Chapter one

Dynamical system

Dynamical Systems are systems, described by one or more equations that evolve over time. For example, the growth of a population can be described by dynamic equations. Time can be understood to be either discrete (day 1, day 2 etc.) or continuous (3.4567... seconds). If we take time to be continuous, dynamical systems will be described by differential equations - equations that involve the derivative (the instantaneous change) of a function. If we take time to be discrete, dynamical systems will be described by difference equations - equations relating the value of a variable at time t + 1 to its value at time t.

Definition of dynamical systems

Definition: A dynamical system may be understood as a mathematical prescription for evolving the state of a system in time.

It is defined by a phase (or state) space D (in this course D $\subset R^n$) and a one-parameter family of mappings, φ_t : D ⟶ D, where t (time) ∈ R.

Time-continuous dynamical systems

Let D $\subset R^n, n \in N, x = (x_1, x_2, \dots, x_n) \in D, t \in R.$

$$\dot{x} = f\big(x(t)\big) = f(x) \qquad (1)$$

The function $f: D \subset R^n \to R^n$, is called a vector field, which can be written as a system of n first order, autonomous, ordinary differential equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n),$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n),$$
$$.$$
$$.$$
$$\frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n). \tag{3}$$

The formal solution of Eq. (2) (if ∃), $x(t) = \varphi_t\big(x(0)\big)$,

is called the trajectory of the vector field.

For examples:

1) Single species growth, the logistic equation

$$\frac{dy}{dt} = by(1 - \frac{y}{k}),$$

Where x population at time , $b > 0$ brith rate and k carrying capacity.

The solution of equation above is $y(t) = \frac{cke^{bt}}{k+ce^{bt}}$.

2) The **Lotka–Volterra equations**, also known as the **predator–prey equations**, are a pair of first-order nonlinear differential equations, frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

where

- x is the number of prey (for example, rabbits);
- y is the number of some predator (for example, foxes);
- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the instantaneous growth rates of the two populations;

- t represents time;
- α, β, γ, δ are positive real parameters describing the interaction of the two species.

Autonomous differential equation: A differential equation of (1), is said to be autonomous, because $\dot{x}$ is determined by x alone .

The solutions of autonomous equations have the following important property.

Definition: the autonomous dynamical system (1),

is said to have a fixed point at $x = a$ if and only if $f(a) = 0$.

Clearly $x = a$ is also a solution of the equation.

Remark: Fixed points are also referred to as critical point, singular point and stationary point.

The qualitative theory of solution curves of (1) is determined by $f(x)$.

When $f(x) \neq 0$, then the solution are either increasing or decreasing, when $f(a) = 0$ there is a solution $x(t) = a$

This is information can be repressed on the x-line (phase line) rather than t, x −plane.

The geometric representation of the qualitative behavior of the differential equation $\dot{x} = f(x)$ is called its phase portrait.

Classification of fixed points

Definition-Stable

A fixed point P is stable if all trajectories that start close to P stay close to P as x increases.

Additionally a fixed point is asymptotically stable if all trajectories to P tends to P as $x \to \infty$.

Definition-unstable

A fixed point P is unstable if it is not stable.

Definition-attractor

For a one dimensional system an asymptotically stable fixed point is called an attractor. And can be graphed in phase line as

P is attractor

Definition-repellor

For a one dimensional system a fixed point P such that all trajectories close to P move away from P, as x increases, is called a repellor. And can be graphed in phase line as

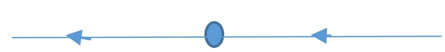
P is repoller

Definition-shunt

For a one dimensional system a fixed point P such that in every neighborhood of P some trajectories are attracted to P and some are repelled by P, is called a shunt. And can be graphed in phase line as

P is shunt

P is shunt



For example: Obtain and classify the fixed points of the following

1)$\dot{x} = x$

2) $\dot{x} = \frac{1}{2}(x^2 - 1)$

3)$\frac{dy}{dx} = y^2(y^2 - 1)$

Definition: two differential equation of the form $\dot{x} = f(x)$ are said to by qualitatively equivalent if they have the same number of fixed points, of the same arranged in the same order along the phase line.

Example: show that the differential equations $\dot{x} = (x+2)(x+1)$ is qualitatively equivalent to $\dot{x} = \frac{1}{2}(x^2 - 1)$.

And the equation $\dot{x} = -(x+2)(x+1)$ is not qualitatively equivalent to $\dot{x} = \frac{1}{2}(x^2 - 1)$.

Solution:

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Example: Arrange the following differential equation in the qualitative equation group

1) $\dot{x} = \cosh x$

2) $\dot{x} = (x - a)^2$

3) $\dot{x} = \cosh x - 1$

4) $\dot{x} = e^x$

Solution:

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Home work

1) Find the fixed points of the following autonomous differential equations

   $a)\ \dot{x} = x + 1$ $\quad b)\ \dot{x} = x - x^3$ $\quad c)\ \dot{x} = \sinh(x^2)$

   $d)\dot{x} = x^4 - x^3 - 2x^2$ $\quad e)\dot{x} = x^2 + 1$

Determine the nature (attractor , repellor or shunt) of each fixed point and hence construct the phase portrait of each equation.

2) Which differential equations in the following list, have the same phase portrait? (qualitatively equivalent)

   a) $\dot{x} = \sinh x$

   b) $\dot{x} = ax, \quad a > 0$

   c) $\dot{x} = \begin{cases} x\ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$

   d) $\dot{x} = \sin x$

   e) $\dot{x} = x^3 - x$

   f) $\dot{x} = \tanh x$

3) Show that the phase portrait of $\dot{x} = (a - x)(b - x)$ is qualitatively the same as that of $\dot{y} = y(y - c)$ for all real $a, b, c: a \neq b\ \&c \neq 0$

Chapter Two

Linear system:

A system $\dot{X} = F(X)$, where X is a vector in R^n, is called a linear system of dimension n if $F: R^n \to R^n$ is a linear mapping.

Since F is linear it can be written in the form

$$F(X)=\begin{pmatrix} f_1(x_1,x_2,\dots,x_n)\\ f_2(x_1,x_2,\dots,x_n)\\ \vdots\\ f_n(x_1,x_2,\dots,x_n)\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}$$

Thus $\dot{X}=F(X)=AX$, where A is the coefficient matrix

Dynamic system in the Plane: consider the form

$$\left.\begin{matrix}\dot{x}=f(x,y)\\ \dot{y}=g(x,y)\end{matrix}\right\}\dots\dots.(1)$$

$(\dot{X}=F(X), X=\begin{pmatrix}x\\y\end{pmatrix}, F=\begin{pmatrix}f\\g\end{pmatrix})$

Where f &g are real valued continuous function for every real value x,y and their partial derivatives are continuous.

Definition: a solution of system (1) is a pair of real valued functions $(U_1(t),U_2(t))$ defined on a common interval I s.t

$$\dot{U_1}(t)=f(U_1(t),U_2(t))$$
$$\dot{U_2}(t)=g(U_1(t),U_2(t))$$

Definition: the graph of a solution on any function $U_1(t),U_2(t)$ defined on an interval I, is the set of all points (t,x,y)which satisfy $x=U_1(t), y=U_2(t)\ ,\forall\ t\in I$

Example: consider $\begin{array}{c} \dot{x_1} = x_2 \\ \dot{x_2} = -x_1 \end{array}$

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Remark: if $x_1(t) = a, x_2(t) = b$ is a solution (1) where a, b are constant. Then $\dot{x_1}(t) = 0$ &$\dot{x_2}(t) = 0 \to f(a,b) = a$ &$g(a,b) = 0$

The graph of this solution is a line $x_1 = a$& $x_2 = b$

i.e the line paralle to the t-axis which intersects the (x_1, x_2) plane in the point (a, b)

Theorem (1) : let $w(t) = (u_1(t), u_2(t))$be a solution of (1) and suppose α is a number. Then $w(t + \alpha)$ is also solution, for any constant α

Proof:

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Remark: in the theorem above it follows that if a solution curve translated parallel to t-axis another solution curve obtained

Theorem(2): let $w_1 = (u_1, v_1)$& $w_2 = (u_2, v_2)$ be solution of (1) and let there are number t_1 &t_2 such that $w_1(t_1) = w_2(t_2)$. Then $w_1(t_1) = w_2(t + t_2 - t_1)$

Proof:

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Trajectory: observe every solution $x = x(t), y = y(t)$ defines a curve in the three dimensional space t, x, y. Thus is , to say set of all points (t, x, y) describe a curve in the three dimensional space.

The geometric theory of the differential equations begins with observation that every solution $x = x(t), y = y(t),\ \ t_0 \leq t \leq t_1$ of

$$\begin{matrix} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{matrix} \dots\dots\dots (1)$$

Also defined a curve in xy −plane, as t runs from $t_0$to t_1, the set of points $\big(x(t), y(t)\big), tracc\ out\ a\ curve\ \Gamma$in the xy −plane. This curve is called trajectory of solution of (1). That is represented parametrically by more than one solution.

Definition: let u, v be a solution of (1), let C be its graph, and let Γ be the projection of C on to xy −plane. The curve Γ is called the trajectory of u, v. Note that (x, y) lies on Γ iff there is a number t_0 s.t the point (t_0, x, y) lies on C

Theorem (classification of trajectory)

Through every point in the xy −plane, there passes a unique trajectory, which is either a point, a simple closed curve or a simple arc. The trajectory of a constant solution is a point. The trajectory of non- constant periodic solution is a simple closed curve. The trajectory of a non- constant and non-periodic solution is a simple arc.

Definition: the trajectory of a constant solution of

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \quad \dots\dots\dots.(1)$$

Is called fixed point (singular) point.

Equivalently, the point (x_0, y_0) is called a fixed point of (1). If

$f(a,b) = g(a,b) = 0$

Definition: the trajectory of periodic solution is called orbit.

Definition: the xy −plane itself is called the phase plane

Definition: the decomposition of the phase plane in to trajectory of system (1) is called phase portrait.

Example: find singular point of the systems

1) $\begin{aligned} \dot{x} &= 3x - 2y \\ \dot{y} &= x + y \end{aligned}$

2) $\begin{aligned} \dot{x} &= x + 2y - 1 \\ \dot{y} &= -2x - 4y + 2 \end{aligned}$

3) $\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= x + y - 1 \end{aligned}$

Solution:

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Theorem: through each point in the xy-plane there passes a unique trajectory. (through any point passing at most one trajectory)

Proof:

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Autonomies system in the plane:

Consider linear (homogeneous) systems

$$\begin{matrix} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{matrix} \dots\dots\dots (1) \quad (\dot{X} = AX, X = \begin{pmatrix} x \\ y \end{pmatrix})$$

Where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is anon singular (invertible) constant matrix.

Since, A is invertible, that $\det(A) = ad - bc \neq 0$, which implies $(0,0)$ is the singular point of the system.

Definition: a linear system $\dot{X} = AX$ is said to be simple, if the matrix A is anon singular $((\det(A) \neq 0)$ and A has non- zero eigenvalues. Has a single isolated fixed point in the phase plane.

Definition: a square matrix A is said to be similar to a matrix B if there exist an invertable matrix P such that $A = P^{-1}BP$

Theorem: let P be anon- singular matrix then the linear change of $X = PY$ transform the linear system $\dot{X} = AX$ into a system $\dot{Y} = JY$ where $J = P^{-1}AP$.

Proof:

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Example: find the matrix representation of the linear system

$$\dot{x_1} = x_1 + 2x_2$$
$$\dot{x_2} = 2x_2$$

Under the change of variables $x_1 = y_1 + 2y_2 \& x_2 = y_2$ is

transformed in to $\begin{matrix} \dot{y_1} = y_1 \\ \dot{y_2} = 2y_2 \end{matrix}$

Solution

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Proposition: let A be areal 2×2matrix then there is a real non-singular matrix P such that $J = P^{-1}AP$ which is one of the following types

1. $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ where $\lambda_1 \& \lambda_2$ are two real distinct eigenvalues of $A, 0 \neq \lambda_1 \neq \lambda_2 \in R$
2. $J = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$ where λ_0 is repeated eigenvalues of $A, ana\ A\ is\ diagonal\ matrix \quad 0 \neq \lambda_0 \in R$
3. $J = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}$ where λ_0 is repeated eigenvalues of $A, ana\ A\ is\ not\ diagonal\ matrix \quad 0 \neq \lambda_0 \in R$
4. $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ where A has complex eigenvalues $\lambda = \alpha \pm i\beta$ where $\alpha, \beta \in R, \beta \neq 0$

The matrix J is said to be the Jordan (Canonical) form of A. the eigenvalues of A(and J) are the values of λ for which

$$P_A(\lambda) = \lambda^2 - tr(A)\lambda + \det(A) = 0$$

Here $tr(A) = a + b$ is the trace of A and

$\det(A) = ad - bc$ is the determinate of A

Thus, the eigenvalues of A are

$\lambda_1 = \frac{1}{2}\left(tr(A) + \sqrt{\Delta}\right)$& $\lambda_2 = \frac{1}{2}\left(tr(A) - \sqrt{\Delta}\right)$

With $\Delta = [\left(tr(A)\right)^2 - 4\det(A)]$ it is the natural of the eigenvalues

Real distinct ($\Delta > 0$)

Real equal ($\Delta = 0$)

Complex ($\Delta < 0$)

Simple Canonical Systems

We consider the following cases

Case1/ A have real distinct eigenvalues λ_1& λ_2

In this case J is given by

$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ then the $\dot{X} = AX$ transforms to $\dot{Y} = JY$

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to\to \begin{matrix} \dot{y_1} = \lambda_1 y_1 \\ \dot{y_2} = \lambda_2 y_2 \end{matrix}$$

$$\left.\begin{matrix} \frac{dy_1}{dt} = \lambda_1 y_1 \to\to \frac{dy_1}{y_1} = \lambda_1 dt \to\to y_1 = c_1 e^{\lambda_1 t} \\ \frac{dy_2}{dt} = \lambda_2 y_2 \to\to \frac{dy_2}{y_2} = \lambda_2 dt \to\to y_2 = c_2 e^{\lambda_2 t} \end{matrix}\right\} \dots\dots\dots\dots (*)$$

If $C_1 = C_2 = 0$ then (*) yields a singular points

If $C_1 = 0$ and $C_2 \neq 0$, we obtain two rays,

$$y_1 = 0, y_2 > 0 (for\ C_2 > 0) and\ y_1 = 0, y_2 < 0 (for\ C_2 < 0)$$

Similarly, when Positive and negative parts of y_1 −axis

If $C_1 \neq 0$ and $C_2 \neq 0$

Now from (*)

$$\frac{y_1}{C_1} = e^{\lambda_1 t} \to\to \lambda_1 t = ln\frac{y_1}{C_1} \to\to t = \frac{1}{\lambda_1}\ln(\frac{y_1}{C_1})$$

$$\frac{y_2}{C_2} = e^{\lambda_2 t} \to\to \lambda_2 t = ln\frac{y_2}{C_2} \to\to t = \frac{1}{\lambda_2}\ln(\frac{y_2}{C_2})$$

$$\frac{1}{\lambda_1}\ln\left(\frac{y_1}{C_1}\right) = \frac{1}{\lambda_2}\ln\left(\frac{y_2}{C_2}\right) \to \ln\left(\frac{y_2}{C_2}\right) = \frac{\lambda_2}{\lambda_1}\ln\left(\frac{y_1}{C_1}\right)$$

$$\ln\left(\frac{y_2}{C_2}\right) = \ln\left(\frac{y_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}} \to\to \left(\frac{y_2}{C_2}\right) = \left(\frac{y_1}{C_1}\right)^{\frac{\lambda_2}{\lambda_1}}$$

$$y_2 = C_2 y_1^{\frac{\lambda_2}{\lambda_1}} C_1^{-\frac{\lambda_2}{\lambda_1}} = k y_1^{\frac{\lambda_2}{\lambda_1}} \dots\dots\dots.(**), k = C_2 C_1^{-\frac{\lambda_2}{\lambda_1}}$$

We have three case, depending on the nature of $\lambda_1 \& \lambda_2$

i) If $\lambda_2 < \lambda_1 < 0$

Note that as t increase (i.e $t \to \infty$) then every trajectory tends to the origin

i.e $y_1(t) \to 0 \quad as\ t \to \infty$

$$y_2(t) \to 0 \quad as\ t \to \infty$$

Since $\frac{\lambda_2}{\lambda_1} > 1$, then the eq(**) $y_2 = k y_1^{\frac{\lambda_2}{\lambda_1}}$ is approximately is parabola

In this case , the origin is called Stable node.

i) If $\lambda_2 \& \lambda_1$ are positive with $0 < \lambda_2 < \lambda_1$

The phase portraits looks exactly the same as above except that now every trajectory tends to origin as $t \to -\infty$ then every trajectory tends to the origin

i.e $y_1(t) \to 0 \quad as\ t \to -\infty$

$$y_2(t) \to 0 \quad as\ t \to -\infty$$

Since $\frac{\lambda_2}{\lambda_1} < 1$, then

The origin is called unstable node

iii) Ifλ_2&λ_1 are positive signs with $\lambda_2 < 0, \lambda_1 > 0$

then $y_1(t) \to 0 \quad as\ t \to -\infty$

$$y_2(t) \to 0 \quad as\ t \to \infty$$

Since $\frac{\lambda_2}{\lambda_1} < 0$, then equation $y_2 = k y_1^{\frac{\lambda_2}{\lambda_1}}$

Is approximately hyperbola

In this case the origin is saddle point

Case b) A has a repeated eigenvalues $\lambda_0 = \frac{trA}{2} \in R$, when $\Delta = 0 \rightarrow$ $\Delta = \sqrt{B^2 - 4AC}$ there are two cases

i. If A is diagonal, then system $\dot{X} = AX$ transform into system $\dot{Y} = JY$, where

$$J = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \&Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Then

$$\left.\begin{matrix} \dot{y_1} = \lambda_0 y_1 \rightarrow y_1 = c_1 e^{\lambda_0 t} \\ \dot{y_2} = \lambda_0 y_2 \rightarrow y_2 = c_2 e^{\lambda_0 t} \end{matrix}\right\} \rightarrow y_2 = \frac{c_2}{c_1} y_1$$

When$c_1 = 0$, then $y_1 = 0$and $y_2 > 0$ if $c_2 > 0$

and $y_2 < 0$ if $c_2 < 0$

Similarly

If $c_2 = 0$, then $y_2 = 0$and $y_1 > 0$ if $c_1 > 0$

and $y_1 < 0$ if $c_1 < 0$

every trajectory tends toward or away from the origin according to $\lambda_0 < 0 \ or \ \lambda_0 > 0$

1) When $\lambda_0 < 0$, every trajectory tends to the origin as $t \rightarrow \infty$ and in this case the origin is called stable star node

2) When $\lambda_0 > 0$,every trajectory ran away from the origin as $t \to +\infty$ and the origin is called unstable star node

ii. If A is not diagonal. in this case the system $\dot{X} = AX$ transform to $\dot{Y} = JY$, where

$$J = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$$

So,

$$\dot{y_1} = \lambda_0 y_1 + y_2 \ldots \ldots \ldots .(1)$$
$$\dot{y_2} = \lambda_0 y_2 \ldots \ldots \ldots \ldots \ldots ..(2)$$

From eq(2) we get $y_2 = c_2 e^{\lambda_0 t}$

Then $\frac{dy_1}{dt} = \dot{y_1} = \lambda_0 y_1 + c_2 e^{\lambda_0 t} \rightarrow \frac{dy_1}{dt} - \lambda_0 y_1 = c_2 e^{\lambda_0 t}$

Is first linear differential equation with $P(t) = -\lambda_0$ and

$Q(t) = c_2 e^{\lambda_0 t}$

So, the solution is given

$$y_1 = \frac{\int e^{-\int \lambda_0 dt} c_2 e^{\lambda_0 t} dt + c_1}{e^{-\int \lambda_0 dt}} = e^{\lambda_0 t}\left[c_2 \int dt + c_1\right]$$
$$= e^{\lambda_0 t}[c_2 t + c_1]$$

Clearly $y_2 = \frac{c_2}{c_2 t + c_1} y_1$

There are two cases depending on the sign of λ_0

Suppose first that $\lambda_0 < 0$, then every trajectory tends to the origin as $t \rightarrow \infty$. Then the origin is called stable improper node

Second if $\lambda_0 > 0$, the phase portrait differs only in the origin of the trajectory, which tends a way from the origin as $t \to \infty$. Then the origin is called unstable improper node

Case 3) if A has complex eigenvalues $\lambda_1, \lambda_2 = \alpha \pm i\beta, \beta \neq 0$ where $\Delta < 0 \to \beta^2 - 4AC < 0$

Then $\dot{X} = AX$ transform to $\dot{Y} = JY$ where

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

Now, we have

$$\left.\begin{aligned}\dot{y_1} &= \alpha y_1 - \beta y_2\\ \dot{y_2} &= \beta y_1 + \alpha y_2\end{aligned}\right\} \dots\dots(1)$$

We now use polar coordinates

Let $y_1 = rcos\theta$ and $y_2 = rsin\theta$with $r > 0$& $r = \sqrt{y_1^2 + y_2^2}$

$$\left.\begin{aligned}\dot{y_1} &= \dot{r}cos\theta - rsin\theta\dot{\theta}\\ \dot{y_2} &= \dot{r}sin\theta + rcos\theta\dot{\theta}\end{aligned}\right\} \dots\dots\dots(2)$$

Solving these two equations for $\dot{r} and\ \dot{\theta}$

Put eq (2), $y_1 = rcos\theta$ and $y_2 = rsin\theta$ in eq(1)

$$\dot{r}cos\theta - rsin\theta\dot{\theta} = \alpha rcos\theta - \beta rsin\theta \dots\dots\dots(3)$$

$$\dot{r}sin\theta + rcos\theta\dot{\theta} = \beta rcos\theta + \alpha rsin\theta \dots\dots\dots(4)$$

Multiply eq(3) by $cos\theta$ and eq(4) by $sin\theta$ and adding the result, so we get

$$\dot{r}(cos^2\theta + sin^2\theta) = \alpha r(cos^2\theta + sin^2\theta)$$

$$\dot{r} = \alpha r \to \frac{dr}{dt} = \alpha r \to \frac{dr}{r} = \alpha dt \to r = c_1 e^{\alpha t}$$

Multiply eq(4) by $cos\theta$ and eq(3) by $(-sin\theta)$ and adding the result, so we get

$$\dot{\theta}r(cos^2\theta + sin^2\theta) = \beta r(cos^2\theta + sin^2\theta)$$

$$\dot{\theta} = \beta \to \frac{d\theta}{dt} = \beta \to d\theta = \beta dt \to \theta = \beta t + c_2 \ \ with\ r = c_1 e^{\alpha t}$$

$y_1 = rcos\theta = c_1 e^{\alpha t}\cos(\beta t + c_2)$

$$y_2 = rsin\theta = c_1 e^{\alpha t} sin(\beta t + c_2)$$

There are several different cases

1) If $\alpha = 0$ then $r = c_1$& $y_1^2 + y_2^2 = c_1^2$ is the equation of circles then trajectories other than the origin are circles centered on the origin, every trajectory other than the origin itself is therefore an orbit all have period $\frac{2\pi}{\beta}$ the time for one revolution around the origin

The origin is called center

2) If $\alpha > 0$, then $r \to 0\ as\ t \to \infty$ and the trajectories are spiral. Which approach to the origin with increasing$t$. we now eliminate t from $\dot{r} and\ \dot{\theta}$ we have

$$\frac{dr}{dt} = \alpha r \& \ \frac{d\theta}{dt} = \beta$$

$$\rightarrow \frac{dr}{d\theta} = \frac{\alpha}{\beta} r \rightarrow \frac{dr}{r} = \frac{\alpha}{\beta} d\theta \rightarrow ln|r| = \frac{\alpha}{\beta}\theta + k \rightarrow r$$

$$= ce^{\frac{\alpha}{\beta}\theta} \ where \ c = e^{k}$$

Is the equation of logarithmic spiral and the origin is called stable focus

3) If $\alpha > 0$, then $r \rightarrow \infty \ as \ t \rightarrow \infty$ and the trajectories are spiral. Then the origin is called unstable focus

Examples determine the type of phase portrait of the corresponding Jordan form of each of the following system

1) $\dot{x_1} = -x_1 + x_2$
$\dot{x_2} = 4x_1 - x_2$

Solution

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2) $\dot{x_1} = x_1 + 2x_2$
$\dot{x_2} = -x_1 + x_2$

Solution

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3) $\dot{x_1} = 2x_1 + x_2$
$\dot{x_2} = -2x_1 + 4x_2$

Solution)

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4) $\dot{x_1} = 3x_1 - x_2$
$\dot{x_2} = x_1 + x_2$

solution

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Trace-determinant diagram

Recall that the polynomial of a square matrix A is defined to be

$$P(\lambda) = \det(A - \lambda I) = |A - \lambda I|$$

For a 2×2 matrix $A,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$P(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+b)\lambda + (ad - bc)$$

$$= \lambda^2 - tr(A)\lambda + det(A)$$

So the eigenvalues are

$\lambda_1 = \frac{1}{2}(tr(A) + \sqrt{\Delta}$ and $\lambda_1 = \frac{1}{2}(tr(A) - \sqrt{\Delta}$

Where $\Delta = (tr(A))^2 - 4\det(A)$

We now explain how the phase portrait of the linear autonomous system depend on trace and determinate of the constant coefficient matrix A

1) If $\det(A) < 0$, the eigenvalues are real and of opposite sign then the phase portrait is a saddle (always is unstable)
2) If $0 < \det(A) < \frac{(tr(A))^2}{4}$ $(i.e\Delta > 0)$, the eigenvalues are real, distinct and of the same sign, the phase portrait is a node as well as stable if $tr(A) < 0$, unstable if $tr(A) > 0$

3) If $0 < \frac{(tr(A))^2}{4} < \det(A)$ $(i.e\Delta < 0)$, then eigenvalues are neither real nor purely imaginary and the phase portrait is sparil, stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) > 0$
4) If $\mathrm{tr}(A) = 0$ *and* $\det(A) > 0$, the eigenvalues are purely imaginary and the phase portrait is a center.
5) If $\Delta = 0$, the eigenvalues are real and equal (repeated) in this case we have two case depend on matrix A:
   i) If A is diagonal, then the phase portrait is a star node, stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) > 0$
   ii) If A is not diagonal, then the phase portrait is a improper node, stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) > 0$

Remark

1) The direction of twist can be identified by the sign of the coefficient C in the original matrix A. so we have the following for cases

2) The direction of trajectories in a case are depend on C

3) The phase portrait of the improper node have the following four cases

Example) determine the type of phase portrait of the corresponding Jordan(Canonical) form of the system $\dot{X} = AX$

1) $A = \begin{pmatrix} -1 & -2 \\ 1 & -1 \end{pmatrix}$

Solution

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2) $A = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$

Solution

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3) $A = \begin{pmatrix} -1 & 1 \\ -3 & -1 \end{pmatrix}$

Solution

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4) $A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

Solution

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5) $A = \begin{pmatrix} 1 & -1 \\ 5 & -1 \end{pmatrix}$

Solution

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6) $A = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix}$

Solution

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7) $A = \begin{pmatrix} 3 & -4 \\ 1 & 4 \end{pmatrix}$

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8) $A = \begin{pmatrix} 8 & 4 \\ -1 & 4 \end{pmatrix}$

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9) $A = \begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}$

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10) $A = \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}$

Solution

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Chapter Three

Non-Linear System in the plane

Definition/ suppose the system can be written in the form

$$\left.\begin{aligned}\dot{x_1} &= ax_1 + bx_2 + g_1(x_1, x_2)\\ \dot{x_2} &= cx_1 + dx_2 + g_2(x_1, x_2)\end{aligned}\right\} \dots \dots (1)$$

Where $\lim_{r \to 0} \frac{g_i(x_1, x_2)}{r} = 0 \;\; as\, r = \sqrt{x_1^2 + x_2^2}$

The linear system $\begin{aligned}\dot{x_1} &= ax_1 + bx_2\\ \dot{x_2} &= cx_1 + dx_2\end{aligned}$ is said to be the linearization or

(linearized system) of eq(1) at the origin.

The vector field $\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$ is called the linear part of

$\begin{pmatrix} ax_1 + bx_2 + g_1(x_1, x_2) \\ cx_1 + dx_2 + g_2(x_1, x_2) \end{pmatrix}$

Example 1/ find the linearization of the following system

i) $\begin{aligned}\dot{x_1} &= x_1^3\\ \dot{x_2} &= x_2 + x_2 sinx_1\end{aligned}$ $\qquad$ $ii)$ $\begin{aligned}\dot{x_1} &= x_1^2 e^{x_2}\\ \dot{x_2} &= x_2^2 e^{x_2 - 1}\end{aligned}$

solution

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Remark/ if $(a,b)$is a fixed point of $\dot{x} = f(x)$,

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $and \ \ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and it is not origin let

$$\begin{matrix} y_1 = x_1 - a \\ y_2 = x_2 - b \end{matrix} \rightarrow \begin{matrix} x_1 = y_1 + a \\ x_2 = y_2 + b \end{matrix}$$

$$\begin{matrix} \dot{y_1} = \dot{x_1} = f_1(y_1 + a, y_2 + b) \\ \dot{y_2} = \dot{x_2} = f_2(y_1 + a, y_2 + b) \end{matrix} \qquad or \quad \begin{matrix} \dot{y_1} = f_1^*(y_1, y_2) \\ \dot{y_2} = f_2^*(y_1, y_2) \end{matrix}$$

Or $\dot{y} = f^*(y) \dots \dots (2)$ where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ &$f^* = \begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix}$

The system (2) has the fixed point at the origin since functions $f_1(x_1,x_2)$ &$f_2(x_1,x_2)$ are continuously differentiable at some neighbourhood of the point (a,b).

By Tayler expansion, then

$$f_1(x_1,x_2) = f_1(a,b) + (x_1-a)\frac{\partial f_i(a,b)}{\partial x_1} + (x_2-b)\frac{\partial f_i(a,b)}{\partial x_2}$$

$$+R_i(x_1,x_2) \qquad\qquad i = 1,2$$

The remainder functions $R_i(x_1,x_2)$ satisfy $\lim\limits_{r\to 0}\frac{R_i(x_1,x_2)}{r} = 0$ where $r = \sqrt{(x_1-a)^2+(x_2-b)^2}$ since (a,b) is a fixed point, then $f_i(a,b) = 0$ and

$$\dot{y_1} = y_1\frac{\partial f_1(a,b)}{\partial x_1} + y_2\frac{\partial f_1(a,b)}{\partial x_2} + R_1(y_1+a, y_2+b)$$

$$\dot{y_2} = y_1\frac{\partial f_2(a,b)}{\partial x_1} + y_2\frac{\partial f_2(a,b)}{\partial x_2} + R_2(y_1+a, y_2+b)$$

The linearization of this system at a fixed point (a,b) is given by

$$\begin{pmatrix}\dot{y_1}\\ \dot{y_2}\end{pmatrix} = \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix}_{(x_1,x_2)=(a,b)} \begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

Because $\lim\limits_{r\to 0}\frac{R_i(y_1+a,y_2+b)}{r} = 0, \quad r = \sqrt{y_1^2+y_2^2}$

Example2/ show that the system $\begin{matrix}\dot{x_1} = e^{x_1+x_2} - x_2\\ \dot{x_2} = -x_1 - x_1x_2\end{matrix}$ has only one fixed point and find the linearization of it at this fixed point.

Solution

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Example3/ find the linearization of the system at the fixed point of

$$\dot{x_1} = -x_2 + x_1 + x_1 x_2$$
$$\dot{x_2} = x_1 - x_2 - x_2^2$$

Solution:

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Def/ a fixed point $(a, b)$of a non- linear system $\begin{matrix} \dot{x_1} = f_1(x_1,\ x_2) \\ \dot{x_2} = f_2(x_1, x_2) \end{matrix}$ is said to be simple if it's linearized system is simple i.e

$$\begin{vmatrix} \frac{\partial}{\partial x_1} f_1(a,\ b) & \frac{\partial}{\partial x_2} f_1(a,\ b) \\ \frac{\partial}{\partial x_1} f_2(a, b) & \frac{\partial}{\partial x_2} f_2(a, b) \end{vmatrix} \neq 0$$

Theorem "linearization theorem "/ let the non-linear system $\dot{x} = f(x)$ have a simple fixed point at $x = 0$. then in a neighbourhood of the region the phase portrait of the system and it's linearization are qualitatively equivalent provided the linearized system is not a center .

Example 4/ use the linearization theorem to determine the phase portrait of the system

$i) \begin{matrix} \dot{x_1} = x_1 + 4x_2 + e^{x_1} - 1 \\ \dot{x_2} = -x_2 - x_2 e^{x_1} \end{matrix}$ at the origin.

$ii) \begin{matrix} \dot{x_1} = x_2 \\ \dot{x_2} = -x_1 - (1 + x_1^2 + x_1^4)x_2 \end{matrix}.$

$iii) \begin{matrix} \dot{x_1} = x_2^2 - 3x_1 + 2 \\ \dot{x_2} = x_1^2 - x_2^2 \end{matrix}.$

Solution:

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Theorem " symmetric condition"/ if the origin is the fixed point of

$$\begin{aligned} \dot{x_1} &= f_1(x_1,x_2) \\ \dot{x_2} &= f_2(x_1,x_2) \end{aligned} \quad \ldots\ldots\ldots (1)$$

and is the center for the linearized system if

$$\begin{aligned} f_1(x_1,-x_2) &= -f_1(x_1,x_2) \\ f_2(x_1,-x_2) &= f_2(x_1,x_2) \end{aligned} \quad \text{or} \quad \begin{aligned} f_1(-x_1,x_2) &= f_1(x_1,x_2) even \\ f_2(-x_1,x_2) &= -f_2(x_1,x_2) odd \end{aligned}$$

Then the origin is the center for eq(1)

Example5/ determine the type of fixed point in the following

$$i)\begin{matrix}\dot{x_1} = 2x_1x_2 - 2x_2 \\ \dot{x_2} = x_1 - x_2^2\end{matrix} \qquad ii)\begin{matrix}\dot{x} = y + 1 \\ \dot{y} = -4x + 8\end{matrix}$$

Solution:

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Remark/ symmetric condition theorem is not necessary and sufficient condition

Non- simple fixed point

Def/ a fixed point $(a, b)$of a non- linear system $\begin{matrix} \dot{x_1} = f_1(x_1,\ x_2) \\ \dot{x_2} = f_2(x_1, x_2) \end{matrix}$ is said to be non- simple if it's linearized system is non-simple i.e

$$\begin{vmatrix} \frac{\partial}{\partial x_1} f_1(a,\ b) & \frac{\partial}{\partial x_2} f_1(a,\ b) \\ \frac{\partial}{\partial x_1} f_2(a, b) & \frac{\partial}{\partial x_2} f_2(a, b) \end{vmatrix} = 0$$

Example6 / sketch the phase portrait of the following

$$i)\begin{matrix} \dot{x_1} = x_1^2 \\ \dot{x_2} = x_2 \end{matrix} \qquad ii)\begin{matrix} \dot{x_1} = x_1 - x_2^2 \\ \dot{x_2} = x_2(x_1 - x_2^2) \end{matrix}$$

Solution:

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Stability of fixed point

We consider the autonomous $\begin{matrix} \dot{x_1} = f_1(x_1, x_2) \\ \dot{x_2} = f_2(x_1, x_2) \end{matrix}$ ……… (1)

Def/ let (a, b)be a fixed point in (1) we say that (a, b) is stable if for every $\varepsilon > 0, \exists \delta > 0,$ s.t every solution u, v which satisfies

$[u(t_o) - a]^2 + [v(t_o) - b]^2 < \delta^2$ for some $t > t_o$ ………..(2)

Also satisfies $[u(t) - a]^2 + [v(t) - b]^2 < \varepsilon^2 \quad \forall t > t_o$

Geometrically/ this means that a fixed point (a, b) of (1) is said to be stable if for every neighbourhood $N$of (a, b)there is a smaller neighbourhood $N' \subseteq N$of (a, b) s.t every trajectory which passes through N' remains in N as t increase

Def/ a fixed point (a, b) of (1) is said to be asymptotically stable if (a, b) is stable and if δ can be chosen so that

$[u(t_o) - a]^2 + [v(t_o) - b]^2 < \delta^2$ for some $t > t_o$ implies that

$(u(t), v(t)) \to (a, b)$at $t \to \infty$

Geometrically/ a fixed point (a, b) of (1) is said to be asymptotically stable if there is a neighbourhood $N$of (a, b) s.t every trajectory passing through N approaches (a, b) as $t \to \infty$.

Remark/ every asymptotically stable fixed point is stable by taking $N' = N,$ but converse is not true.

Example7/ show that the system $\begin{matrix} \dot{x_1} = x_2 \\ \dot{x_2} = -x_1^3 \end{matrix}$is stable at the origin but not asymptotically stable.

Solution :

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Def/ a fixed point of the system (1) which is stable but not asymptotically stable is said to be weakly or neutrally stable

Example/ center is stable but not asymptotically stable so it is called weakly stable

Def/ a fixed point of system (1) which is not stable is said to be unstable

Geometrically/ this means that there is a neighbourhood N of the fixed point (a, b) such that for every neighbourhood $N' \subset N$ there is at least one trajectory which passes through N' and does not remain in N

Example/ saddle point is unstable fixed point

unstable node is unstable fixed point

Remark/ the fixed point of a linear system $\dot{X} = AX$ with $|A| \neq 0$ is stable if no eigen value of Ahas positive real part asymptotically stable if both eigen values have negative real part, and weakly stable if the eigen values are purely imaginary thus stable nodes and focus are asymptotically stable saddle point ,unstable node, focus are unstable and center are weakly stable.

Example8/ show that the two systems

$$\left.\begin{aligned} \dot{x_1} &= -x_2 + x_1(x_1^2 + x_2^2) \\ \dot{x_2} &= x_1 + x_2(x_1^2 + x_2^2) \end{aligned}\right\} \dots\dots\dots(1)$$

$$\left.\begin{aligned} \dot{x_1} &= -x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x_2} &= x_1 - x_2(x_1^2 + x_2^2) \end{aligned}\right\} \dots\dots\dots(2)$$

Both have the some linearized systems at the origin, but their phase portraits are qualitatively different

Solution:

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