Question Bank:

Exercise:

1. Show that the random variables X_1 and X_2 with joint pdf

$$f(x_1, x_2) = \begin{cases} 12x_1x_2(1-x_2) & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

are independent.

2. If the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = 2e^{-x_1-x_2}$, $0 < x_1 < x_2$, $0 < x_2 < \infty$, zero elsewhere, show that X_1 and X_2 are dependent.

3. Let $p(x_1, x_2) = \frac{1}{16}$, $x_1 = 1, 2, 3, 4$, and $x_2 = 1, 2, 3, 4$, zero elsewhere, be the joint pmf of X_1 and X_2. Show that X_1 and X_2 are independent.

4. Find $P(0 < X_1 < \frac{1}{3}, 0 < X_2 < \frac{1}{3})$ if the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = 4x_1(1-x_2)$, $0 < x_1 < 1$, $0 < x_2 < 1$, zero elsewhere.

5. Find the probability of the union of the events $a < X_1 < b$, $-\infty < X_2 < \infty$, and $-\infty < X_1 < \infty$, $c < X_2 < d$ if X_1 and X_2 are two independent variables with $P(a < X_1 < b) = \frac{2}{3}$ and $P(c < X_2 < d) = \frac{5}{8}$.

Exercise:

Q1/ Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random exactly 2 will be defective by using (a) the binomial distribution and (b) the Poisson approximation to the binomial distribution.

Q2/ Between the hours of 2 and 4 pm. the average number of phone calls per minute coming into the switchboard of a company is 2.5. Find the probability that during one particular minute there will be (a) 0, (b) 1, (c) 2, (d) 3, (e) 4 or fewer, and (f) more than 6 phone calls.

Q3/ Suppose that X is a random variable with moment-generating function $m_X(t) = e^{3.2(et-1)}$. What is the distribution of X ?

EXERCISES

1. If

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw,$$

show that $\Phi(-z) = 1 - \Phi(z)$.

2. If X is $N(75, 100)$, find $P(X < 60)$ and $P(70 < X < 100)$ by using either Table II or the R command `pnorm`.

3. If X is $N(\mu, \sigma^2)$, find b so that $P[-b < (X - \mu)/\sigma < b] = 0.90$, by using either Table II of Appendix D or the R command `qnorm`.

4. Let X be $N(\mu, \sigma^2)$ so that $P(X < 89) = 0.90$ and $P(X < 94) = 0.95$. Find μ and σ^2.

5. Show that the constant c can be selected so that $f(x) = c2^{-x^2}$, $-\infty < x < \infty$, satisfies the conditions of a normal pdf.
Hint: Write $2 = e^{\log 2}$.

6. If X is $N(\mu, \sigma^2)$, show that $E(|X - \mu|) = \sigma\sqrt{2/\pi}$.

7. Show that the graph of a pdf $N(\mu, \sigma^2)$ has points of inflection at $x = \mu - \sigma$ and $x = \mu + \sigma$.

8. Evaluate $\int_2^3 \exp[-2(x-3)^2]\, dx$.

9. Determine the 90th percentile of the distribution, which is $N(65, 25)$.

10. If e^{3t+8t^2} is the mgf of the random variable X, find $P(-1 < X < 9)$.

11. Let the random variable X have the pdf

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad 0 < x < \infty, \quad \text{zero elsewhere.}$$

(a) Find the mean and the variance of X.

(b) Find the cdf and hazard function of X.

Hint for (a): Compute $E(X)$ directly and $E(X^2)$ by comparing the integral with the integral representing the variance of a random variable that is $N(0, 1)$.

Exercise: Let X have a gamma distribution with $\alpha = \frac{r}{2}$, where r is a positive integer, and $\beta > 0$. Define the random variable $Y = \frac{2X}{\beta}$.

Exercise: A) show that the mean and the variance of **Γ- distribution** are:

$\mu = E(x) = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$

A) Show that the mean and the variance of ***β*- distribution** are:

$\mu = E(x) = \frac{\alpha}{\alpha+\beta}$ and $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Exercise: A) if $\alpha = \beta = \frac{1}{2}$, then what is the value of $\beta\left(\frac{1}{2},\frac{1}{2}\right)$ =??

A) prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Exercise:

Let X be a random variable such that $E(X^m) = (m+1)!2^m$, $m = 1, 2, 3, \ldots$. Determine the mgf and the distribution of X.

Hint: Write out the Taylor series[6] of the mgf.

Show that

$$\int_\mu^\infty \frac{1}{\Gamma(k)} z^{k-1} e^{-z}\, dz = \sum_{x=0}^{k-1} \frac{\mu^x e^{-\mu}}{x!}, \quad k = 1, 2, 3, \ldots.$$

This demonstrates the relationship between the cdfs of the gamma and Poisson distributions.

Hint: Either integrate by parts $k-1$ times or obtain the "antiderivative" by showing that

$$\frac{d}{dz}\left[-e^{-z}\sum_{j=0}^{k-1} \frac{\Gamma(k)}{(k-j-1)!} z^{k-j-1}\right] = z^{k-1}e^{-z}.$$

Let X_1, X_2, and X_3 be iid random variables, each with pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere.

(a) Find the distribution of $Y = \text{minimum}(X_1, X_2, X_3)$.
Hint: $P(Y \le y) = 1 - P(Y > y) = 1 - P(X_i > y, i = 1, 2, 3)$.

(b) Find the distribution of $Y = \text{maximum}(X_1, X_2, X_3)$.