot be written as an elementary function.

The normal distribution is very important for various applications; however, it is seldom used for loss modelling as it is symmetric. Also, the density function decays faster than any power of x, and this is another reason why the normal distribution is not suitable as a loss distribution.

Example: Compute the moment generating function of N(μ, σ2)

$$MGF = M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-[(x-\mu)^2-2\sigma^2 tx]}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-[(x-\mu)^2-2\sigma^2 tx]}{2\sigma^2}} e^{\mu t+\frac{\sigma^2 t^2}{2}} e^{-\mu t-\frac{\sigma^2 t^2}{2}} dx$$

$$E(e^{tx}) = \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-[(x-\mu)^2-2\sigma^2 t(x-\mu)+\sigma^4 t^2]}{2\sigma^2}} dx$$

$$= \frac{e^{t\mu+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-[(x-\mu)-\sigma^2 t]^2}{2\sigma^2}} dx$$

Therefore, $M_t(x) = E(e^{tx}) = e^{t\mu+\frac{\sigma^2 t^2}{2}}$.

IS Normal distribution good model?

Normal distributions do not realistically model loss distributions because

- they are symmetric about the mean;
- the tails, ie P(X > x) decays faster than any power of x,
  x^nP(X > x) → 0, as x → ∞ for all n.

1.2 Log-normal distribution:

Definition.

The random variable Y is said to have the log-normal distribution with parameters μ ∈ R and $\sigma^2 > 0$, if log(Y) ~ N(μ, σ^2).
The density function log-normal distribution log N(μ, σ^2) can be obtained from N(μ, σ^2) by the substitution x → log x

$$f(x)=\frac{1}{\sqrt{2\pi}x} e^{\frac{-(\log(x)-\mu)^2}{2\sigma^2}}, \quad x > 0.$$

The density of the log-normal distribution

Suppose we have the density of X and we want to find the density of Y = g(X). g must be monotonic. In that case we have

$$f_Y(y) = \left|\frac{1}{\acute{g}(g^{-1}(y))}\right| f_X\big(g^{-1}(y)\big).$$

In our case Y = e^x, where X ~ N(μ,σ^2), and thus

$$f_Y(y) = \left|\frac{1}{e^{\ln(y)}}\right| \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(y)-\mu)^2/2}{2\sigma^2}\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}y} \exp\left(-\frac{(\ln(y)-\mu)^2/2}{2\sigma^2}\right)$$

Log-normal distribution:

- Skewness positive–not symmetric;
- but tails still rapidly decaying, ie for all α > 0 the product f(x)x^a tends to zero as x tends to infinity.
- The moment generating function cannot be explicitly integrated.
- Mean is $e^{\mu+\sigma^2/2}$
- variance is $(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$

1.3 Exponential Distribution:

The cumulative probability function of the exponential distribution E(λ), with parameter λ > 0, is

$$P[T \le t] = 1 - e^{-\lambda t}, \ x \ge 0.$$

This corresponds to the density function.

$$f(x, \tfrac{1}{\lambda}) = \lambda e^{-\lambda x}, \quad Or \quad f(x,\lambda) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}, \ X \ge 0$$

- Mean is $\frac{1}{\lambda}$.
- variance is $\frac{1}{\lambda^2}$.
- Moment generating function

$$MGF(t) = \int_0^{+\infty} \lambda e^{-\lambda x} e^{tx}\, dx = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda.$$

1.4 Gamma Function:

Some facts about Gamma function

- For α > 0 we have
- $\Gamma(\alpha) = \int_0^{\infty} t^{a-1} e^{-t} dt.$
- For n a positive integer
- $\Gamma(n) = (n - 1)!$
- Also for all α > 0
- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$

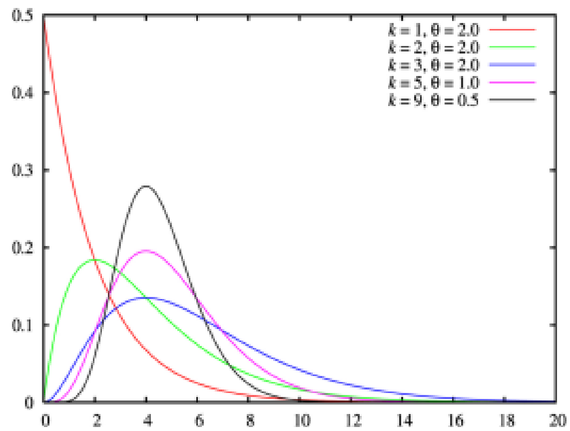

Figure 2: Pdf for different parameter choices. *Source: wikipedia*

1.4.1 Gamma distribution:

i. The Gamma distribution Γ (α, β) depends on two positive parameters α, β and has the density function

$$f(x) = \frac{\beta^{\propto}}{\Gamma(x)} e^{-\beta x} x^{\propto-1} .$$

ii. When α = 1, it degenerates to the exponential distribution $\varepsilon(\beta)$. The cumulative distribution can be expressed through elementary functions provided α is integer.

iii. The mean and variance of Γ (α, β) are $\frac{\propto}{\beta}$ and $\alpha \frac{\propto}{\beta^2}$

iv. MGF of Gamma

$$\text{MGF} = \left(\frac{\beta}{\beta - t}\right)^{\propto}, \quad \text{for } t < \beta.$$

v. Remark that the gamma distribution has positive skewness but light tails, as the density function is rapidly descending

1.5 Cauchy Distribution:

The Cauchy distribution is an example of a heavy tailed distribution and has the density function

$$f(x) = \frac{\theta}{\pi[\theta^2 + (x - M)^2]} \quad , x \in R \, ,$$

where $\theta > 0$ and $M \in R$ are the parameters.

- For large $x > 0$, $P(X > x) \sim C/x$—FAT TAILS.
- In fact, tails are so fat that no moments(of order ≤ 1 are defined).
- However it is symmetric around its median, which is equal to M.
- Remember: mean does NOT exist, variance does NOT exist, MGF does NOT exist.

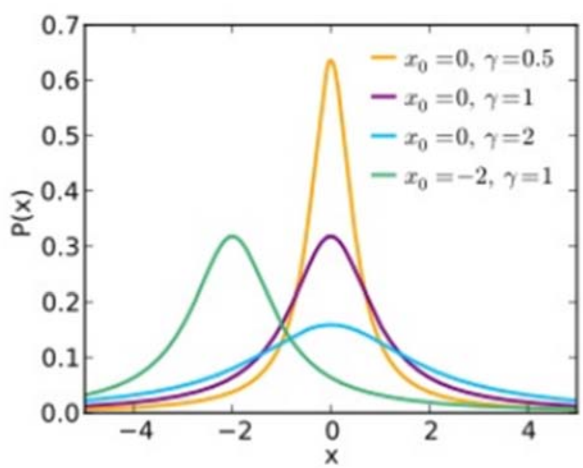

Figure 2: Pdf for different parameter choices. *Source: wikipedia*

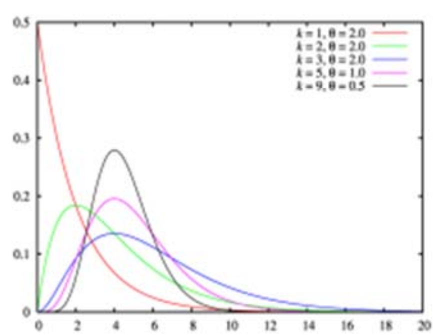

Figure 3: Pdf for different parameter choices. *Source: wikipedia*

1.6 Pareto distribution:

A random variable X follows Pareto distribution PA(α,b) if its probability density function is

$$f(x|a,b) = \frac{ba^b}{(x)^{b+1}}, \quad x \geq a > 0, b > 0.$$

where the parameters α and b are positive real numbers.

The cumulative distribution function is

$$F(x) = P[X < x] = 1 - \frac{a^b}{x^b}, x \geq a.$$

- the n-th moment is finite only if α > n.
- mean $\frac{ab}{b-1}, b > 1.$
- Variance $\frac{ba^2}{(b-1)^2(b-2)}$, b > 2
- Since it cannot possibly have all moments, MGF defined only

on negative half-axis.

1.7 Weibull distribution.

The Weibull cumulative probability distribution is

$$F(x) = P[X \leq x] = 1 - e^{-\lambda xy},$$

Density function is

$$f(x) = \frac{Yx^{Y-1}}{\lambda^Y} e^{-\left(\frac{x}{\lambda}\right)^Y}, x \geq 0,$$

where λ and Y are positive real numbers.

When γ = 1 exponential; lighter tail than E(λ) if γ < 1 and thicker if γ > 1.

- Mean: $\lambda \Gamma(1 + 1/Y)$
- Variance: $\lambda^2\Gamma(1 + 2/Y) - [\Gamma(1 + 1/Y)]^2$

For the moments we change variables

$$\mathbf{y=\lambda x^{\gamma}, \ dy=\lambda\gamma x^{\gamma-1}dx \ , \ x=\lambda^{-\frac{1}{\gamma}} y^{\frac{1}{\gamma}}}$$

$$\text{Mn} = \lambda \gamma \int_0^{\infty} x^n x^{\gamma-1} e^{-\lambda x\gamma} dx = \lambda\gamma \int_0^{+\infty} x^{n+\gamma-1} e^{-\lambda x\gamma} dx$$

$$= \lambda\gamma \int_0^{+\infty} (\lambda^{\frac{-1}{\gamma}} y^{\frac{1}{\gamma}})^{n+\gamma-1} e^{-y} (\lambda\gamma)^{-1} (\lambda y)^{\frac{\gamma-1}{\gamma}} dy$$

$$= \lambda^{-\frac{n}{\gamma}} \int_0^{+\infty} y^{\frac{n}{\gamma}} e^{-y} dy = \lambda^{-\frac{n}{\gamma}} \Gamma(\frac{n}{\gamma} + 1).$$

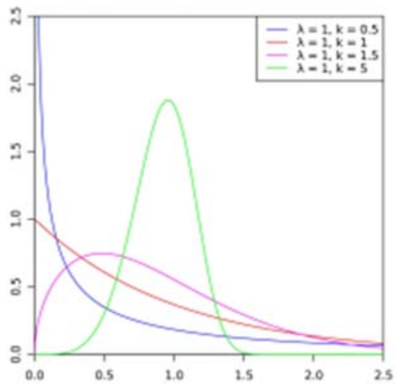

Figure 4: Pdf of Weibull for different parameters

1.8 Burr distribution:

The Burr distribution can be viewed as a one parameter extension of the Pareto distribution and has the cumulative probability function

$$F(x) = P[X \leq x] = 1 - (\frac{\lambda}{\lambda + x\gamma})^{\propto}.$$

The Pareto case is obtained by the specialization
$\gamma = 1$.

The density function is

$$f(x) = \propto \left(\frac{\lambda}{\lambda+x\gamma}\right)^{\propto-1} \frac{\lambda}{(\lambda+x\gamma)^2} \gamma x^{\gamma-1}$$

$$= \frac{\propto\gamma}{\lambda} (\frac{\lambda}{\lambda+x\gamma})^{\propto+1} x^{\gamma-1}.$$

Its asymptotic behaviour at large x is $\sim \frac{\propto\gamma\lambda^{\propto}}{x^{\propto\gamma+1}}$;

So it decays more rapidly than the Pareto density function if γ > 1 and vice versa.

The moment generating function does not exist as in the case of Pareto distribution. The n-th moment exists if and only if αγ > n.

Some Examples

For log-normal distribution

The random variable Y=log X N(10,4) distribution , Find

(a) The pdf of X
(b) Mean & Variance of X
(c) P(X$\leq$ 1000)

Part (a)

$$f(x) = \begin{cases} \frac{1}{2x\sqrt{2\pi}} e^{\left(-\frac{(lnx-10)^2}{8}\right)} & x \geq 0 \\ 0 & otherwise \end{cases}$$

Part(b)

$$\begin{aligned} E(X) &= e^{\left(\mu_Y+\frac{\sigma_Y{}^2}{2}\right)} \\ &= e^{10+(\frac{4}{2})} \\ &= e^{12} \\ &\approx 162.754 \end{aligned}$$

$$\begin{aligned} Var(X) &= [e^{\sigma_Y{}^2} - 1]\, e^{(2\mu_Y+\sigma_Y{}^2)} \\ &= (e^4 - 1)\, e^{(20+4)} \\ &= (e^4 - 1)\, e^{(24)} \\ &\approx 53.598 \end{aligned}$$

Part (c)

$$\begin{aligned} P[X\leq 1000] &= P[\log X \ \leq \log 1000] \\ &= P[Y \leq \log 1000] \\ &= P\left[Z \leq \frac{\log 1000-10}{2}\right] \\ &= P(Z \leq -1.55) \\ &= \Phi(-1.55) \end{aligned}$$

For Pareto Distribution

To compute probabilities: Enter the values of the parameters a, b, and x; [P(X <= x)].

Example When a = 2, b = 3, and the value of x = 3.4,Find

(a) Pdf of x

(b) Mean & variance

(c) $P(X \leq 3.4)$ and $P(X > 3.4)$

Part (a)

$$f(x) = \begin{cases} \frac{24}{(x)^4}, & x \geq 2 > 0, 3 > 0 \\ 0 & otherwise \end{cases}$$

Part (b)

$$E(x) = \frac{2\times3}{2}$$

$$= 3$$

$$Var(x) = \frac{3\times2^2}{(3-1)^2(3-2)}$$

$$= \frac{12}{4}$$

$$= 3$$

Part (c)

$P(X \leq 3.4) = 0.796458$ and

$P(X > 3.4) = 0.203542$.

For Weibull distribution.

To compute probabilities: Enter the values of m, c, b, and the cumulative probability; click [P(X <= x)].

Example : When m = 0, Y = 2.3, λ = 2, and x = 3.4,

(a) Pdf of x

(b) Mean & variance

(c) P(X ≤ 3.4) and P(X > 3.4)

Part(a)

$$f(x) = \frac{2x^{1.3}}{2^{2.3}} e^{-\left(\frac{x}{2}\right)^{2.3}} \quad , x \geq 0$$

Part(b)

$$E(x) = 2.3\, \Gamma(\tfrac{3}{2})$$

$$V(x) = 5.29\, \Gamma(2) - \left(\Gamma(\tfrac{3}{2})\right)^{2.3}$$

Part(c)

$$P(X \leq 3.4) = 0.966247 .$$

$$P(X > 3.4) = 0.033753.$$

Statistical Inference:

Methods of Estimation

We shall describe here two classical methods of estimation, namely, the moment estimation and the method of maximum likelihood estimation. Let $X_1, \dots, X_n$ be a sample of observations from a population with the distribution function F(x|$\theta_1, \dots, \theta_k$), where $\theta_1, \dots, \theta_k$ are unknown parameters to be estimated based on the sample.

Moment Estimation

Let $f(x|\theta_1, \dots, \theta_k)$ denote the pdf or pmf of a random variable X with cdf $F(x|\theta_1, \dots, \theta_k)$. The moments about the origin are usually functions of $\theta_1, \dots, \theta_k$.

Notice that $E(X_i^k) = E(X_i^k)$, i = 2, . . . , n, because the X_i 's are identically distributed. The moment estimators can be obtained by solving the following

system of equations for $\theta_1, \dots, \theta_k$

$$\frac{1}{n}\sum_{i=1}^{n} X_i = E(X_1)$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E(X_1^2)$$

.

.

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k = E\left(X_1^k\right),$$

Where

$$E\left(X_1^j\right) = \int_{-\infty}^{\infty} x^j f(x|\,\theta_1, \dots, \theta_k)\, dx\,,\;\; j = 1,2, \dots., k.$$

Maximum Likelihood Estimation:

For a given sample x = $(x_1, \ldots, x_n)$, the function defined by

$$L(\theta_1, \ldots, \theta_k | x_1, \ldots, x_n) = \prod_{i=1}^{n} f(\, x_i | \theta_1, \ldots, \theta_k)$$

is called the likelihood function. The maximum likelihood estimators are the values of $\theta_1, \ldots, \theta_k$ that maximize the likelihood function.

Chapter Two

2.1 Applications to reinsurance:

Like any participant within the financial market and like any of its customers, an insurance company is subject to a risk, specifically, the risk of large claims. Unexpectedly large claims may destroy the finance of the company, so it might also want to protect itself by sharing this risk with another insurance company. Then it becomes itself a policyholder. Such a strategy is called reinsurance. Thus, the original gross premium and gross claim amount for the direct insurer are reduced as a result of reinsurance. The actual premium gained and claim paid off by the direct insurer are called respectively net premium and net claim amount.

There are two major types of reinsurance, proportional and non-proportional. The proportional reinsurance implies that the reinsurer covers a percentage of each claim. As a consequence, the insurer pays to the reinsurer a proportion of each premium from each policy, typically, the same percentage as that of the claim covered by the reinsurer. If the proportion of the premium is the same for all risks, it is called quota share reinsurance. If the proportion varies from risk to risk, it is called surplus reinsurance.

In the non-proportional insurance, the claim amount is divided into intervals (layers), and the reinsurer is liable to cover a certain amount of claim, once it falls into a particular layer. Typically, the reinsurer covers the claim above certain level called retention limit, and maybe below upper level. There are two possible strategies to impose the retention limit: either on each individual claim or on the total claim amount of a specified group of policies. The first case is called individual excess of loss while the second is stop loss reinsurance.

We consider the two types of reinsurance in some detail. Let X be the amount of claim to be paid off; let Y denote the claim amount paid by the direct insurer. It is convenient to introduce the variable $\tilde{Y} = X - Y$, which has the meaning of the claim amount paid by the reinsurer.

2.2 Excess of loss:

In this type of reinsurance there is a level U settled for each individual claim, above which the excess is covered by the reinsurer. By definition,

Y = min{X, U}, $\tilde{\boldsymbol{Y}}$ = max{X, U} − U. …….. (1)

Let f(x) be the probability density function for X. Then

$$E[Y] = \int_0^\infty \min\{X, U\}\, f(x)dx = \int_0^U x\, f(x)dx + U\int_U^\infty f(x)dx\,,$$

$$E[\tilde{Y}] = \int_0^\infty (\max\{x, U\} - U) f(x)dx = \int_U^\infty (x - U) f(x)dx.$$

The means are calculated with respect to the distribution f(x). Since Y + $\tilde{Y}$= X, we have E[Y] + E[$\tilde{Y}$] = E[X]. One consequence of reinsurance is that E[Y] ≤ E[X] As E[X] is the mean claim amount without reinsurance, the latter allows the direct insurer to reduce risk by E[$\tilde{Y}$]. The moment generating function of the insurer is

$$MGF = E[e^{tY}] = \int_0^U e^{tx} f(x)dx + e^{tU}\int_U^\infty f(x)dx.$$

As usual, the discrete case is obtained by replacing integrals by summations.

The reinsurance company comes into play only if the claim amount exceeds U. Hence its probability of claim events is conditional given X>U. Using standard rules of computing conditional probability distributions, we find the density function of the reinsurer:

$$\tilde{f}(x) = \frac{f(x)}{P[X > U]} I_{[U,+\infty]},$$

where $P[X > U] = \int_U^{+\infty} f(x)dx$ (in discrete case the density function is replaced by probabilities). The portion of claim borne by the reinsurer is $\tilde{Y}$. Its expected value is then equal to

$$E[\tilde{Y}] = \frac{\int_U^{+\infty}(x-U)f(x)dx}{\int_U^{+\infty} f(x)dx} = \frac{\int_0^{+\infty} xf(x+U)dx}{\int_0^{+\infty} f(x+U)dx}.$$

The moment generating function reads

$$MGF = E\left[e^{t\tilde{Y}}\right] = \frac{\int_U^{+\infty} e^{t(x-U)}f(x)dx}{\int_U^{+\infty} f(x)dx}$$

$$= \frac{\int_0^{+\infty} e^{tx}f(x+U)dx}{\int_0^{+\infty} f(x+U)dx}.$$

2.3 Proportional reinsurance

In the proportional case the direct insurer pays a fixed proportion α of every claim:

$$Y = \alpha X,$$

$$\tilde{Y} = (1-\alpha)X,$$

The coefficient α is called retention factor.

The probability density functions of the direct insurer and reinsurer are found by simple rescaling:

$$g(x) = \frac{1}{a} f\left(\frac{x}{a}\right),$$

$$\tilde{g}(x) = \frac{1}{1-a} f\left(\frac{x}{1-a}\right).$$

As usual, the discrete case is obtained by replacing integrals by summations.

Example: We illustrate the present section on the example of exponential distribution, that is, $f(x) = \lambda e^{-\lambda x}$.

In this case, the moment generating functions of the direct insurer and reinsurer are

$$E[e^{tY}] = \lambda \int_0^U e^{tx} e^{-x\lambda} dx + \lambda e^{tU} \int_U^\infty e^{-x\lambda} = \frac{1-e^{-U(\lambda-t)}}{1-\frac{t}{\lambda}} + e^{-U(\lambda-t)},$$

$$E\left[e^{t\tilde{Y}}\right] = \lambda \frac{\int_0^{+\infty} e^{t(x)} e^{-(x+U)\lambda} dx}{\int_0^{+\infty} \lambda\, e^{-(x+U)\lambda} dx} = \frac{\int_0^{+\infty} e^{-x\lambda+tx} dx}{\int_0^{+\infty} e^{-x\lambda} dx} = \frac{\lambda}{\lambda - t}.$$

It is interesting to note that the generating function of the reinsurer is independent of U.

2.4 Excesses

Sometimes the insurer establishes a lower limit L for the claim X to be covered. In that case only the part

$$\max\{X, L\} - L$$

is paid to the policyholder. Comparing this with (1), we conclude that the insurer acts like a reinsurer with replacement of U by L. Hence all the methods developed above are applicable. The insurance with excesses is often applied in the situation when there is a significant probability of minor risk, like in car insurance, and the numerous financially insignificant claims are difficult to operate. It is obvious, that excesses should reduce the premium paid by policyholders.

Question:

A risk has a Pareto distribution P a(α, c). Assuming deductible L, derive the expected claim amount.

Answer:

A lower limit L $\qquad Y = \max\{X, L\} - L$

$$f(x|a, c) = \frac{ac^a}{(c+x)^{a+1}}, \quad x \geq c > 0, a > 0.$$

$$E[Y] = \int_0^{\infty} (\max\{X, L\} - L) \frac{ac^a}{(c+x)^{a+1}}\, dx$$

$$= \int_L^{\infty} (x - L) \frac{ac^a}{(c+x)^{a+1}}\, dx$$

$$= \int_L^{\infty} (x - L) \frac{ac^a}{(c+x)^{a+1}}\, dx$$

$$= ac^a \int_L^\infty \frac{(x- L)}{(c+x)^{a+1}}\, dx$$

Let $c + x = t$, $dx = dt.$

$$= ac^a \int_L^\infty \frac{t - c - L}{(t)^{a+1}}\, dt$$

$$= ac^a \int_L^\infty \left(\frac{t}{(t)^{a+1}} - \frac{c}{(t)^{a+1}} - \frac{L}{(t)^{a+1}}\right) dt$$

$$= ac^a \int_L^\infty \frac{1}{t^a} dt - \int_L^\infty \frac{c}{(t)^{a+1}} - \int_L^\infty \frac{L}{(t)^{a+1}}\ dt$$

Use $\frac{1}{x^n} dx = -\frac{1}{(n-1)x^{n-1}}$, $n \neq 1$

$$= ac^a \left(-\frac{1}{at^{a-1} - t^{a-1}} + \frac{c}{at^a} + \frac{L}{at^a}\right)\Big|_L^\infty$$

Subs $c + x = t$,

$$= ac^a \left(-\frac{1}{a(c+x)^{a-1} - (c+x)^{a-1}} + \frac{c}{a(c+x)^a} + \frac{L}{a(c+x)^a}\right)\Big|_L^\infty$$

$$= ac^a \left(\frac{c+L}{a(c+x)^a} - \frac{1}{(a-1)(c+x)^{a-1}}\right)\Big|_L^\infty$$

$$= -ac^a \left(\frac{c+L}{a(c+L)^a} - \frac{1}{(a-1)(c+L)^{a-1}}\right)$$

$$= -ac^a \left(\frac{1}{a(c+L)^{a-1}} - \frac{1}{(a-1)(c+L)^{a-1}}\right)$$

$$= \frac{-ac^a}{(c+L)^{a-1}} \left(\frac{1}{a} - \frac{1}{(a-1)}\right)$$

$$= \frac{c^a}{(c+L)^{a-1}(a-1)}$$

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پوختە

ئەم توێژینەوەیە هەوڵێکە بۆ وەسفکردنی دابەشکردنی زیان. لەم پرۆژەیەدا سەرەتا پێناسەیەک بۆ دابەشکردنی زیانەکان دەخەینەڕوو. ئێمە هەوڵدەدەین مەترسییەکان لە بازرگانی بیمەدا بزانین. جیاوازییەکانی نێوان هەردوو بیمەدەر و خاوەن بیمە بە ئامانجی تێگەیشتنێکی قووڵتر لە هەردوو بیمەدەر و خاوەن بیمەکە تیشک دەخرێنە سەر. ئامانجی دووەمی ئەم پرۆژەیە ئەوەیە کە نیشان بدەین کە ئێمە چەقبەستوویی ئەرێنیمان هەیە و هەروەها باس لەو دابەشکردنانە دەکەین کە بە شێوەیەکی بەرفراوان بۆ مۆدێلکردنی زیان لە بیمەدا بەکاردەهێنرێن. لە ئەنجامدانی ئەوەدا ئەو دابەشکردنانەی کە بە دابەشکردنی زیانەکان ڕاهاتوون کە تیشکیان خراوەتە سەر. پاشان، ئێمە دێینە سەر بۆ باسکردن لەسەر جۆرەکانی دابەشکردن وەک:

gamma, exponential, Pareto, normal, lognormal, .Weibull and Burr