Chapter Two

Probability Theory

Random Experiment: any experiment when the results are unknown is called random experiment and it is denoted by R.E.

Sample Space: Is the set of all possible outcomes of a random experiment and denoted by S.

Example: R.E. In the toss of a coin, let the outcome tails be denoted by T and let the outcome heads be denoted by H. $S = \{T, H\}$.

Example: R.E. In the toss of a die twice, the sample space consists of the 36 ordered pairs:

$S = \{(1,1), \dots, (1,6), (2,1), \dots, (2,6), \dots, (6,6)\}$.

Events: Any subset of sample space is called event and denoted by E. There are two types of event: simple event and compound event.

Probability:

Suppose that an event E can happen in h ways out of a total of n possible equally likely ways. Then the probability of occurrence of the event (called its *success*) is denoted by

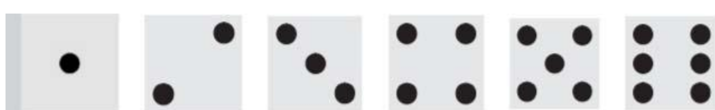
$$p = P_r\{E\} = \frac{h}{n}$$

The probability of nonoccurrence of the event (called its *failure*) is denoted by

$$q = P_r\{notE\} = 1 - \frac{h}{n} = 1 - p$$

Thus $p + q = 1, or$ $\Pr\{E\} + P_r\{notE\} = 1$.

Example: When a die is tossed, there are 6 equally possible ways in which the die can fall:



The event E, that a 3 or 4 turns up, is: $P_r\{E\} = \frac{2}{6} = \frac{1}{3}$. $\quad P_r\{notE\} = 1 - P_r\{E\} = 2/3$

Example: toss a coin twice. $S = \{HH, TT, HT, TH\}$

$P(E_1) = P(HH) = \frac{1}{4}$, $P(E_2) = P(TT) = \frac{1}{4}$, $P(E_3) = P(HT) = \frac{2}{4}$.

Example: toss a coin three times. Write a sample space and find the probability for each event.

Definition: (***probability function)***: Let S be a sample space and let B be the set of events. Let P be a real-valued function defined on B ($P{:}B \to [0,1]$). Then P is a probability set function if P satisfies the following three conditions:

1) $0 \leq P(A) \leq 1,\ for\ all\ A\ \in\ B.$
2) $P(S) = 1.$
3) $If\ \{E_n\}\ is\ a\ sequence\ of\ events\ in\ B\ where\ \ E_m\ \cap\ E_n\ =\ \emptyset\ for\ all\ m\ \neq n,$ then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

Theorem 1: For each event $E\ \in\ B,\ P(E) = 1\ -\ P(E^c).$

Prove: We have $S\ =\ E\ \cup\ E^c$ and $E \cap E^c = \emptyset$, by def. of probability condition 2 and 3, we get

$1 = P(E) + P(E^c), then\ P(E) = 1\ -\ P(E^c).$

Theorem 2: The probability of the null set is zero; that is , $P(\emptyset) = 0$

Proof: By above theorem, take A = φ so that A^c = S.

Accordingly, we have P(φ)=1 − P(S)=1 − 1=0 .

Second method: Let A be any event on S. $E \cap \emptyset = \emptyset\ \ and\ E \cup \emptyset = E$

Then $P(E \cup \emptyset) = P(E)$

$P(E) + P(\emptyset) = P(E), then\ P(\emptyset) = 0$

Theorem 3: For each event $E\ \in\ B,\ P(E) = 1\ -\ P(E^c).$

Prove: We have $S\ =\ E\ \cup\ E^c$ and $E \cap E^c = \emptyset$, by def. of probability condition 2 and 3, we get

$1 = P(E) + P(E^c), then\ P(E) = 1\ -\ P(E^c).$

Theorem 4: If A and B are events such that A ⊂ B, then P(A) ≤ P(B).

Proof: Now B = A ∪ (A^c ∩ B) and A ∩ (A^c ∩ B) = ∅.

Hence, from condition (3) of Definition probability, P(B) = P(A) + P(A^c ∩ B).

From condition (1) of Definition probability, P(A^c ∩ B) ≥ 0.

Hence, P(B) ≥ P(A).

Example: Toss a die one time. then S={1, 2, 3, 4, 5, 6}

A=the occurrence number is 3 ={3} , , $P(A) = \frac{1}{6}$

B= the occurrence numbers are odd ={1, 3, 5} , $P(B) = \frac{3}{6}$

$$\mathrm{A}\ \subset\ \mathrm{B}, \text{then } \mathrm{P(A)} <\ \mathrm{P(B)}$$

Theorem 5: For each $A \in B, 0 \le P(A) \le 1$.

Proof: Since $\varphi \subset A \subset S$, we have by Theorem 4 that $P(\varphi) \le P(A) \le P(S) or$

$$0 \le P(A) \le 1,$$

Theorem 6: If A and B are events in S such that $A \cap B \neq \emptyset$,

then P(A ∪ B) = P(A) + P(B) − P(A ∩ B).

Proof: Each of the sets A ∪ B and B can be represented, respectively as follows:

$$A \cup B = A \cup (A^c \cap B) \ \ and \ \ B = (A \cap B) \cup (A^c \cap B).$$

$$P(A \cup B) = P(A) + P(A^c \cap B) \, and$$

$$P(B) = P(A \cap B) + P(A^c \cap B).$$

If the second of these equations is solved for P(A^c∩B) and this result is substituted in the first equation, we obtain $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example: Toss a die one time. then S={1, 2, 3, 4, 5, 6}

A={1, 3, 5} $P(A) = \frac{3}{6}$, B= { 3, 6} , $P(B) = \frac{2}{6}$

$A \cap B = \{3\}, then \ P(A \cap B) = \frac{1}{6}$

$A \cup B = \{1, 3, 5, 6\}, \qquad then \ P(A \cup B) = \frac{4}{6}$

So by above theorem, we obtain $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6}$$

Counting Rules

We discuss two major counting rules that are usually discussed in an elementary algebra course.

Permutation: the number of deferent arrangement (order) of r things formed out of n things is called Permutation. Or is the number of ways that we can select r things from n things with order and denoted by P_r^n and is given by

$$P_r^n = n(n-1)\cdots\big(n-(r-1)\big) = \frac{n!}{(n-r)!}, \quad r \le n$$

Example: how many number of selecting two letters from letters A, B, and C?

$$P_r^n = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 6$$

Example: 1. how many 3-digit number can be formed from 3, 4, 6, 8, 5, 1 (without replacement)?

2. how many 3-digit number can be formed from 3, 4, 6, 8, 5, 1 (with replacement)?

Rule 1: if $r = n$ then the definition is still true. Then number of n units taken n time is

$$n! = P_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Rule 2: if n objects are to be distributed on k cells such that n_1 objects to first cells, n_2 is objects to second cell, and n_k objects to last cell. Then we can do that by

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1 * n_2 * \cdots * n_k} \text{ time such that } n = n_1 + n_2 + \cdots + n_k$$

Example: how many words can be obtained from the word "Department"?

Combination: is unorder selection of k things taken from a set of n things which is denoted by

$$\binom{n}{k} = C_k^n = \frac{n!}{k!\,(n-k)!}$$

Example: how many ways we can select a committee of 4 person from 8 person?

$$C_4^8 = \frac{8!}{4!(8-4)!} = 70 \; ways$$

It is interesting to note that if we expand the binomial series,

$(a + b)^n = (a + b)(a + b) \cdots (a + b)$, we get

$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, because we can select the k factors from which to take a in $\binom{n}{k}$ ways. So $\binom{n}{k}$ is also referred to as a binomial coefficient.

Conditional probability and Independence:

let A and B be two events such that $P(A) \neq 0$. Then the probability of event B given the knowledge of event A is denoted by

$$P(B|A) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(A)}.$$

From the definition of the conditional probability set function, we observe that

$$P(\mathrm{A} \cap \mathrm{B}) = P(A)P(B|A)$$

Definition: Two events A and B are ***independent*** if the knowledge of the occurrence of one event does not effect of the other event occurring. That is,

Let A and B be two events. We say that A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

It means, when P(A) > 0, P(B|A) = P(B).

Example: A bowl contains ten balls. Five of the balls are red, two of the ball are white, and the remaining three are blue. If we select two balls, what is the probability that the second ball is red given the knowledge that the first ball is red too?

If we select two balls, what is the probability that the second ball is white given the knowledge that the first ball is red too? (H.W)

Solution: $P(B|A) = \frac{P(A\cap B)}{P(A)}$, Let $A = P(1Red) = \frac{5}{10}$, $P(\text{A and B}) = \frac{5}{10}.\frac{4}{9}$.

$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{5}{10}.\frac{4}{9}}{\frac{5}{10}} = \frac{4}{9}$.

Example: A standard deck of playing cards consists of 52 cards :

- Four suits : Hearts , Diamonds (red) , and Spades , Clubs (black) .
- Each suit has 13 cards, whose denomination is 2 , 3 , · · · , 10 , Jack , Queen , King , Ace
- The Jack , Queen , and King are called face cards .

Suppose we draw a card from a shuffled set of 52 playing cards.

1- What is the probability of drawing a Queen, given that the card drawn is of suit Hearts ?
2- What is the probability of drawing a Queen, given that the card drawn is a Face card ?

Answer: 1- $P(Q|H) = \frac{P(Q\cap H)}{P(H)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}$

2- $P(Q|F) = \frac{P(Q\cap F)}{P(F)} = \frac{P(Q)}{P(F)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3}$. $\quad (Here\ Q \subset F\ , so\ that\ Q \cap F = Q\ .)$

Example: (H.W) A hand of five cards is to be dealt at random without replacement from an ordinary deck of 52 playing cards. The conditional probability of an all spade hand (B), relative to the hypothesis that there are at least four spades in the hand (A), is,

since $A \cap B = B$

Example: A bowl contains eight chips. Three of the chips are red and the remaining five are blue. Two chips are to be drawn successively, at random and without replacement. We want to compute the probability that the first draw results in a red chip (A) and that the second draw results in a blue chip (B). It is reasonable to assign the following probabilities:

$$P(A) = \frac{3}{8} \quad and \quad P(B|A) = \frac{5}{7}$$

we $have\ P(A \cap B) = (3\ 8) \times (5\ 7) = 15/56$ 0.2679.

Theorem (Total probability): let $A_1, A_2, , \cdots A_n$ be n mutually events in sample space S such that $S = \cup_{i=1}^{n} A_i$, and let B be any event in S such that $P(B) \neq 0$. Then

$P(B) = \sum_{i=1}^{n} P(B|A_i).\mathrm{P}(A_i)$

Bayes' Theorem

Theorem: **(Bayes' Theorem**) Let $A_1, A_2, , \cdots A_n$ be a collection of mutually events in sample space S and let B be any event in S such that $P(B) > 0$. Then

$$P(A_k|B) = \frac{P(B|A_k).\mathrm{P}(A_k)}{\sum_{i=1}^{n} P(B|A_i).\mathrm{P}(A_i)}.$$

Proof: Based on the definition of conditional probability, we have

$P(A_k|B) = \frac{P(A_k \cap \mathrm{B})}{P(B)}$, then $P(A_k \cap \mathrm{B}) = P(A_k|B)P(B) - - - - - (1)$

$P(B|A_k) = \frac{P(\mathrm{B} \cap A_k)}{P(A_k)}$, then $P(\mathrm{B} \cap A_k) = P(B|A_k)P(A_k) - - - - - -(2)$

Since $P(A_k \cap \mathrm{B}) = P(\mathrm{B} \cap A_k)$, then from equation (1) and (2), we get

$$P(A_k|B)P(B) = P(B|A_k)P(A_k)$$

Then $P(A_k \cap \mathrm{B}) = P(B|A_k)P(A_k) - - - - - -(3)$

Then from equation (3) and (1), we get

$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B)}$,

By Total probability theorem, we get $P(A_k|B) = \frac{P(B|A_k).\mathrm{P}(A_k)}{\sum_{i=1}^{n} P(B|A_i).\mathrm{P}(A_i)}$.

Example: Say it is known that $bowl\ A1$ contains 5 red, 2 white, and 4 black balls and $bowl\ A2$ contains 4 red, 5 white, and 1 black ball. If we selected one bowl only and select one white ball on it, what is the probability that the white ball is selected from $bowl\ A2$.

Solution: By Bayes' Theorem, we have $P(A_k|B) = \frac{P(B|A_k).\mathrm{P}(A_k)}{\sum_{i=1}^{n} P(B|A_i).\mathrm{P}(A_i)}$., then

$$P(bowl\ A2|White) = \frac{P(White|bowl\ A2).\,\mathrm{P}(bowl\ A2)}{P(White|bowl\ A1)\mathrm{P}(bowl\ A1) + P(White|bowl\ A2)\mathrm{P}(bowl\ A2)}.$$

$$P(bowl\ A2|White) = \frac{\frac{5}{10}\times\frac{1}{2}}{\frac{2}{11}\times\frac{1}{2}+\frac{5}{10}\times\frac{1}{2}} = \frac{\frac{5}{20}}{\frac{2}{22}+\frac{5}{20}} = 0.73.$$

Example: let is we have three boxes, Box -1- contain 9 balls numbered from 1 to 9, Box -2- contain 5 balls numbered from 3 to 7, and Box 3 contain 9 balls numbered from 2 to 10. The player toss a die if the occur number is 3 the player select the ball from box -1-. If the occur number 2 or 5 the player select a ball from box -2-, otherwise the player select the ball from box -3-

1) What is the probability that select ball is even numbered ball?
2) if the select ball is odd numbered ball, what is the probability that it select from box-3-?

Solution:

1) $P(\text{even numbered ball}) = P(even\ |box\ 1)P(box1) + P(even\ |box\ 2)P(box2) + P(even\ |box\ 3)P(box3) = \frac{4}{9} \times \frac{1}{6} + \frac{2}{5} \times \frac{2}{6} + \frac{5}{9} \times \frac{3}{6} = ?$
2) $P(box3\ |odd\ numbered\ ball) = \frac{\frac{4}{9} \times \frac{3}{6}}{\frac{5}{9} \times \frac{1}{6} + \frac{3}{5} \times \frac{2}{6} + \frac{4}{9} \times \frac{3}{6}} = ?$

Example: Say it is known that bowl A1 contains three red and seven blue chips and bowl A2 contains eight red and two blue chips. All chips are identical in size and shape. A die is cast and bowl A1 is selected if five or six spots show on the side that is up; otherwise, bowl A2 is selected. For this situation, it seems reasonable to assign $P(A1) = \frac{2}{6}$ and $P(A2) = \frac{4}{6}$. The selected bowl is handed to another person and one chip is taken at random. Say that this chip is red, an event which we denote by B. By considering the contents of the bowls, it is reasonable to assign the conditional probabilities $P(B|A1) = \frac{3}{10}$ and $P(B|A2) = \frac{8}{10}$. Thus the conditional probability of bowl A1, given that a red chip is drawn, is

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} = \frac{(\frac{2}{6})(\frac{3}{10})}{(\frac{2}{6})(\frac{3}{10}) + (\frac{4}{6})(\frac{8}{10})} = \frac{3}{19}.$$

Similarly, we have $P(A_2|B) = \frac{16}{19}$

Discrete and continuous Random Variables:

Definition (**Random Variable**): Consider a random experiment with a sample space S. A function X, which assigns to each element $c \in S$ one and only one number $X(c) = x$, is called a random variable. that is; $X: S \to D \subset \mathbb{R}$

The space or range of X is the set of real numbers $D = \{x : x = X(c), c \in S\}$. D generally, is a countable set or an interval of real numbers.

Definition (***Discrete Random Variable***). We say a random variable is a discrete random variable if its space or range is either finite or countable.
Discreate probability distribution is the table of all value of discreate random variable with it's probability of each random variable and it's denoted by $p(x)$.

Probability mass function (pmf)

Definition (**probability mass function**) **(pmf)**: Let X be a discrete random variable with space D. The probability mass function (pmf) of X is given by

$$P_X(x) = P[X = x] \quad \forall x \in D$$

Note that pmfs satisfy the following two properties:

1) $p_X(x) \geq 0 \quad \forall x \in D$
2) $\sum_{x \in D} p_X(x) = 1$

Example: toss a coin twice, then $S = \{HH, TH, HT, TT\}$

If X=number occurrence of H, then X=0, 1, and 2.

X	0	1	2
S	TT	HT, HH	HH
p(x)	1/4	1/4	1/4

$P[X = x] = \sum_{X=0}^{2} p_X(x) = p_X(0) + p(1) + p(2) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$. Hence $p_X(x)$ is pmf.

Example: Toss a die twice. The sample space is $S = \{(i, j): 1 \le i, j \le 6\}$. X is the sum of occurrence number. The random variable X is $X(i, j) = i + j$. The space of X is $D = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. the discreate probability function of X is given by

X	S	$p(x)$	X	S	P(x)
2	(1, 1)	1/36	8	(2, 6) (3, 5) (5, 3) (6, 2) (4, 4)	5/36
3	(1, 2) (2, 1)	2/36	9		4/36
4	(1, 3) (3, 1) (2, 2)	3/36	10	(5, 5)(4, 6) (6, 4)	3/36
5		4/36	11	(5, 6), (6, 5)	2/36
6		5/36	12	(6, 6)	1/36
7		6/36			

$\sum_{X=2}^{12} p(x) = p(2) + p(3) + p(4) + \cdots + p(12) = \frac{36}{36} = 1$. Hence $p_X(x)$ is pmf.

Example: let $p_X(x) = \begin{cases} 1/4 & if\ x = 1, 3 \\ 2/4 & if\ x = 2 \\ 0 & otherwise \end{cases}$ is $p(x)$ is pmf?

1) $p(1) = 1/4 > 0,\ p(2) = 2/4 > 0,\ p(3) = 1/4 > 0, p(X\backslash x = 1, 2, 3) = 0$

Therefore $p_X(x) \ge 0 \quad for\ all\ x$

2) $\sum_{all\ x} p_X(x) = p(1) + p(2) + p(3) + p(X\backslash x = 1, 2, 3) = 1/4 + 2/4 + 1/4 + 0 = 1$

Hence $p_X(x)$ is pmf.

Example: Let $p_X(x) = \begin{cases} x/6 & if\ x = 1,2,3 \\ 0 & otherwise \end{cases}$ is $p_X(x)$ is pmf? (H.W)

Cumulative Distribution Function:

The cumulative distribution function (cdf) for a random variable X is a real valued function F defined by $F_X(t) = P_X(x \le t)$, for each real $t \in \mathbb{R},\ and\ \ 0 \le F_X(t) \le 1$

For discrete random variable X cdf is given by $F_X(t) = \sum_{t<X}^{X} p_X(t)$.

Example: find cumulative distribution function (cdf)for

$$p_X(x) = \begin{cases} x/6 & if\ x = 1,2,3 \\ 0 & otherwise \end{cases}$$

since $p_X(x)$ is pmf, then we can calculate *cdf* for $p_X(x)$.

$$F(x)=\begin{cases}0 & x<1\\ 1/6 & x\le 1\ <2\\ \frac{3}{6} & 2\le\ x<3\\ 1 & x\ge 3\end{cases}$$

$$since\ F_X(t)=\sum_{t<X}^{X} p_X(t)$$

$$F_2(t)=\sum_{t<2}^{X} f(t)=f(t<1)+f(t=1)+f(t<2)=0+\frac{1}{6}+\frac{2}{6}=\frac{3}{6}$$

Example: let $f_X(x) or\ p_X(x)=\begin{cases}1/4 & if\ x=1,3\\ 2/4 & if\ x=2\\ 0 & otherwise\end{cases}$

$$F(x)=\begin{cases}0 & x<1\\ \frac{1}{4} & 1\le x<2\\ \frac{3}{4} & 2\le x<3\\ \frac{4}{4} & x\ge 3\end{cases}$$

$$F_1(t)=\sum_{t<1}^{X} f(t)=f(t<1)+f(t=1)=0+\frac{1}{4}$$

$$F_2(t)=\sum_{t<2}^{X} f(t)=f(t<1)+f(t=1)+f(t=2)=0+\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$$

$$F_3(t)=\sum_{t<3}^{X} f(t)=f(t<1)+f(t=1)+f(t=2)+f(t=3)=0+\frac{1}{4}+\frac{2}{4}+\frac{1}{4}=1$$

$$F_4(t)=\sum_{t<4}^{X} f(t)=f(t<1)+f(t=1)+f(t=2)+f(t=3)+f(t=4)$$

$$=\ 0+\frac{1}{4}+\frac{2}{4}+\frac{1}{4}+0=1$$

Properties:

- $F_X(x)$ is a non-decreasing function of x.
- $\lim_{x\to-\infty} F_X(x)=\ 0$ and $\lim_{x\to\infty} F_X(x)=\ 1$.
- $P(a\ <\ X\ \le\ b)\ =\ F_X(b)\ -\ F_X(a)$.

Continuous Random Variables:

Definition: *A random variable X is called continues random variable if it's range is continuous.*

Or A continuous random variable is a function $X\ (\ s\ )$ from an uncountably infinite sample space S to the real numbers R , $X\ :\ S\ \to\ R$.

probability density function (pdf)

Definition (probability density function) (pdf): Suppose that X is a random variable and there is a function $f(x)$ such that

1) $f(x) \geq 0 \;\; \forall x$
2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$
3) $for\ every\ events\ A, then\ \ P(A) = P(X\ in\ A) = \int_A \ f(x)dx$
4) For every interval [a, b], $P(a \leq X \leq b) = \int_a^b f(x)\, dx$

Then X is said to be a continuous random variable with probability density function (pdf) $f(x)$.

Example: $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases}$

1) $f(x) \geq 0 \;\; \forall x$
2) $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{\infty} f(x)\, dx$
   $= \int_{-\infty}^{0} 0\, dx + \int_{0}^{1} 2(1-x)\, dx + \int_{1}^{\infty} 0\, dx = 0 + 2x - \frac{2x^2}{2} |_0^1 + 0 = (2-1) - 0 = 1$
   Hence we get $f(x)$ is pdf

Cumulative distribution function (cdf)

$$F(x) = P(x \leq a) = \int_{-\infty}^{\infty} f(x)\, dx$$

if X is continuous random variable.

Theorem: Let X be a random variable with cumulative distribution function $F(x)$. Then

(a) For all a and b, if $a < b, then\ F(a) \leq F(b)$ (F is nondecreasing).
(b) $\lim_{x \to -\infty} F_X(x) = 0$ (the lower limit of F is 0).
(c) $\lim_{x \to \infty} F_X(x) = 1$ (the upper limit of F is 1)
(d) For $a < b\ , P(a < X \leq b) = F_X(b) - F_X(a)$.

Notation: The probability density function is the derivative of the probability distribution function:

$$f_X(x) \equiv F'_X(x) \equiv \frac{d}{dx} F_X(x) .$$

Example: Let X be the lifetime in years of a mechanical part. Assume that X has the cdf

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

Find the pdf of X, and find the probability that a part has a lifetime between one and three years.

Solution: since pdf is the derivative of the probability distribution function, $\frac{d}{dx} F_X(x)$ is

$$f_X(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & elsewhere \end{cases}$$

The probability that a part has a lifetime between one and three years is given by

$$p(1 < X \leq 3) = F_X(3) - F_X(1) = \int_1^3 e^{-x} dx = e^{-1} - e^{-3} = 0.318.$$

Example: $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases}$

Find $F(0.35), F(3), and \ F(10)$?

Solution: $F(0.35) = \int_0^{0.35} 2(1-x)\, dx =?$, and $F(3) = \int_{-\infty}^{3} 2(1-x)\, dx = 1$

Example: suppose X has a p.d.f

$$f_X(x) = \begin{cases} cx^3 & 0 < x < 2 \\ 0 & \text{elsewhere,} \end{cases}$$

for a constant c. Then

$$1 = \int_0^2 cx^3\, dx = c\left[\frac{x^4}{4}\right]_0^2 = 4c,$$

and, hence, $c = 1/4$. For illustration of the computation of a probability involving X, we have

$$P\left(\frac{1}{4} < X < 1\right) = \int_{1/4}^{1} \frac{x^3}{4}\, dx = \frac{255}{4096} = 0.06226. \blacksquare$$

Example: suppose X has a pdf

$$f(x) = \begin{cases} x+1 & -1 < x < 0 \\ 2x-6 & 3 < x < c, \quad c > 3 \\ 0 & otherwise \end{cases}$$

Determined c such that $f(x)$ is pdf of continuous random variable X. (H.W)

Mathematical expectation:

Expectation of a random variable: If the random variables $x_i = x_1, x_2, \cdots, x_n$, with probability $f(x_1), f(x_2), \ldots, f(x_n)$ respectively, then the expected value of x is denoted by $E(x)$ and is given by

$$E(x) = x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n) = \sum_{i=1}^{n} x_i f(x_i).$$

If the probability distribution is discreate, then $E(x)=\sum_{i=1}^{n} x_i f(x_i)$.

If the probability distribution is continues, then $E(x)=\int_{-\infty}^{\infty} xf(x)\,dx$.

Example 1: Let the random variable X of the discreate type have the pmf given by the table

x	0	1	2	3
p(x) or f(x)	1/8	3/8	3/8	1/8

$E(x)=\sum_{i=1}^{n} x_i f(x_i) = 0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=\frac{12}{8}$.

Example 2. **Let X have the p.d.f.**

$$f(x) = 4x^3, \quad 0 < x < 1,$$
$$= 0 \quad \text{elsewhere.}$$

Then

$$E(X) = \int_0^1 x(4x^3)\,dx = \int_0^1 4x^4\,dx = \left[\frac{4x^5}{5}\right]_0^1 = \frac{4}{5}.$$

Example 3. **Let X have the p.d.f.**

$$f(x) = 2(1-x), \quad 0 < x < 1,$$
$$= 0 \quad \text{elsewhere.}$$

Then

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 (x)2(1-x)\,dx = \tfrac{1}{3},$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2f(x)\,dx = \int_0^1 (x^2)2(1-x)\,dx = \tfrac{1}{6},$$

and, of course,

$$E(6X + 3X^2) = 6(\tfrac{1}{3}) + 3(\tfrac{1}{6}) = \tfrac{5}{2}.$$

Example: let X have a p.d.f

$$p(x) = \begin{cases} \frac{x}{6} & x = 1,2,3 \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$E(6X^3 + X) = 6E(X^3) + E(X) = 6\sum_{x=1}^{3} x^3p(x) + \sum_{x=1}^{3} xp(x) = \frac{301}{3}.$$

Example: let X have a p.d.f. $f(x) = \begin{cases} \frac{x+2}{18} & < -2\ x < 4 \\ 0 & otherwise \end{cases}$

Find $E(X), E(X+2)^3, E(6X-2((X+2)^3))$. (H.W)