## Numerical Analysis CHAPTER ONE <br> Approximations and Errors

The difference between exact solution and numerical solution is that numerical solution gives the answer with some "error", because numerical solution involved an approximation.

For many engineering problems, we cannot obtain analytical solution; therefore, we cannot compute the errors associated with numerical solution.

In professional practice, errors can be costly and sometimes catastrophic. If a structure or device fails, lives can be lost.

1. Types of Errors: There are two major types of errors:
2. Round off error: which is due to computers, (e.g. $\pi=3.14160$ instead of $\pi=3.14159253589 \ldots$...
3. Truncation error is the difference between a truncated value and the actual value. A truncated quantity is represented by a numeral with a fixed number of allowed digits, with any excess digits.

As an example of truncation error, consider the speed of light in a vacuum. The official value is $299,792,458$ meters per second. In scientific (power-of-10) notation, that quantity is expressed as $2.99792458 \times 10^{8}$. Truncating it to two decimal places yields 2.99 x $10^{8}$. The truncation error is the difference between the actual value and the truncated value, or $0.00792458 \times 10^{8}$. Expressed properly in scientific notation, it is $7.92458 \times 10^{5}$.

1. Significant digits: They are the number of digits that can be used with confidence. They correspond to the certain digits plus one estimated digit. The figure shows a car speedometer that reads $48.5 \mathrm{~km} / \mathrm{h}$, where (48) are significant digits and (5) is estimated.


## 2. Accuracy and Precision:

1. Accuracy refers to how closely a computed or measured values agrees with true value.
2. Precision refers to how closely measured values agree with each other (i.e. how many significant digits). Numerical methods should be significantly accurate to meet requirements, and precise enough for good engineering design.

3. Absolute Error (or true error):It is the relationship between the exact result (or true value) and approximate value.
i.e. $\quad$ Approximate value $=$ True value $+\operatorname{Error}\left(\mathrm{E}_{\mathrm{t}}\right)$

Where $\left(E_{t}\right)$ is the exact value of error.
2. Relative Error: It is the normalized error to the true value as in: Relative error $\left(\mathbf{G}_{\mathbf{r}}\right)=$ True error $\left(\mathrm{E}_{\mathrm{t}}\right) /$ True value

Or in $\%, \quad \mathbf{\epsilon}_{\mathbf{r}} \%=\left(\right.$ True error $\left(\mathrm{E}_{\mathrm{t}}\right) /$ True value $) * 100 \%$
3. Approximate Error $\left(\epsilon_{\mathrm{a}}\right)$ : If we do not know the true value, and the true error also unknown, then we use an approximate error $\left(\epsilon_{a}\right)$, which can be defined as:
$\mathbf{\epsilon}_{\mathbf{a}}=$ (Approximate error / approximate value) * $100 \%$
Where, $\left[\mathbf{E}_{\mathbf{a}}=\right.$ Approximate error $=$ Present value - Previous value $]$
This definition is used mainly with iterative methods. If a specified value of error $\left(\boldsymbol{\epsilon}_{\boldsymbol{s}}\right)$ is wanted, then the computation is generally repeated until $\left|\epsilon_{\mathrm{a}}\right|<\mathrm{\epsilon}_{\mathrm{s}}$.

## 4. Absolute error in algebraic operations:

1. Absolute error in summation and subtraction: If the absolute error $\left(\mathrm{E}_{\mathrm{a}}\right)$, then if two numbers are added or subtracted, then the magnitude of total absolute error is equal to the sum of individual errors. i.e.

$$
\mathrm{E}_{\mathrm{a}}=\mathrm{E}_{\mathrm{a} 1} \pm \mathrm{E}_{\mathrm{a} 2} \pm \mathrm{E}_{\mathrm{a} 3} \pm \ldots
$$

2. Absolute error in product: If we have two numbers "A" and "B". Let the absolute error in " A " is " $\mathrm{E}_{\mathrm{a}}$ ", and absolute error in " B " is " $\mathrm{E}_{\mathrm{b}}$ ", then
Approximate value of $\quad$ "A" $=\mathrm{A}+\mathrm{E}_{\mathrm{a}}$, and approximate value of " $\mathrm{B} "=\mathrm{B}+\mathrm{E}_{\mathrm{b}}$
Then abs. error in product $\left(\mathrm{E}_{\mathrm{ap}}\right)=\left(\mathrm{A}+\mathrm{E}_{\mathrm{a}}\right) *\left(\mathrm{~B}+\mathrm{E}_{\mathrm{b}}\right)-\mathrm{AB}$

$$
\begin{aligned}
& =\mathrm{AB}+\mathrm{A} * \mathrm{~Eb}+\mathrm{B} * \mathrm{Ea}+\mathrm{Ea} * \mathrm{~Eb}-\mathrm{AB} \\
& \quad \approx \mathrm{~A} * \mathrm{E}+\mathrm{B} * \mathrm{Ea}
\end{aligned}
$$

3. Absolute error in division: For the same above example, the absolute error will be:
Absolute error in division $\left(\mathrm{E}_{\mathrm{ad}}\right)=\frac{A+\mathrm{Ea}}{B+\mathrm{Eb}}-\frac{A}{B} \approx \frac{B * \mathrm{Ea}-A * \mathrm{~Eb}}{B^{2}}$
4. Error propagation: When error is introduced in a variable, it propagates in other variables because of computations. This amount of error depends upon the mathematical or numerical operation performed. Consider the function,

$$
f(x)=\frac{1}{1-x^{2}}
$$

Let's calculate $\mathrm{f}(\mathrm{x})$ for $\mathrm{x}=0.9$. Then exact value will be (5.2631579).
Let's assume that approximate value of ( x ) is $\left(0.900005\right.$,i.e. an error of $\left.5 * 10^{-6}\right)$. With this value of (x), the approximate value of $f(x)$ will be (5.2634072), so the error will be $\left(0.000025\right.$,i.e. an error of $\left.25^{*} 10^{-6}\right)$. This is called error magnification (or error propagation since the error propagate from the $6^{\text {th }}$ digit to the $5^{\text {th }}$ digit.

Under such condition, the numerical method or computation procedure is said to be (numerically unstable). To avoid this instability, the numerical process is rearranged or some other method is used.

## 2. Solved examples:

Example1.1- Calculate the absolute and relative error in the following cases:
a) True value $=1 \times 10^{-6}$, Approximate value $=0.5 \times 10^{-6}$
b) True value $=1 \times 10^{6}$, Approximate value $=0.99 \times 10^{6}$

## Solution:

1. $\quad$ Absolute error $=\mid$ True value - Approximate value $\mid$
$=1 \times 10^{-6}-0.5 \times 10^{-6}=0.5 \times 10^{-6}$
Relative $\operatorname{error}\left(\epsilon_{\mathrm{r}}\right)=\frac{\text { Absolute error }}{\text { True value }}=\frac{0.5 \times 10^{-6}}{1 \times 10^{-6}}=0.5$
Percentage relative error $\left(\epsilon_{\mathrm{r}} \%\right)=\epsilon_{\mathrm{r}} \times 100 \%=0.5 \times 100$

$$
=50 \%
$$

2. $\quad$ Absolute error $=\mid$ True value - Approximate value $\mid$
$=1 \times 10^{6}-0.99 \times 10^{6}=0.01 \times 10^{6}=10000$
Relative error $\left(\mathrm{C}_{\mathrm{r}}\right)=\frac{\text { Absolute error }}{\text { True value }}=\frac{10000}{1 \times 10^{6}}=0.01$

$$
\left(\epsilon_{\mathrm{r}} \%\right)=\mathrm{C}_{\mathrm{r}} \times 100 \%=0.01 \times 100=1 \%
$$

Example1.2- $\mathrm{f}(\mathrm{x}=0.4000)$ is correct to 4 significant digits, find the relative error.

## Solution:

( x ) is correct to 4 significant digits, this means there will be error in the (fifth) digit. The maximum value of this error will be: $\mathrm{Et}=0.00005$ ( 5 is the max. value of the $5^{\text {th }}$ digit)

Relative error $\left(\epsilon_{\mathrm{r}}\right)=\frac{\text { Absolute error }}{\text { True value }}=\frac{0.00005}{0.4000}=0.000125$

## Example1.3-

Find the approximate maximum error in ( $5.43 \times 27.2$ ).

## Solution:

Here we have to calculate error in product.
Let $\mathrm{A}=5.43$ and $\mathrm{B}=27.2$
The error in A is $\mathrm{E}_{\mathrm{a}}=0.005$, and error in B is $\mathrm{E}_{\mathrm{b}}=0.05$
$\therefore$ Product absolute error $\left(\mathrm{E}_{\mathrm{ap}}\right)=\mathrm{A} * \mathrm{E}_{\mathrm{b}}+\mathrm{B} * \mathrm{E}_{\mathrm{a}}$

$$
=5.43 * 0.05+27.2 * 0.005=0.4075
$$

Example1.4- Determine the value of $\left(\mathrm{e}^{0.5}\right)$ correct to three significant digits using the expansion $\mathrm{e}^{\mathrm{x}}=1+\mathrm{x}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots \ldots$, if true value $=1.648720$.

Solution:

| No. of terms | Total value | Absolute error | Relative error ( $\mathrm{C}_{\mathrm{r}}$ ) |
| :---: | :---: | :---: | :---: |
| 1 | $\mathrm{e}^{\mathrm{x}}=\mathrm{e}^{0.5}=1$ | $\begin{gathered} 1.64872-1= \\ 0.64872 \\ \hline \end{gathered}$ | $\frac{0.64872}{1.64872}=0.393$ |
| 2 | $\mathrm{e}^{\mathrm{x}}=1+\mathrm{x}=1+0.5=1.5$ | $\begin{gathered} 1.64872-1.5= \\ 0.14872 \end{gathered}$ | $\frac{0.14872}{1.64872}=0.090$ |
| 3 | $\begin{gathered} \mathrm{e}^{\mathrm{x}}=1+\mathrm{x}+\mathrm{x}^{2} / 2! \\ =1+0.5+(0.5)^{2} / 2! \\ =1.625 \end{gathered}$ | $\begin{gathered} 1.64872-1.625= \\ 0.02372 \end{gathered}$ | $\frac{0.02372}{1.64872}=0.01438$ |
| 4 | $\begin{gathered} \mathrm{e}^{\mathrm{x}}=1+\mathrm{x}+\mathrm{x}^{2} / 2!+\mathrm{x}^{3} / 3! \\ =1+0.5+(0.5)^{2} / 2!+ \\ (0.5)^{3} / 3! \\ =\mathbf{1 . 6 4 5 8 3} \\ \hline \end{gathered}$ | $\begin{gathered} \hline 1.64872-1.645833= \\ 0.002887 \end{gathered}$ | $\frac{0.002887}{1.64872}==0.001751$ |
| 5 | $\mathrm{e}^{\mathrm{x}}=1.6484375$ | $\begin{gathered} \hline 1.64872-1.6484375= \\ 0.0002825 \\ \hline \end{gathered}$ | $\frac{0.0002825}{1.64872}=0.0001486$ |

$\therefore$ five terms are needed for determine the value of $\left(\mathbf{e}^{0.5}\right)$ correct for three significant digits

Example1.5: The quotient $\frac{25.4}{12.37}$ gives the result 2.05335489. Find the maximum error.

Solution: Let $\mathrm{a}=25.4$, then max. absolute error $E_{a}=0.05$.
Let $\mathrm{b}=12.37$, then max. absolute error $E_{b}=0.005$.
The absolute error in division is given as,

$$
E_{d}=\frac{B * E_{a}-A * E_{b}}{b^{2}}
$$

Putting the values in above equation,

$$
\begin{aligned}
& \quad \varepsilon d=(12.37 * 0.05-25.4 * 0.005) /(12.37)^{2} \\
& =0.003212
\end{aligned}
$$

Hence the true quotient will have the value of: $(2.053 \pm 0.003212)$

