Second rule :

$$y'' + a_1 y' + a_2 y = Q(x)$$

1)The first case when $Q(x) = \cos ax$ $\quad or\ Q(x) = \sin ax$ then The particular integral is defined by Express P(D) as a function of D^2 say $\varphi(D^2)$ and then replace $D^2\ by\ -a^2$.

If $\varphi(-a^2) \neq 0$ then we use the following result

$$P(D)y = (D^2 + a_1 D + a_2)y = Q(x)$$

$$y_p = \frac{1}{P(D)} \cos ax = \frac{1}{\varphi(D^2)} \cos ax = \frac{1}{\varphi(-a^2)} \cos ax$$

Example : Find the general solution of $y'' - y = \cos 2x$

Solution :The general solution is of the form $Y = y_c + y_p$

we find the complementary solution y_c of $y'' - y = 0$

$$(D^2 - 1)y = 0$$

$$(D^2 - 1) = 0$$

$$(m^2 - 1) = 0$$

$$m_{1,2} = \pm 1$$

Then the complementary solution is $y_c = c_1 e^x + c_2 e^{-x}$

To find the particular solution y_p of the non-homogenous diff. equation

$$y_p = \frac{1}{(D^2 - 1)} \cos 2x = \frac{1}{(-2^2 - 1)} \cos 2x$$

$$= \frac{1}{-5} \cos 2x$$

Then the general solution is $y = c_1 e^x + c_2 e^{-x} - \frac{1}{5} \cos 2x$

Remark 1: Sometimes we cannot form $\varphi(D^2)$ .Then we shall try to get $\varphi(D, D^2)$,that is ,a function of D & D^2 in such cases we proceed as follows :

Example: Solve $y'' - 2y' + y = \cos 3\,x$

Solution : we find the complementary solution of $y'' - 2y' + y = 0$

$$(D^2 - 2D + 1)y = 0$$

$$(D^2 - 2D + 1) = 0$$

$$(m^2 - 2m + 1) = 0$$

$$(m - 1)^2 = 0$$

$$m_{1,2} = 1$$

Then the complementary solution is $y_c = c_1 e^x + c_2 x e^x$

To find the particular solution of the non-homogenous diff. equation

$$(D^2 - 2D + 1)y = \cos 3x$$

$$y_p = \frac{1}{(D^2 - 2D + 1)} \cos 3x = \frac{1}{(-3^2 - 2D + 1)} \cos 3x$$

$$= \frac{-1}{2} \frac{1}{(D + 4)} \cos 3x = \frac{-1}{2} \frac{(D - 4)}{(D + 4)(D - 4)} \cos 3x$$

$$= \frac{-1}{2} \frac{(D - 4)}{(D^2 - 4^2)} \cos 3x = \frac{-1}{2} (D - 4) \frac{1}{(-3^2 - 4^2)} \cos 3x$$

$$= \frac{1}{50} (D - 4) \cos 3x = \frac{1}{50} (D \cos 3x - 4 \cos 3x)$$

$$= \frac{1}{50} (-3 \sin 3x - 4 \cos 3x)$$

Then the g.s. is $y = c_1 e^x + c_2 x e^x + \frac{1}{50} (-3 \sin 3x - 4 \cos 3x)$

Remark 2:If $\varphi(-a^2) = 0$, then we shall use the formula $y_p = \frac{1}{D^2+a^2}\cos ax$

$$y_p = \frac{1}{-a^2 + a^2}\cos ax$$

$$y_p = \frac{1}{D^2 + a^2}\cos ax = real\ part\ of\ (\frac{1}{D^2 + a^2}e^{iax})$$

$$e^{ibt} = \cos bt + i\sin bt$$

$$= real\ part\ of\ \left(\frac{1}{(D+ia)(D-ia)}e^{iax}\right)$$

$$= real\ part\ of\ \left(\frac{1}{(ia+ia)(D-ia)}e^{iax}\right)$$

$$= real\ part\ of\ \left(\frac{1}{(2ia)(D-ia)}e^{iax}\right)$$

$$= real\ part\ of\ \left(\frac{x}{(2ia)}e^{iax}\right)$$

$$= real\ part\ of\ \left(\frac{-ix}{(2a)}(\cos ax + i\sin ax)\right)$$

$$= real\ part\ of\ \left(\frac{-ix}{(2a)}(\cos ax) + \frac{x}{(2a)}\sin ax)\right)$$

$$y_p = \frac{x}{2a}\sin ax$$

Also for $Q(x) = \sin ax$

$$y_p = \frac{1}{D^2 + a^2}\sin ax = Imaginary\ part\ of\ (\frac{1}{D^2 + a^2}e^{iax})$$

$$= Imaginary\ part\ of\ \left(\frac{1}{(D+ia)(D-ia)}e^{iax}\right)$$

$$= Imaginary\ part\ of\ \left(\frac{1}{(ia+ia)(D-ia)}e^{iax}\right)$$

$$= Imaginary\ part\ of\ \left(\frac{1}{(2ia)(D-ia)}e^{iax}\right)$$

$$= Imaginary\ part\ of\ \left(\frac{x}{(2ia)}e^{iax}\right)$$

$$= Imaginary\ part\ of\ \left(\frac{-ix}{(2a)}(\cos ax + i\sin ax)\right)$$

$$= Imaginary\ part\ of\ \left(\frac{-ix}{(2a)}(\cos ax) + \frac{x}{(2a)}\sin ax)\right)$$

$$y_p = \frac{-x}{2a}\cos ax$$

Example : Find the general solution of $y'' + 4y = 8\cos 2x$

Solution: First we find y_c of diff.eq.

$$y'' + 4y = 0$$

$$(D^2 + 4)y = 0 \Rightarrow (m^2 + 4) = 0$$

$$m^2 = -4 \Rightarrow m = \pm 2i$$

$$y_c = c_1\cos 2x + c_2\sin 2x$$

$$y_p = \frac{1}{D^2+2^2}8\cos ax$$

Now to find the $y_p$of diff. eq.

$$y_p = 8\frac{1}{D^2+2^2}\cos 2x = 8\frac{1}{-2^2+2^2}\ \cos 2x$$

$$y_p = \frac{1}{D^2+2^2}8\cos ax = 8\ real\ part\ of\ (\frac{1}{D^2+2^2}e^{2ix})$$

$$= 8\ real\ part\ of\ \left(\frac{1}{(D+2i)(D-2i)}e^{2ix}\right)$$

$$= 8\ real\ part\ of\ \left(\frac{1}{(4i)(D-2i)}e^{2ix}\right)$$

$$= 8\, real\ part\ of\ \left(\frac{x}{(4i)} e^{2ix}\right)$$

$$= 8\, real\ part\ of\ \left(\frac{i\,x}{(4i * i)} e^{2ix}\right)$$

$$= 8\, real\ part\ of\ \left(\frac{-ix}{(4)} (\cos 2x + i \sin 2x)\right)$$

$$= 8\, real\ part\ of\ \left(\frac{-ix}{(4)} (\cos 2x) + \frac{x}{(4)} \sin 2x)\right)$$

$$y_p = 8\ \frac{x}{4} \sin ax$$

$$y_p = 2x \quad \sin 2x$$

Exercise: Find the general solution of the following differential equations

1. $y'' + y' - 2y = e^x$ $\qquad Ans: y = c_1 e^x + c_2 e^{-2x} + \frac{1}{3} x\, e^x$
2. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{-x}$ $\qquad Ans: y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{8} x\, e^{-x}$
3. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$ $\qquad Ans: y = c_1 e^x + c_2 e^{2x} + \frac{1}{12}\, e^{5x}$
4. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$ $\qquad Ans: y = (c_1 + c_2 x) e^{-2x} - \frac{1}{2}\, x^2 e^{-2x}$
5. $y'' + y = \sin 2x$ $\qquad Ans: y = (c_1 \cos x + c_2 \sin x) - \frac{1}{3} \sin 2x$
6. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin 3x$ $\qquad Ans: y = c_1 e^x + c_2 e^{2x} + \frac{1}{130} (9 \cos 3x - 7 \sin 3x)$
7. $y'' + y' = \cos 2x$ $\qquad Ans: y = (c_1 \cos x + c_2 \sin x) - \frac{1}{10} (\sin 2x + 2 \cos 2x)$
8. $y'' - y = \cos x$ $\qquad Ans: y = (c_1 e^x + c_2 e^{-x}) - \frac{1}{2} \cos x$
9. $y'' + 16y = \cos 4x$ $\qquad Ans: y = (c_1 \cos 4x + c_2 \sin 4x) + \frac{x}{8} \sin 4x$
10. $y'' - y = \cos^2 3x$ $\qquad Ans: y = (c_1 e^x + c_2 e^{-x}) - \frac{1}{2} - \frac{39}{74} \cos 6x$

Rule 3:If $Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ be a polynomial to find the particular solution . Take out the lowest degree term from P(D), so as to reduce in the form $[1 + \psi(D)]^n$. We now take it to numerator and get $[1 + \psi(D)]^{-n}$ which is then expanded with the help of binomial theorem

$$(1+t)^n = 1 + nt + \frac{n(n-1)}{2!}t^2 + \frac{n(n-1)(n-2)}{3!}t^3 + \cdots$$

Example: Find the particular solution of $y'' + y' - 6y = x$

Solution: $D^2y + Dy - 6y = x$

$$(D^2 + D - 6)y = x$$

$$y_p = \frac{1}{D^2 + D - 6}x = \frac{1}{-6 + \left(\frac{-6(D^2 + D}{-6}\right)}x = -\frac{1}{6}\frac{1}{1 + \left(\frac{-D^2 - D}{6}\right)}$$

By using

$$(1+t)^n = 1 + nt + \frac{n(n-1)}{2!}t^2 + \frac{n(n-1)(n-2)}{3!}t^3 + \cdots$$

$$= -\frac{1}{6}\left[1 + \left(\frac{-D^2 - D}{6}\right)\right]^{-1} x$$

$$= -\frac{1}{6}\left(1 + (-1)\left(\frac{-D^2 - D}{6}\right) + \frac{(-1)(-2)}{2!}\left(\frac{-D^2 - D}{6}\right)^2 + \cdots\right)x$$

$$= -\frac{1}{6}\left(x + (-1)\left(\frac{-D^2x - Dx}{6}\right)\right) = -\frac{1}{6}\ (x + \frac{1}{6})$$

Then $y_p == -\frac{1}{6}\ (x + \frac{1}{6})$

Example: Find the general solution of $y'' - 4y' + 4y = x^2 + x$ (*)

Solution:the general solution is $y = y_c + y_p$

to find y_c of eq (*)

$$y'' - 4y' + 4y = 0$$

$$(D^2 - 4D + 4)y = 0 \Rightarrow (m^2 - 4m + 4) = 0$$

$$(m-2)^2=0 \Rightarrow \quad m1=2, m2=2$$

Then $y_c=(c_1+c_2x)e^{2x}$

To find y_p of eq(*)

$$(D^2-4D+4)y=x^2+x$$

$$y_p=\frac{1}{(D^2-4D+4)}(x^2+x)$$

$$y_p=\frac{1}{(D-2)^2}(x^2+x)=\frac{1}{(-2)^2\left(1+\frac{-D}{2}\right)^2}(x^2+x)$$

$$=\frac{1}{4}\left(1+\frac{-D}{2}\right)^{-2}(x^2+x)$$

$$=\frac{1}{4}\left(1+(-2)\left(\frac{-D}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(-\frac{D}{2}\right)^3+\cdots\right)(x^2+x)$$

$$=\frac{1}{4}\left(1+D+\frac{3}{4}D^2+\frac{1}{2}D^3+\cdots\right)(x^2+x)$$

$$=\frac{1}{4}\left((x^2+x)+D(x^2+x)+\frac{3}{4}D^2(x^2+x)+\frac{1}{2}D^3(x^2+x)+\cdots\right)$$

$$=\frac{1}{4}\left((x^2+x)+2x+1+\frac{3}{4}.2+\frac{1}{2}.0+\cdots\right)=\frac{1}{4}\left((x^2+x)+2x+1+\frac{3}{2}\right)$$

$$y_p=\frac{1}{4}\left(x^2+3x+\frac{5}{2}\right)$$

$$y=(c_1+c_2x)e^{2x}+\frac{1}{4}\left(x^2+3x+\frac{5}{2}\right)$$

Exercise: Find the general solution of the following differential equations

1. $y''+y'-6y=x$ $\qquad$ $Ans: y=c_1e^{2x}+c_2e^{-3x}-\frac{1}{36}(6x+1)$

2. $y'' - 4y = x^2$ $\quad\quad$ $Ans: y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}(2x^2 + 1)$

3. $y'' + 2y' + y = 2x + x^2$ $\quad\quad$ $Ans: y = (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$

Rule 4: when $Q(x) = e^{kx}\, V$ where V is a function of $\cos ax$ $\;or\;$ $\sin ax$ or polynomial function as $b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$

Then we use the formula $y_p = \frac{1}{P(D)} e^{kx}\, V = e^{kx} \frac{1}{P(D+k)} V$

Example:Solve $y'' + 3y' + 2y = e^{2x} \sin x$

Solution: we have $y_c = c_1 e^{-2x} + c_2 e^{-x}$ H.W.

To find the y_p

$$D^2 y + 3Dy + 2y = e^{2x} \sin x$$

$$(D^2 + 3D + 2)y = e^{2x} \sin x$$

$$y_p = \frac{1}{D^2 + 3D + 2}\, e^{2x} \sin x$$

$$y_p = e^{2x} \frac{1}{(D+2)^2 + 3(D+2) + 2} \sin x$$

$$y_p = e^{2x} \frac{1}{D^2 + 4D + 4 + 3D + 6 + 2} \sin x$$

$$y_p = e^{2x} \frac{1}{D^2 + 7D + 12} \sin x$$

$$y_p = e^{2x} \frac{1}{-1^2 + 7D + 12} \sin x$$

$$y_p = e^{2x} \frac{1}{7D + 11} \sin x$$

$$y_p = e^{2x} \frac{1}{7D + 11} \quad \frac{7D - 11}{7D - 11} \sin x$$

$$y_p = e^{2x} \frac{7D - 11}{49D^2 - 121} \sin x$$

$$y_p = e^{2x} \frac{7D - 11}{49(-1^2) - 121} \sin x \Rightarrow y_p = -\frac{e^{2x}}{170} \ (7D - 11) \sin x$$

$$y_p = -\frac{e^{2x}}{170} \ (7D \sin x - 11 \sin x) \Rightarrow y_p = -\frac{e^{2x}}{170} \ (7 \cos x - 11 \sin x)$$

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{e^{2x}}{170} \ (11 \sin x - 7 \cos x)$$

Example: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$

Solution: general solution is $y = y_c + y_p$

Then $y_c = (c_1 + c_2 x)e^x \qquad H.W.$

To find y_p then $\quad D^2 y - 2Dy + y = x^2 e^{3x}$

$$(D^2 - 2D + 1)y = x^2 e^{3x}$$

$$y_p = \frac{1}{D^2 - 2D + 1} \ x^2 e^{3x} \Rightarrow y_p = e^{3x} \frac{1}{(D+3)^2 - 2(D+3) + 1} \ x^2$$

$$y_p = e^{3x} \frac{1}{D^2 + 6D + 9 - 2D - 6 + 1} \ x^2 \Rightarrow y_p = e^{3x} \frac{1}{D^2 + 4D + 4} \ x^2$$

$$y_p = e^{3x} \frac{1}{(D + 2)^2} \ x^2 = \frac{e^{3x}}{4} \quad \frac{1}{\left(1 + \frac{D}{2}\right)^2} \ x^2$$

$$y_p = \frac{e^{3x}}{4} \quad \left(1 + \frac{D}{2}\right)^{-2} \ x^2$$

$$= \frac{e^{3x}}{4} \quad \left(1 + (-2)\frac{D}{2} + \frac{(-2)(-3)}{2!}\left(\frac{D}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{D}{2}\right)^3 + \cdots \right) x^2$$

$$= \frac{e^{3x}}{4} \quad \left(1 - D + 3\frac{D^2}{4} - 4\frac{D^3}{8} + \cdots\right) x^2$$

$$= \frac{e^{3x}}{4} \quad \left(x^2 - Dx^2 + 3\frac{D^2 x^2}{4} - 4\frac{D^3 x^2}{8} + \cdots\right) = \frac{e^{3x}}{4} \quad \left(x^2 - 2x + \frac{3}{2} - 0\right)$$

$$y_p = \frac{e^{3x}}{4} \quad \left(x^2 - x + \frac{3}{2}\right)$$

Then $y = (c_1 + c_2 x)e^x + \frac{e^{3x}}{4} \quad \left(x^2 - 2x + \frac{3}{2}\right)$

Exercise: Solve the following differential equations

1.$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = x\, e^{-x}\, , given\ that\ y = 0\ \&\ \frac{dy}{dx} = 0\ when\ x = 0$

$Ans: y = \frac{1}{216}(e^{5x} - e^{-x}) - \frac{1}{12}e^{-x}(x^2 + \frac{x}{3})$

2. $y'' - y = \cosh x\ \cos x$

$Ans: y = c_1 e^x + c_2 e^{-x} + \frac{2}{5}\sin x \sinh x - \frac{1}{5}\cos x\ \cosh x$

3. $y'' = e^x \cos x$ $\qquad Ans \qquad y = c_1 + c_2 x + \frac{1}{2}e^x \sin x$

4. $y'' + 4y' - 12y = (x-1)e^x$ $\qquad Ans:\ y = c_1 e^{2x} + c_2 e^{-6x} + \frac{1}{64}e^{2x}(4x^2 - 9x)$

5. $y'' - 2y' + 5y = e^{2x} \sin x$

$$Ans:\ y = e^x(c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{10}e^{2x}(\cos x - 2\sin x)$$

Rule 5: when $Q(x) = (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0)\,V$ where V is a function of $\cos ax \quad or \quad \sin ax$

Then we use the formula

$$y_p = \frac{1}{P(D)}(b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0)V$$

$$= \frac{1}{P(D)} (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0) e^{iax}$$

$$= e^{iax} \frac{1}{P(D+ia)} (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0)$$

Then If $V = \cos ax \ then \ y_p \ take \ the \ real \ part$ & $V = \sin ax \ then \ y_p \ take \ the \ Imaginary \ part$

Example :Find particular solution of $y'' + 2y' + y = x \cos x$

Solution: To find y_p

$$D^2 y + 2Dy + y = x \cos x$$

$$(D^2 + 2D + 1) y = x \cos x$$

$$y_p = \frac{1}{(D+1)^2} x \cos x = real\ part\ of\ (\frac{1}{(D+1)^2} x e^{ix})$$

$$= e^{ix} \frac{1}{(D+i+1)^2} x$$

$$= \frac{e^{ix}}{(i+1)^2} \frac{1}{\left(1 + \frac{D}{i+1}\right)^2} x = \frac{e^{ix}}{2i} \left(1 + \frac{D}{i+1}\right)^{-2} x$$

$$= \frac{e^{ix}}{2i} (1 + (-2)(\frac{D}{i+1}) + \frac{(-2)(-3)}{2!} \left(\frac{D}{i+1}\right)^2 + \cdots) x$$

$$= \frac{e^{ix}}{2i} \left(1 - \frac{2D}{i+1}\right) x = \frac{e^{ix}}{2i} \left(x - \frac{2}{i+1}\right)$$

$$= -i \frac{e^{ix}}{2} \left(x - \frac{2}{i+1} \frac{i-1}{i-1}\right) = -i \frac{e^{ix}}{2} (x + i - 1)$$

$$= -i \frac{(\cos x + i \sin x)}{2} (x + i - 1) = -\frac{(i \cos x - \sin x)}{2} (x - 1 + i)$$

$$= \frac{1}{2} ((x-1) \sin x + \cos x) + i \frac{1}{2} ((x-1) \cos x + \sin x)$$

$$y_p = real\ part\ of(\frac{1}{2}((x-1)\sin x+\cos x)+i\frac{1}{2}((x-1)\cos x+\sin x))$$

$$y_p = \frac{1}{2}((x-1)\sin x+\cos x))$$

Exercise :Solve the following differential equations

1. $y''+y'=x\cos x$ Ans $y=c_1+c_2e^{-x}+\frac{1}{2}x(\sin x-\cos x)+\frac{1}{2}\sin x$
2. $y''+4y=x\sin x$ Ans $y=c_1\cos 2x+c_2\sin 2x+\frac{1}{9}(3x\sin x-2\cos x)$
3. $y''-y=x^2\cos x$ Ans $y=c_1e^x+c_2e^{-x}+\frac{1}{2}(1-x^2)\cos x+x\sin x$
4. $y''+4y=x\ sin^2x$ Ans $y=c_1\cos 2x+c_2\sin 2x+\frac{1}{8}x(1+2\sin 2x)$
5. $y''-4y'+4y=8x^2e^{2x}\sin x$
   Ans $y=(c_1+c_2x)e^{2x}+e^{2x}(3\sin 2x-2x^2\sin 2x-4x\cos 2x)$

Example: Find particular solution of $y''-4y'+4y=x^2+e^x+\cos 2x$

Solution:$D^2y-4Dy+4y=x^2+e^x+\cos 2x$

$$(D^2-4D+4)y=x^2+e^x+\cos 2x$$

$$y=\frac{1}{(D^2-4D+4)}(x^2+e^x+\cos 2x)$$

$$y=\frac{1}{(D^2-4D+4)}x^2+\frac{1}{(D^2-4D+4)}e^x+\frac{1}{(D^2-4D+4)}\cos 2x$$

Particular solution of part

$$y_{p1}=\frac{1}{(D^2-4D+4)}x^2=\frac{1}{(D-2)^2}x^2$$

$$=\frac{1}{4}\frac{1}{\left(1+(\frac{D}{-2})\right)^2}x^2=\frac{1}{4}\left(1+(\frac{D}{-2})\right)^{-2}x^2$$

$$=\frac{1}{4}\left(1+(-2)\left(\frac{D}{-2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{D}{-2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{D}{-2}\right)^3+\cdots\right)x^2$$

$$=\frac{1}{4}\left(1+D+\frac{3}{4}D^2-\frac{1}{2}D^3+\cdots\right)x^2=\frac{1}{4}\left(x^2+Dx^2+\frac{3}{4}D^2x^2-\frac{1}{2}D^3x^2+\cdots\right)$$

$$=\frac{1}{4}\left(x^2+2x+\frac{3}{4}2-\frac{1}{2}0\right)=\frac{1}{4}(x^2+2x+\frac{3}{2})$$

Particular solution of part

$$y_{p2}=\frac{1}{(D^2-4D+4)}e^x=\frac{1}{(1^2-4*1+4)}e^x=e^x$$

Particular solution of part

$$y_{p3} = \frac{1}{(D^2-4D+4)}\cos 2x = \frac{1}{(-2^2-4D+4)}\cos 2x$$

$$= \frac{1}{-4D}\cos 2x = -\frac{1}{4}\frac{1}{D}\cos 2x = -\frac{1}{4}\frac{1}{2}\sin 2x = -\frac{1}{8}\sin 2x$$

$$y_p = y_{p1} + y_{p2} + y_{p3} = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x - \frac{1}{8}\sin 2x$$

Example: Find particular solution of $y'' + y' + y = x^2 + e^{-x}\sin 2x$

Solution: $D^2y + Dy + y = x^2 + e^{-x}\sin 2x$

$$(D^2 + D + 1)y = x^2 + e^{-x}\sin 2x$$

$$y = \frac{1}{(D^2 + D + 1)}(x^2 + e^{-x}\sin 2x)$$

$$y = \frac{1}{(D^2 + D + 1)}x^2 + \frac{1}{(D^2 + D + 1)}(e^{-x}\sin 2x)$$

Particular solution of part

$$y_{p1} = \frac{1}{(D^2+D+1)}x^2 = (1 + D + D^2)^{-1}x^2$$

$$= \left(1 + (-1)(D + D^2) + \frac{(-1)(-2)}{2!}(D + D^2)^2 + \frac{(-1)(-2)(-3)}{3!}(D + D^2)^3 + \cdots\right)x^2$$

$$= (1 - D - D^2 + D^2 + 2D^3 + D^4 - D^3 - 3D^4 - 3D^5 - D^6 + \cdots)x^2$$

$$= (1 - D + D^3 - 2D^4 - 3D^5 - D^6 + \cdots)x^2$$

$$= (x^2 - Dx^2 + D^3x^2 - 2D^4x^2 - 3D^5x^2 - D^6x^2 + \cdots)$$

$$= (x^2 - 2x + 0) = (x^2 - 2x)$$

Particular solution of part

$$y_{p2} = \frac{1}{(D^2+D+1)}(e^{-x}\sin 2x) = e^{-x}\frac{1}{((D-1)^2+(D-1)+1)}\sin 2x$$

$$= e^{-x}\frac{1}{(D^2-2D+1+D-1+1)}\sin 2x = e^{-x}\frac{1}{(D^2-D+1)}\sin 2x$$

$$= e^{-x}\frac{1}{(-2^2 - D + 1)}\sin 2x = e^{-x}\frac{1}{(-3 - D)}\sin 2x$$

$$= e^{-x}\frac{1}{(-3 - D)}\frac{(-3 + D)}{(-3 + D)}\sin 2x = e^{-x}\frac{(-3 + D)}{(9 - D^2)}\sin 2x$$

$$= e^{-x}\frac{(-3 + D)}{(9 - (-2^2))}\sin 2x = \frac{e^{-x}}{13}(-3 + D)\sin 2x$$

$$= \frac{e^{-x}}{13}(-3\sin 2x + D\sin 2x) = \frac{e^{-x}}{13}(-3\sin 2x + 2\cos 2x)$$

$$y_p = y_{p1} + y_{p2} = x^2 - 2x + \frac{e^{-x}}{13}(-3\sin 2x + 2\cos 2x)$$