Example Let X_1 and X_2 have the joint pdf

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 10x_1x_2^2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Suppose $Y_1 = X_1/X_2$ and $Y_2 = X_2$. Hence, the inverse transformation is $x_1 = y_1y_2$ and $x_2 = y_2$, which has the Jacobian

$$J = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2.$$

The inequalities defining the support $\mathcal{S}$ of (X_1, X_2) become

$$0 < y_1y_2,\ \ y_1y_2 < y_2,\ \text{and}\ \ y_2 < 1.$$

These inequalities are equivalent to

$$0 < y_1 < 1 \ \text{and}\ \ 0 < y_2 < 1,$$

which defines the support set $\mathcal{T}$ of (Y_1, Y_2). Hence, the joint pdf of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2) = 10y_1y_2y_2^2|y_2| = 10y_1y_2^4\ , \quad (y_1,y_2) \in \mathcal{T}.$$

The marginal pdfs are

$$f_{Y_1}(y_1) = \int_0^1 10y_1y_2^4\,dy_2 = 2y_1, \quad 0 < y_1 < 1,$$

zero elsewhere, and

$$f_{Y_2}(y_2) = \int_0^1 10y_1y_2^4\,dy_1 = 5y_2^4, \quad 0 < y_1 < 1,$$

zero elsewhere.

Example: Let X_1 and X_2 have independent gamma distribution with parameters $\alpha, \theta\ and\ \beta, \theta$ respectively.
If $Z_1 = \frac{x_1}{x_1+x_2}$ and $Z_2 = X_1 + X_2$, find the joint pdf $Z_1\ and\ Z_2$.

Solution: - $X_1 \sim Gamma(\alpha, \theta)$
$X_2 \sim Gamma(\beta, \theta)$
$f(x_1) = \frac{1}{\Gamma(\alpha)\theta^\alpha} \ \ x_1^{\alpha-1}\, e^{\frac{-x_1}{\theta}}, 0 < x_1 < \infty$
$f(x_2) = \frac{1}{\Gamma(\beta)\theta^\beta}\, x_2^{\beta-1}\, e^{\frac{-x_2}{\theta}}, 0 < x_2 < \infty$
Since X_1 and X_2 are independents, then:
$f(x_1, x_2) = f(x_1) * f(x_2)$

$$f(x_1,x_2)=\left\{\frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}}\quad x_1^{\alpha-1}\ x_2^{\beta-1}\ e^{\frac{-(x_1+x_2)}{\theta}}\quad ,0<x_1<\infty\ ,0<x_2<\infty\right\}$$

and $z_1=\frac{x_1}{x_1+x_2}$, $z_2=x_1+x_2\Rightarrow$

$$x_1=z_1z_2\ and\ x_2=z_2(1-z_1)$$

$$|J|=\begin{vmatrix}\frac{dx_1}{dz_1} & \frac{dx_1}{dz_2}\\ \frac{dx_2}{dz_1} & \frac{dx_2}{dz_2}\end{vmatrix}=\begin{vmatrix}z_2 & z_1\\ -z_2 & (1-z_1)\end{vmatrix}=z_2$$

The joint pdf of $z_1\ and\ z_2\ is\ f(z_1,z_2)=|J|.f(h_1^{-1}(z_1,z_2),\dots,h_n^{-1}(z_1,z_2)$

$$=|z_2|.\frac{1}{\Gamma(\alpha)\Gamma(\beta)\ \theta^{\alpha+\beta}}(z_1z_2)^{\alpha-1}(z_2(1-z_1))^{\beta-1}e^{-\frac{z_2}{\theta}}\quad ,\quad 0<z_2<\infty\ ,0<z_1<1$$

2.2.2 Transformations for Several Random Variables:

Let$(X_1, X_2, X_3, \dots, X_n)$ be n- dimensional continuous random variable with joint pdf $f(x_1, x_2, \dots, x_n)$ and suppose that $Y_i = h_i(X_1, X_2, X_3, \dots, X_n), i = 1, 2, \dots, n$.

Consider an integral of the form

$$\int \cdots \int_A f(x_1, x_2, \dots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

taken over a subset A of an n-dimensional space $\mathcal{S}$. Let

$$y_1 = u_1(x_1, x_2, \dots, x_n), \quad y_2 = u_2(x_1, x_2, \dots, x_n), \dots, y_n = u_n(x_1, x_2, \dots, x_n),$$

together with the inverse functions

$$x_1 = w_1(y_1, y_2, \dots, y_n), \quad x_2 = w_2(y_1, y_2, \dots, y_n), \dots, x_n = w_n(y_1, y_2, \dots, y_n)$$

define a one-to-one transformation that maps $\mathcal{S}$ onto $\mathcal{T}$ in the $y_1, y_2, \dots, y_n$ space and, hence, maps the subset A of $\mathcal{S}$ onto a subset B of $\mathcal{T}$. Let the first partial derivatives of the inverse functions be continuous and let the n by n determinant (called the Jacobian)

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

not be identically zero in $\mathcal{T}$. Then

$$\int \cdots \int_A f(x_1, x_2, \dots, x_n)\, dx_1 dx_2 \cdots dx_n$$
$$= \int \cdots \int_B f[w_1(y_1, \dots, y_n), w_2(y_1, \dots, y_n), \dots, w_n(y_1, \dots, y_n)] |J|\, dy_1 dy_2 \cdots dy_n.$$

The joint pdf of the random variables
$Y_1 = h_1(x_1, x_2, \dots, x_k)$, and $Y_2 = h_2(x_1, x_2, \dots, x_k), \dots, Y_k = h_k(x_1, x_2, \dots, x_k)$
Is given by
$g(y_1, y_2, \dots, y_n) = f(h_1^{-1}(y_1, y_2, \dots, y_n), h_2^{-1}(y_1, y_2, \dots, y_n) \dots, h_n^{-1}(y_1, y_2, \dots, y_n))|J|$

Example Let X_1, X_2, X_3 have the joint pdf

$$f(x_1, x_2, x_3) = \begin{cases} 48x_1x_2x_3 & 0 < x_1 < x_2 < x_3 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

If $Y_1 = X_1/X_2$, $Y_2 = X_2/X_3$, and $Y_3 = X_3$, then the inverse transformation is given by

$$x_1 = y_1y_2y_3, \ x_2 = y_2y_3, \text{ and } x_3 = y_3.$$

The Jacobian is given by

$$J = \begin{vmatrix} y_2y_3 & y_1y_3 & y_1y_2 \\ 0 & y_3 & y_2 \\ 0 & 0 & 1 \end{vmatrix} = y_2y_3^2.$$

Moreover, inequalities defining the support are equivalent to

$$0 < y_1y_2y_3, \ y_1y_2y_3 < y_2y_3, \ y_2y_3 < y_3, \text{ and } y_3 < 1,$$

which reduces to the support $\mathcal{T}$ of Y_1, Y_2, Y_3 of

$$\mathcal{T} = \{(y_1, y_2, y_3) : \ 0 < y_i < 1, i = 1, 2, 3\}.$$

Hence the joint pdf of Y_1, Y_2, Y_3 is

$$\begin{aligned} g(y_1, y_2, y_3) &= 48(y_1y_2y_3)(y_2y_3)y_3|y_2y_3^2| \\ &= \begin{cases} 48y_1y_2^3y_3^5 & 0 < y_i < 1, i = 1, 2, 3 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

EXERCISES

1. Let X_1, X_2, X_3 be iid, each with the distribution having pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere. Show that

$$Y_1 = \frac{X_1}{X_1 + X_2}, \quad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \quad Y_3 = X_1 + X_2 + X_3$$

are mutually independent.

2. If $f(x) = \frac{1}{2}$, $-1 < x < 1$, zero elsewhere, is the pdf of the random variable X, find the pdf of $Y = X^2$.

3. If X has the pdf of $f(x) = \frac{1}{4}$, $-1 < x < 3$, zero elsewhere, find the pdf of $Y = X^2$.
Hint: Here $\mathcal{T} = \{y : 0 \leq y < 9\}$ and the event $Y \in B$ is the union of two mutually exclusive events if $B = \{y : 0 < y < 1\}$.

4. Let X_1, X_2, X_3 be iid with common pdf $f(x) = e^{-x}$, $x > 0$, 0 elsewhere. Find the joint pdf of $Y_1 = X_1$, $Y_2 = X_1 + X_2$, and $Y_3 = X_1 + X_2 + X_3$.

2.3 Student t-distribution:

Let W be a random variable having pdf. W$\sim N(0,1)$ and V a random variable with pdf V$\sim x^2_{(r)}$ and both Wand V are independent random variable. Then the PDF. of $T = \frac{W}{\sqrt{\frac{v}{r}}}$ is known as $\boldsymbol{t}-$ **distribution** with r degree freedom (df) and given by

$$g(t) = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2})\ \sqrt{r\pi}}\ (1+\frac{t^2}{r})^{\frac{-(r+1)}{2}} \quad ; \ -\infty < t < \infty\ , r > 0.$$

Proof:- We have

W$\sim N(0,1)$

V$\sim \boldsymbol{\chi}^2_{(\boldsymbol{r})}$

$$f_1(w) = \frac{1}{\sqrt{2\pi}}\ e^{-\frac{w^2}{2}} \qquad ; \ -\infty < w < \infty \ ,$$

$$f_2(v) = \frac{v^{\frac{r}{2}-1}\ e^{\frac{-v}{2}}}{\Gamma(\frac{r}{2})\ 2^{(\frac{r}{2})}}\ , v > 0$$

Since v and w are independent, then

$$f(w,v) = f(w).f(v)$$

$$f(w,v) = \frac{1}{\sqrt{2\pi}}\ e^{-\frac{w^2}{2}}\ .\frac{v^{\frac{r}{2}-1}\ e^{\frac{-v}{2}}}{\Gamma(\frac{r}{2})\ 2^{(\frac{r}{2})}}$$

$$f(w,v) = \frac{1}{\sqrt{2\pi}\ \Gamma\frac{r}{2}\ 2^{\ \frac{r}{2}}}\ \ v^{\frac{r}{2}-1}\ e^{\frac{-(w^2+v)}{2}} \quad -\infty < w < \infty \ \ , \ v > 0$$

Space of w and v:

$$A = \{(w,v); -\infty < w < \infty; v > 0\}$$

$$t = \frac{W}{\sqrt{\frac{V}{r}}} \qquad let\ v = u \text{ , then}$$

$$t = \frac{W}{\sqrt{\frac{u}{r}}} \Rightarrow w = t\sqrt{\frac{u}{r}}$$

V=u $\Rightarrow$ $w^2 = t^2\left(\frac{u}{r}\right) = \frac{t^2\ u}{r}$

Space of t and u:

$$B = \{(t,u); -\infty < t < \infty; u > 0\}$$