CHAPTER TWO

Solution of Equations

3.1- Introduction:

Finding the roots of equations is one of the oldest problems in mathematics, since it is required in a great variety of applications.

Consider the simple quadratic equation:

$ax^2 + bx + c = 0$

we say that $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \ldots\ldots \quad (3.1)$$

are the roots of this equation, because for these values of (x), the quadratic equation is satisfied.

Functions can be divided into two types:

1. **Polynomials**: which can be written in the form of:

   $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots + a_nx^n$

Example : $f(x) = 1 - 2.37x + 7.5x^2$

And $f(x) = 5x^2 - x^3 + 7x^6$

2. **Transcendental Functions**: that is non-algebraic, which include trigonometric , exponential, logarithmic . . etc.

Example : $f(x) = e^{-x} - x$

$F(x) = \text{Sin } x$

$F(x) = \ln x^2 - 1$

3.2- Descart's rule of signs:

The number of positive real roots of $f_n(x) = 0$ cannot exceed the number of (+ve) sign changes in $f_n(x)$. And the number of (-ve) real roots of $f_n(-x)$.

Example 3.1: Find the number of positive and negative real roots in the equation:
$f_5(x) = 8x^5 + 12x^4 - 10x^3 + 17x^2 - 18x + 5 = 0$

Solution: Let x=1 then $f_5(1) = +8 + 12 - 10 + 17 - 18 + 5$

∴number of (+ve) real roots are (four).

Let x = -1 then $f_5(x) = -8 + 12 + 10 + 17 + 18 +5$

∴number of (-ve) real roots is (one).

```
>> p1=[8 12 -10 17 -18 5]


p1 =

    8    12   -10    17   -18     5

>> roots_p1=roots(p1)

roots_p1 =

  -2.5000

   0.0000 + 1.0000i

   0.0000 - 1.0000i

   0.5000 + 0.0000i

   0.5000 - 0.0000i
```

Example 3.2: Use Descart's rule of signs to the number of real roots of: $f(x) = x^5 + x^4 + 4x^3 + 3x^2 + x + 1$

Solution: Let's find first $F(1) = +x^5 + x^4 + 4x^3 + 3x^2 + x + 1$

There are no sign changes, so there are no (+ve) real roots.

Now let's find $f(-1) = - x^5 + x^4 - 4x^3 + 3x^2 - x + 1$

There are (5) sign changes, so there are five (-ve) roots.

Example 3.3: $f(x) = 2x^4 – x^3 + 4x^2 – 5x + 3$

Example 3.4: $f(x) = x^5 – 3x^3 + 8x – 10$

Sol:

```
>> f1=[1  0  -3  0  8  -10]
```

```
f1 =

     1     0    -3     0     8   -10

>> roots_f1=roots(f1)

roots_f1 =

  -1.6792 + 0.9601i
  -1.6792 - 0.9601i
   1.5383 + 0.0000i
   0.9100 + 0.9536i
   0.9100 - 0.9536i
```

Example 3.5: $f(x) = 6x^3 + 7x^2 - 3x + 1$

```
>> f1=[6  7  -3  1];
>> roots_f1=roots(f1)
roots_f1 =
  -1.5566 + 0.0000i
   0.1950 + 0.2628i
   0.1950 - 0.2628i
```

3.3- Methods of Solutions: There are two groups of methods to solve the functions: - Bracketing methods

- Open methods

3.3.1- Bracketing methods: It is named because all the methods used within a certain interval (or range) of the function. It includes:

- Graphical method
- Bisection method
- False position method

3.3.1.1- Graphical method:

It is a simple method to find the roots, by making a table for assumed values of (x)for a certain range (min. value is called (x_{lower} or (x_l) and max. value that is call x_{upper} (x_u)), they are substituted in the function to find the value of (f(x) or y), then making a plot of the function and observe where it crosses the x-axis (i.e. root). Graphical techniques are of limited practical value, because they are not precise.

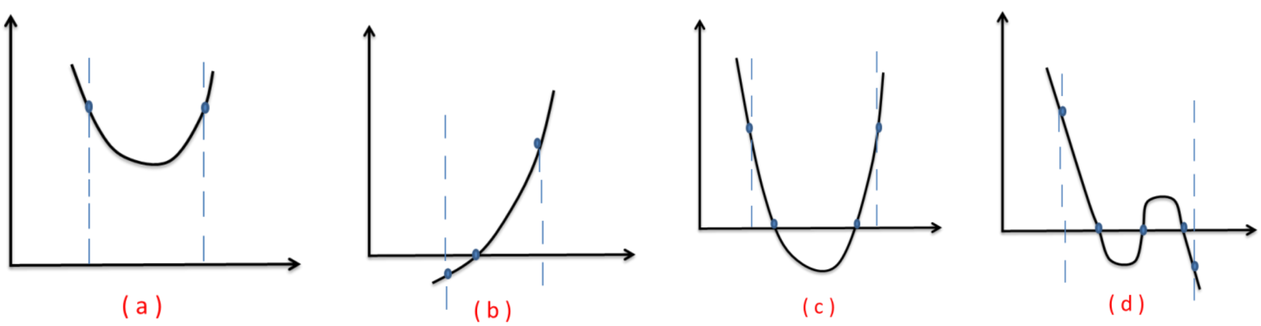


Fig. 3.1

In fig 3.1a,c , indicate that $f(x_l)$ and $f(x_u)$have the same sign, either there will be **no** roots, or there will be **even** number of roots within the interval.

In fig 3.1 b, d, indicates that the function have different signs at the end points, so there will be an **odd** number of roots in the interval.

Generally if $f(x_u) * f(x_l) < 0$, means there is at least **one** root between them.

3.3.1.2- **Bisection** method:

It is one type of incremental search method which the interval is always divided in half. The location of the root is then determined as laying at the midpoint of the subinterval within which the sign change occurs. The following are the steps for actual computation:

Step #1: Choose the lower (x_l=a) and upper (x_u=b) guesses for the roots so that the function change sign.

Step #2: An estimate of the root (x_m=c) is determined by:

$$x_m = \frac{x_l + x_u}{2} \ldots\ldots\ldots\ldots. (3.2)$$

Step #3: Make the following evaluation to determine which subinterval the root is:

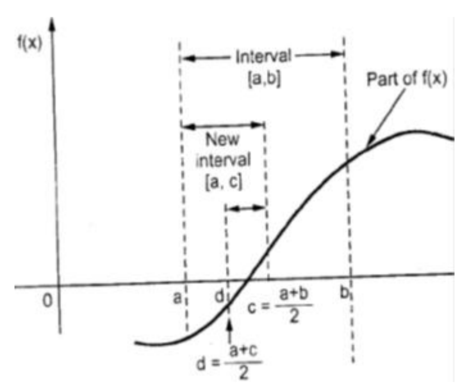
1. $F(x_l)*f(x_m)<0$ the root lies in the lower subinterval
2. $F(x_l)*f(x_m)>0$ the root lies in the upper subinterval
3. $F(x_l)*f(x_m)=0$ the root equals (x_m)[terminate computation]

Example3.6 : Use the method of bisection to find the root of the equation, $f(x) = x^4 + 2x^3 - x - 1 = 0$ lying in the interval [0, 1] at the end of sixth iteration. How many iterations are required if the permissible error is $\epsilon s = 0.0005$?

Sol. : Assume $x_l = a$, $x_u = b$ and $x_m = c$
The given interval is [a, b] = [0, 1]
Iteration No. 1 a= 0, b= 1

$\therefore$ $c = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$

f(a) = f(0) = -1,
f(b) = f(1) = 1,
f(c) = f(0.5) = -1.1875



Since f(b) * f(c) < 0, root lies between 'b' and 'c'. Hence 'a' will be replaced by 'c'. Therefore new interval is [a, b] = [0.5, 1]

Iteration No. 2 a= 0.5, b =1 .'. $c = \frac{0.5+1}{2} = 0.75$

f(a) = f(0.5)=-1.1875,
f(b) = f(l) = 1,
f(c) = f(0.75) = - 0.5898
Since f(b) * f(c)< 0, root lies between 'b' and 'c'. Hence 'a' will be replaced by 'c', and new interval will be [a, b] = [0.75, 1]

Iteration No. 3 a = 0.75, b = 1 .-. $c = \frac{0.75+1}{2} = 0.875$

f(a) = f(0.75) = -0.5898,
f(b) = f(1) = 1,
f(c) = f(0.875) = 0.0510254
f(a) * f(c) < 0, root lies between 'a' and 'c'
'b' will be replaced by ' c' and New interval [a, b] = [0.75, 0.875]

Iteration No. 4 a= 0.75, b = 0.875 .-. $c = \frac{0.75+0.875}{2} = 0.8125$

f(a) = f(0.75) = -0.5898
f(b) = f(0.875) = 0.0510254,
f(c) = f(0.8125) = - 0.3039398
f(b) * f(c) < 0, root lies between 'b' and 'c'
'a' will be replaced by 'c' and new interval [a, b] = [0.8125, 0.875]

Iteration No. 5 a = 0.8125, b = 0.875 , $c = \frac{0.8125+0.875}{2} = 0.84375$

f(a) = f(0.8125) = - 0.3039398,
f(b) = f(0.875) = 0.0510254,
f(c) = f(0.84375) = - 0.1355733
f(b) * f(c) < 0, root lies between 'b' and 'c'
'a' will be replaced by 'c' and new interval [a, b] = [0.84375, 0.875]

Iteration No. 6 a = 0.84375, b = 0.875, $c = \frac{0.84375+0.873}{2} = 0.859375$

f(a) = f(0.84375) = - 0.1355733,
f(b) = f(O.875) = 0.0510254,
f(c) = f(0.859375) = - 0.0446147
f(b) * f(c) < 0, root lies between 'b' and 'c'

'a' will be replaced by 'c' and new interval [a, b] = [0.859375, 0.875]

Hence root will be = $\frac{0.859375+0.875}{2}$ = 0.8671875

Thus root = 0.8671875 at the end of 6th iteration.

To determine number of iterations for permissible error Єs = 0.0005, for the given interval [a, b] = [0, 1]

Permissible error Єs = 0.0005

`Number of iterations are given from the equation:

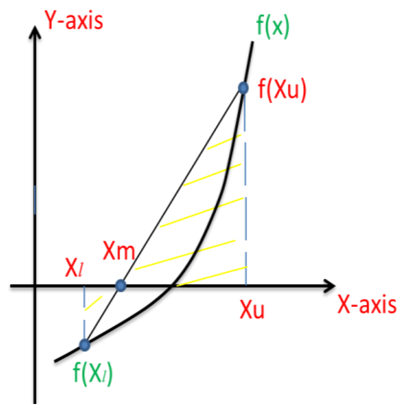
$$n \geq \frac{\log(b-a)-\log(\epsilon_s)}{\log 2} \geq \frac{\log(1-0)-\log(0.0005)}{\log 2} \geq \frac{0-(-3.30103)}{0.30103} \geq 10.96$$

Hence (n = 11) iterations are required to get the error less than permissible error.

3.3.1.3- False position method or Regula Falsi method: It is an improved version of the bisection method. An alternative way from halving the distance is to join the points by a straight line. The intersection of this line with the x-axis represents an improved estimate of the root.

From the figure, the intersection of the straight line with the x-axis can be estimated as in the formula according to the two symmetrical triangles:

$$\frac{y}{x} = \frac{f(x_l)}{x_m - x_l} = \frac{f(x_u)}{x_m - x_u}$$



$$\therefore x_m = x_u - \frac{f(x_u)*(x_l - x_u)}{f(x_l) - f(x_u)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3)$$

This is the false position formula

Example 3.7:

Find the root of $f(x) = e^x - 4x = 0$ using False position method, correct to three decimal places.

Solution: $f(x) = e^x - 4x = 0$, $f(0)=1$, $f(1)=-1.281718$

Hence, since $f(0) * f(1) < 0$, $\therefore$ *the root between* (0 *and* 1)

Let's take $x_l = x_0 = 0$, and $x_u = x_1 = 1$

Using the following relation we can find the next approximation of the root;

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

Iteration No.1

$$x_2 = 1 - \frac{1 - 0}{(-1.281718) - 1} * (-1.281718) = 0.438266$$

f(x_2) = $e^{0.438266}$ - 4*(0.438266) = - 0.203047

Since f(x_0) * f(x_2) < 0, then the root lies between [0, 0.438266].
Hence we take initial approximation for second iteration as,

x_1= 0 , x_2 = 0.438266

Iteration No. 2:

With initial approximations of x_1 = 0 and x_2 = 0.438266 from the previous iteration, we find next approximation x_3 to the root as:

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$x_3 = 0.438266 - \frac{0.438266 - 0}{(-0.203047) - 1} * (-0.203047) = 0.364297$$

f(x_3) = $e^{0.364297}$ - 4*(0.364297) = - 0.017686

Since f(x_1) * f(x_3) < 0, root lies in the interval [0, 0.364297]

Iteration No.3:

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$x_4 = 0.364297 - \frac{0.364297 - 0}{(-0.017686) - 1}(-0.017686) = 0.357966$$

∴ f(x_4) = -0.001447

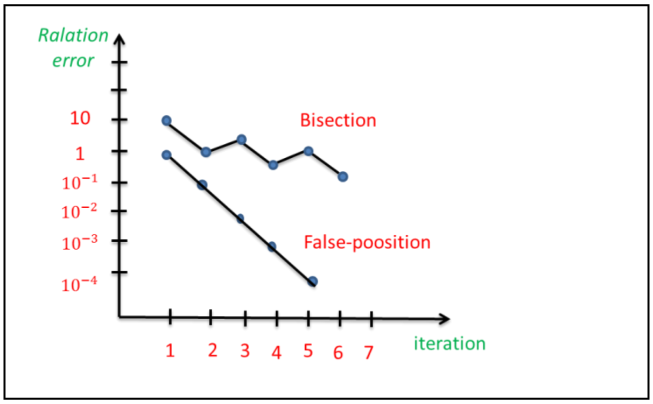
Since f(x_2) * f(x_4) < 0, root lies in the interval [0, 0.357966]

Iteration No. 4: $x_5 = 0.357449$

Since three decimal digits repeat in successive approximation, the approximation to the root is correct up to 3 decimal places.

$$\therefore \ Answer = 0.357449$$

A comparison between the two methods (Bisection and False- position). It is noted that the error for false position decreases much faster than for bisection because of the more efficient scheme for root location in the false-position method.



3.4- Home work:

Determine the real roots of the functions:

1. $F(x) = -0.9\,x^2 + 1.7x + 2.5$ [take x_l=2.8, x_u=3.0]
2. $F(x) = -2 + 6.2x - 4x^2 + 0.7x^3$ [take x_l=0.4, x_u=0.6]
1. Graphically
2. Using quadratic formula (for (1)).
3. Bisection method.
4. False-position.

Compute the estimated error

5. $F(x) = 2x - \log_{10}(x) - 7 = 0.$ Ans.:0.567203
6. $F(x) = x^3 - 2x - 5 = 0.$ Ans.:2.09451

Example 1

Consider finding the root of $f(x) = x^2 - 3$. Let $\varepsilon_{step} = 0.01$, $\varepsilon_{abs} = 0.01$ and start with the interval [1, 2].

Table 1. Bisection method applied to $f(x) = x^2 - 3$.

a	*b*	f(*a*)	f(*b*)	*c* = (a + b)/2	f(*c*)	Update	new b − a
1.0	2.0	-2.0	1.0	1.5	-0.75	a = c	0.5
1.5	2.0	-0.75	1.0	1.75	0.062	b = c	0.25
1.5	1.75	-0.75	0.0625	1.625	-0.359	a = c	0.125
1.625	1.75	-0.3594	0.0625	1.6875	-0.1523	a = c	0.0625
1.6875	1.75	-0.1523	0.0625	1.7188	-0.0457	a = c	0.0313
1.7188	1.75	-0.0457	0.0625	1.7344	0.0081	b = c	0.0156
1.71988/td>	1.7344	-0.0457	0.0081	1.7266	-0.0189	a = c	0.0078

Thus, with the seventh iteration, we note that the final interval, [1.7266, 1.7344], has a width less than 0.01 and $|f(1.7344)| < 0.01$, and therefore we chose b = 1.7344 to be our approximation of the root.

Example 2

Consider finding the root of $f(x) = e^{-x}(3.2 \sin(x) - 0.5 \cos(x))$ on the interval [3, 4], this time with $\varepsilon_{step} = 0.001$, $\varepsilon_{abs} = 0.001$.

Table 1. Bisection method applied to $f(x) = e^{-x}(3.2 \sin(x) - 0.5 \cos(x))$.

a	*b*	f(*a*)	f(*b*)	*c* = (a + b)/2	f(*c*)	Update	new b − a
3.0	4.0	0.047127	-0.038372	3.5	-0.019757	b = c	0.5
3.0	3.5	0.047127	-0.019757	3.25	0.0058479	a = c	0.25
3.25	3.5	0.0058479	-0.019757	3.375	-0.0086808	b = c	0.125
3.25	3.375	0.0058479	-0.0086808	3.3125	-0.0018773	b = c	0.0625
3.25	3.3125	0.0058479	-0.0018773	3.2812	0.0018739	a = c	0.0313
3.2812	3.3125	0.0018739	-0.0018773	3.2968	-0.000024791	b = c	0.0156
3.2812	3.2968	0.0018739	-0.000024791	3.289	0.00091736	a = c	0.0078
3.289	3.2968	0.00091736	-0.000024791	3.2929	0.00044352	a = c	0.0039
3.2929	3.2968	0.00044352	-0.000024791	3.2948	0.00021466	a = c	0.002
3.2948	3.2968	0.00021466	-0.000024791	3.2958	0.000094077	a = c	0.001
3.2958	3.2968	0.000094077	-0.000024791	3.2963	0.000034799	a = c	0.0005

Thus, after the 11th iteration, we note that the final interval, [3.2958, 3.2968] has a width less than 0.001 and $|f(3.2968)| < 0.001$ and therefore we chose b = 3.2968 to be our approximation of the root.

Example 3

Apply the bisection method to $f(x) = \sin(x)$ starting with [1, 99], $\varepsilon_{step} = \varepsilon_{abs} = 0.00001$, and comment.

After 24 iterations, we have the interval [40.84070158, 40.84070742] and $\sin(40.84070158) \approx 0.0000028967$. Note however that $\sin(x)$ has 31 roots on the interval [1, 99], however the bisection method

neither suggests that more roots exist nor gives any suggestion as to where they may be.

Adding and subtracting $X_u f(X_u)$- $X_u f(X_u)$

3.5- **Open methods**:

Previous methods called bracketing methods, because it takes only a short bounded part of the graph and check for roots. There are other methods that is called "Open Methods" that require a single starting value or two values that are not necessary bracket a root. We shall study three of them:

1. - Simple One-Point Iteration
2. - Newton-Raphson method
3. - Secant method

3.5.1-**Simple one-point iteration(Fixed- point iteration)**:

By modifying or rearranging the function so that variable (x) is on the left-hand side of the equation

$$X = g(x)$$

This will provide a formula to predict a value of (x) as a function of (x).Thus, given an initial guess at the root (x_i) in above formula can be used to compute a new estimate (x_{i+1}), as expressed by the iterative formula

$$X_{i+1} = g(x_i) \quad . . . \quad (3.4)$$

The percentage relative error for this equation can be determined using the error estimator

$$\epsilon_r\% = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| * 100\%$$

There are two ways to modify function equation to be in the form of (3.4):

1. By algebraic manipulation as: $f(x) = x^2-2x+3=0$

$$\therefore x = \frac{x^2+3}{2}$$

2. By adding (x) to both sides as: f(x)=Sin(x)=0 then

x=Sin(x) + x.

Note :- $|g'(x)|<1$ $\quad |g'(2)| < 1$

Example 3.8: Use simple one-point iteration to locate the root

$F(x) = e^{-x} - x$.

Solution: The function can be expressed in the form of (3.4)as:

$$x_{i+1} = e^{-x_i}$$

Starting with an initial guess of ($x_0 = 0$) to get $x_1 = 1$, as the first iteration, then substitute again to get x_2 and so on.
The result is as in the table. Notice that with each iteration it becomes closer to the real root (0.56714329).

iteration	X_i	$\epsilon_a\%$
0	0	
1	1.00	100%
2	0.367879	171
3	0.692201	46.9
4	0.500473	38.3
5	0.606244	17.4
6	0.545396	11.2
7	0.579612	5.9
8	0.560115	3.48

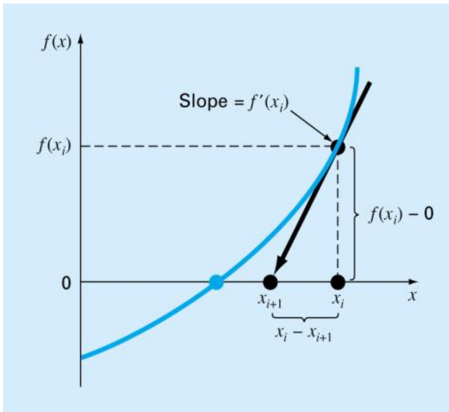
Home Work:

Recalculate the previous error to find ($\epsilon_t\%$) since you have the true value.

3.5.2- Newton-Raphson method (NRM):

In the figure it can be seen that an initial guess at point (x_i), a tangent can be extended from the point[x_i, $f(x_i)$]. This point where this tangent crosses the x-axis usually improved estimate of the root.
The basic formula:

$$f'(x_i) = \frac{f(x_i) - 0}{(x_i - x_{i+1})}$$



Which can be rearranged and rewritten as:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.5)$$

This is called Newton-Raphson formula.

Example 3.9: Use the (NRM) to estimate the root of (f(x) = e^{-x} – x) employing an initial guess of x_0 = 0.

Solution: The first derivative of the function can be evaluated as :

$f'(x) = -e^{-x} - 1$

Which can be substituted along with the original function in the Newton-Raphson formula to give:

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{x_i} - 1}$$

Starting with an initial guess of x_0=0, the result will be as in the table. It is noted that the approach rapidly converge on the true root. Notice that the relative error at each iteration decreases much faster than it does in simple one-point iterative method.

iteration	X_i	**ϵ_t%**
0	0	100%
1	0.5	11.8
2	0.566311	0.147
3	0.567143	$0.22*10^{-4}$
4	0.5671432	$<10^{-8}$

Note:

Two points to be followed using(NRM):

1. $f'(x_i) \neq 0$. If it is "0", change value of x_0.
2. For better convergence, select x_0 such that:

$$f(x_0) * f''(x_0) > 0$$

Example 3.10:

Find the real root of the equation: $x^3 + 2x - 5 = 0$, by applying (NRM) for five iterations. xsin(x)+cos(x)=0

Solution:

We have $f(x) = x^3 + 2x - 5 = 0$,

$$f'(x) = 3x^2 + 2$$
$$f''(x) = 6x$$

f(0) =0 + 0 -5 = -5,

f(1)=1 +2 -5 = -2,

f(2)= 2^3 +2*2 -5 =7

Since F(1) * f(2) < 0, the root lies between 1 & 2.

For better convergence select initial value of x_0 such that:

$$f(x_0) * f''(x_0) > 0$$

Here, $f(2) * f''(2) = 7 * 12 > 0$

So we select $X_0 = 2$.

Using NRM formula: $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Iteration No.1: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{7}{14} = 1.5$

Iteration No.2: :$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{1.375}{8.75} = 1.342857$

Iteration No.3: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.342857 - \frac{0.107242}{7.409796}$

$= 1.328384$

Iteration No.4: $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.328384 - \frac{0.000841}{7.293813}$

$= 1.328269$

Iteration No.5: $x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.328269 - \frac{0.000007}{7.292895}$

$= 1.328269$

Hence the root at the end of 5^{th} iteration is correct to 6 decimal places.

Home work 1

Find the real root of the equation: xsin(x)+cos(x)=0, by applying (NRM) for five iterations.

Example

A root of $f(x) = x^3 - 10x^2 + 5 = 0$ lies close to $x = 0.7$. Compute this root with the Newton–Raphson method.

Solution The derivative of the function is $f'(x) = 3x^2 - 20x$, so that the Newton–Raphson formula in Eq. (4.3) is

$$x \leftarrow x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 10x^2 + 5}{3x^2 - 20x} = \frac{2x^3 - 10x^2 - 5}{x(3x - 20)}$$

It takes only two iterations to reach five decimal place accuracy:

$$x \leftarrow \frac{2(0.7)^3 - 10(0.7)^2 - 5}{0.7[3(0.7) - 20]} = 0.73536$$

$$x \leftarrow \frac{2(0.73536)^3 - 10(0.73536)^2 - 5}{0.73536[3(0.73536) - 20]} = 0.73460$$

3.5.3- **The Secant method**:

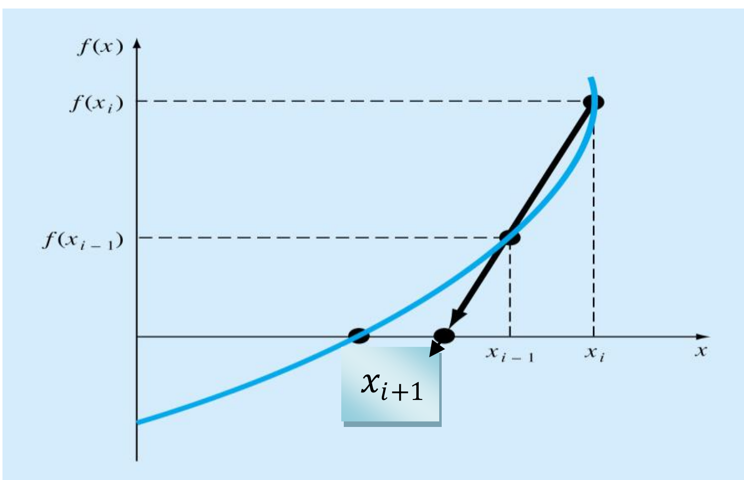
Some functions are difficult to evaluate its derivatives. For these cases the derivatives can be approximated by a finite divided difference:

$$f'(x) = \frac{f(x_{i-1}) - f(x_i)}{(x_{i-1} - x_i)}$$

This approximation can be substituted in equation (3.5) to yield the following iterative equation:

$$x_{i+1} = x_i - \frac{[x_{i-1} - x_i]}{[f(x_{i-1}) - f(x_i)]} * f(x_i) \ldots\ldots (3.6)$$

This is called the secant formula, which needs two arbitrary initial estimates of (X).



Example 3.10: Use the secant method to estimate the root of:
$f(x) = e^{-x} - x$ with initial estimates of [x_{-1}=0 and x_0=1].
Solution:
First iteration: $x_{-1} = 0$, $f(x_{-1}) = 1$
$x_0 = 1.0$, $f(x_0) = -0.63212$

$$x_{i+1} = x_i - \frac{[x_{i-1} - x_i]}{[f(x_{i-1}) - f(x_i)]} * f(x_i)$$

$$\therefore \quad x_1 = 1 - \frac{0 - 1}{1 - (-0.63212)}(-0.63212) = 0.6127$$

Second iteration:
$x_0 = 1$, $f(x_0) = -0.63212$
$x_1 = 0.6127$, $f(x_1) = -0.07081$

$$\therefore x_2 = 0.6127 - \frac{(1 - 0.6127)}{(-0.63212) - (-0.07081)} * (-0.07081) = 0.56384$$

Third iteration:
$x_1 = 0.6127$, $f(x_1) = -0.07081$
$x_2 = 0.56384$, $f(x_2) = 0.00518$

$$\therefore x_3 = 0.56384 - \frac{(0.6127 - 0.56384)}{(-0.07081) - (0.00518)} * (0.00518) = 0.56717$$

If the exact root is (0.56714329) so the percentage absolute true error (ϵ_t) will be:

$$\epsilon_t\% = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| * 100\%$$

1st iteration

$$\epsilon_t\% = \frac{0.56714329 - 0.6127}{0.56714329} * 100\% = 8.03\%$$

By the same way the

2nd iteration $\epsilon_t\% = 0.58\%$ &

3^{rd}iteration $\qquad \epsilon_t\% = 0.0048\%$

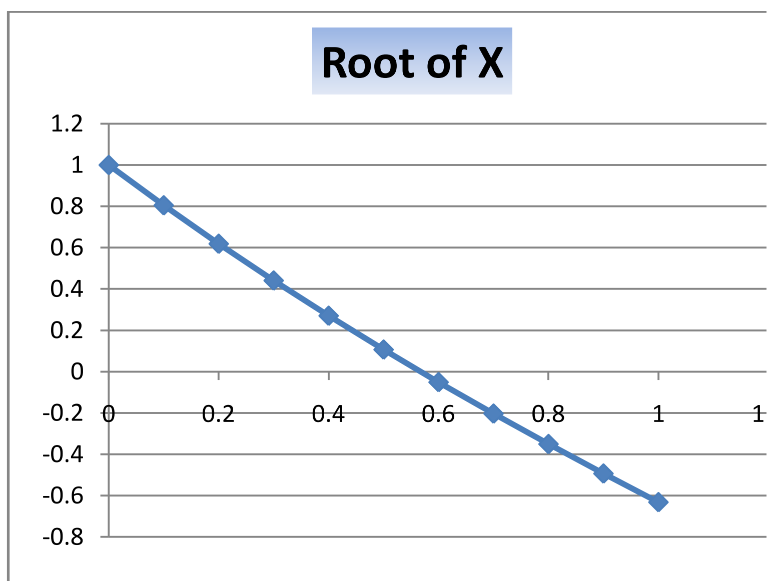
Example 1

Determine the real root of $f(x) = e^{-x} - x$:

a) Graphically.
b) Using the bisection method (three iterations).
c) Using the secant method (three iterations).
d) Using the false position method (three iterations).
e) Using the Newton-Raphson method (three iterations).

Solution. **a)** The graphical approach for determining the roots of an equation.

x	*f*(*x*)
0	1
0.1	0.804837
0.2	0.618731
0.3	0.440818
0.4	0.27032
0.5	0.106531
0.6	-0.05119
0.7	-0.20341
0.8	-0.35067
0.9	-0.49343
1	-0.63212



The root is $x \approx 0.55; \quad f(x) = 0.02695 \cong 0$.

b) *Bisection method.*

Using bisection, the results can be summarized as

Iteration, *i*	a_i	b_i	x_i	$f(a_i)$	$f(b_i)$	$f(x_i)$
0	0	1	0.5	1	-0.63212	0.106531
1	0.5	1	0.75	0.106531	-0.63212	-0.277633
2	0.5	0.75	0.625	0.106531	-0.277633	-0.089738
3	0.5	0.625	0.5625	0.106531	-0.089738	0.007283

Thus, after three iterations the root is $x \approx 0.5625$, $f(x) = 0.007283 \cong 0$,

$$|\varepsilon_a| = \left|\frac{0.5625 - 0.625}{0.5625}\right| \times 100\% = 11.1\%.$$

c) *Secant method*.

Use the secant method to find the root with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$.

First iteration:

$$x_{-1} = 0; \quad f(x_{-1}) = 1.0; \qquad x_0 = 1; \quad f(x_0) = -0.63212.$$

$$x_1 = 1 - \frac{-0.63212 \cdot (0-1)}{1-(-0.63212)} = 0.61270; \quad f(x_1) = -0.070814.$$

Second iteration:

$$x_0 = 1; \quad f(x_0) = -0.63212; \quad x_1 = 0.61270; \quad f(x_1) = -0.070814.$$

Note that both estimates are now on the same side of the root.

$$x_2 = 0.61270 - \frac{-0.070814 \cdot (1-0.61270)}{-0.63212-(-0.070814)} = 0.56384;$$

$$f(x_2) = 0.0051798; \quad |\varepsilon_a| = \left|\frac{0.56384 - 0.61270}{0.56384}\right| \times 100\% = 8.7\%.$$

Third iteration:

$$x_1 = 0.6127; f(x_1) = -0.070814; \quad x_2 = 0.56384; f(x_2) = 0.00518.$$

$$x_3 = 0.56384 - \frac{0.00518 \cdot (0.6127 - 0.56384)}{-0.070814-(-0.00518)} = 0.567696;$$

$$f(x_3) = -8.66 \cdot 10^{-4}; \quad |\varepsilon_a| = \left|\frac{0.567696 - 0.56384}{0.567696}\right| \times 100\% = 0.7\%.$$

a) *False position method*.

Use the false position method with guesses of $a_0 = 0$ and $b_0 = 1$.

First iteration:

$$a_0 = 0; \quad f(a_0) = 1; \qquad b_0 = 1; \quad f(b_0) = -0.63212.$$

$$p_0 = 1 - \frac{-0.63212 \cdot 1}{-0.63212 - 1} = 0.61270; \quad f(p_0) = -0.070814.$$

Second iteration:

Therefore, the root lies in the first subinterval, and p_1 becomes:

$$a_1 = 0; \quad f(a_1) = 1; \qquad b_1 = 0.61270; \quad f(b_1) = -0.070814.$$

$$p_1 = 0.6127 - \frac{-0.070814 \cdot (-0.6127)}{1 - (-0.070814)} = 0.57218; \; f(p_1) = -0.007888.$$

$$|\varepsilon_a| = \left| \frac{0.57218 - 0.6127}{0.57218} \right| \times 100\% = 7.1\%.$$

Third iteration: $f(a_1) \cdot f(p_1) < 0$.

Therefore, the root lies in the first subinterval:

$$a_2 = 0; \quad f(a_2) = 1; \quad b_2 = 0.57218; \quad f(b_2) = -0.007888.$$

$$p_2 = 0.57218 - \frac{-0.007888 \cdot (-0.57218)}{1 - (-0.007888)} = 0.567703;$$

$$f(p_2) = -8.771 \cdot 10^{-4}; \quad |\varepsilon_a| = \left| \frac{0.567703 - 0.57218}{0.567703} \right| \cdot 100\% = 0.8\%.$$

b) *Newton-Raphson method*.

The first derivative of the function $f(x) = e^{-x} - x$ can be evaluated as

$f'(x) = -e^{-x} - 1$ which can be substituted along with the original function into equation (see table 4.2) to give:

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}.$$

Starting with an initial guess of x_0 = 1, this iterative equation can be applied to compute

i	x_i	$f(x_i)$	ε_a, %
0	1	-0.63212	
1	0.537883	0.0461	85.9
2	0.566987	0.000245	5.1

3	0.567143	$4.541 \cdot 10^{-8}$	$2.8 \cdot 10^{-2}$

Thus, the method rapidly converges on the true root.

Notice *that the percent relative error decreases at each iteration much faster than it does in another methods*.

Home work:

1. Determine the smallest positive real root of:
   $f(x) = x^3 - 4.8x^2 + 7.56x - 3.92$
   a. Graphically
   b. Using most efficient method, employ initial guesses of: $X_l = x_{i-1} = 0.5$ and $X_u = x_i = 1.5$ and perform the computation to within $\epsilon_s = 15\%$.
2. Determine the roots of:
   $f(x) = x^3 - 7x^2 - 3.75 + 12.5$
   a. Graphically
   b. Using the most efficient method to within $\epsilon s = 0.1\%$.

3. Determine the real roots of $f(x) = \ln x - x^2 + 7x - 8$:
   (a) graphically,
   (b) using the Newton-Raphson method, and
   (c) using the secant method. Compare and discuss the rate of convergence.
4. Locate the first positive root of $f(x) = \sin(x) + x^2 - 9x + 14$. Use four iterations of
   (a) the bisection method, and
   (b) the false position method.
   Discuss and also perform an error check of your final answer.

5.The location $\bar{x}$ of the centroid of a circular sector is given by :

$$\bar{x} = \frac{2rsin\theta}{3\theta}$$

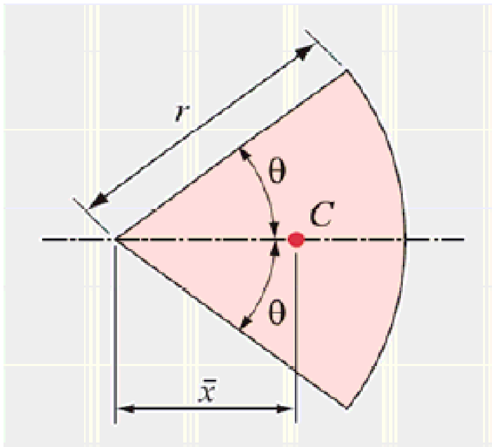
Determine the angle θ for which $\bar{x} = \frac{r}{2}$.

First, derive the equation that must be solved and then determine the root using the following methods:

(*a*) Use the bisection method. Start with $a = 1$ and $b = 2$, and carry out the first five iterations.

(*b*) Use the secant method. Start with the two points $x_1 = 1$ and $x_2 = 2$, and carry out the first five iterations.

(*c*) Use Newton's method. Start at $x_1 = 1$ and carry out the first five iterations.



Solution

Using 5 significant digits.

(a) $f(\theta) = \dfrac{2\sin\theta}{3\theta} - \dfrac{1}{2}$

$i = 1$, $a = 1$, $b = 2$, $f(1) = \dfrac{2\sin(1)}{3 \cdot 1} - \dfrac{1}{2} = 0.06098$, $f(2) = \dfrac{2\sin(2)}{3 \cdot 2} - \dfrac{1}{2} = -0.1969$,

$x_{NS1} = \dfrac{1+2}{2} = 1.5$, $f(1.5) = \dfrac{2\sin(1.5)}{3 \cdot 1.5} - \dfrac{1}{2} = -0.056669$

$i = 2$, $a = 1$, $b = 1.5$, $x_{NS2} = \dfrac{1+1.5}{2} = 1.25$, $f(1.25) = \dfrac{2\sin(1.25)}{3 \cdot 1.25} - \dfrac{1}{2} = 0.006125$

$i = 3$, $a = 1.25$, $b = 1.5$, $x_{NS3} = \dfrac{1.25+1.5}{2} = 1.375$, $f(1.375) = \dfrac{2\sin(1.375)}{3 \cdot 1.375} - \dfrac{1}{2} = -0.24415$

$i = 4$, $a = 1.25$, $b = 1.375$, $x_{NS4} = \dfrac{1.25+1.375}{2} = 1.3125$,

$f(1.3125) = \dfrac{2\sin(1.3125)}{3 \cdot 1.3125} - \dfrac{1}{2} = -0.0089135$

$i = 5$, $a = 1.25$, $b = 1.3125$, $x_{NS5} = \dfrac{1.25+1.3125}{2} = 1.2813$

(b) $\theta_{i+1} = \theta_i - \dfrac{f(\theta_i)(\theta_{i-1} - \theta_i)}{f(\theta_{i-1}) - f(\theta_i)}$

$\theta_1 = 1$, $\theta_2 = 2$, $f(\theta_1) = \dfrac{2\sin(1)}{3 \cdot 1} - \dfrac{1}{2} = 0.06098$, , $f(\theta_2) = \dfrac{2\sin(2)}{3 \cdot 2} - \dfrac{1}{2} = -0.1969$

$i = 2$ $\theta_3 = 2 - \dfrac{(-0.1969)(1-2)}{0.06098 - -0.1969} = 1.2365$, $f(\theta_3) = \dfrac{2\sin(1.2365)}{3 \cdot 1.2365} - \dfrac{1}{2} = 0.0093093$

$i = 3$ $\theta_4 = 1.2365 - \dfrac{0.0093093(2 - 1.2365)}{(-0.1969) - 0.0093093} = 1.271$, $f(\theta_4) = \dfrac{2\sin(1.271)}{3 \cdot 1.271} - \dfrac{1}{2} = 0.001126$

$i = 4$ $\theta_5 = 1.271 - \dfrac{0.001126(1.2365 - 1.271)}{0.0093093 - 0.001126} = 1.2757$, $f(\theta_5) = \dfrac{2\sin(1.2757)}{3 \cdot 1.2757} - \dfrac{1}{2} = -4.5369 \times 10^{-7}$

$i = 5$ $\theta_6 = 1.2757 - \dfrac{(-4.5369 \times 10^{-7})(1.271 - 1.2757)}{0.001126 - (-4.5369 \times 10^{-7})} = 1.2757$

(c) $f(\theta) = \dfrac{2\sin\theta}{3\theta} - \dfrac{1}{2}$, $f'(\theta) = \dfrac{6\theta\cos\theta - 6\sin\theta}{9\theta^2}$, $\theta_{i+1} = \theta_i - \dfrac{f(\theta_i)}{f'(\theta_i)} = \theta_i - \dfrac{(4\sin\theta_i - 3\theta_i)\theta_i}{4(\theta_i\cos\theta_i - \sin\theta_i)}$

$i = 1$, $\theta_1 = 1$, $\theta_2 = 1 - \dfrac{[4\sin(1) - (3 \cdot 1)]1}{4(1\cos(1) - \sin(1))} = 1.3037$

$i = 2$, $\theta_3 = 1.3037 - \dfrac{[4\sin(1.3037) - (3 \cdot 1.3037)]1.3037}{4(1.3037\cos(1.3037) - \sin(1.3037))} = 1.2759$

$i = 3$, $\theta_4 = 1.2759 - \dfrac{[4\sin(1.2759) - (3 \cdot 1.2759)]1.2759}{4(1.2759\cos(1.2759) - \sin(1.2759))} = 1.2757$

$i = 4$, $\theta_5 = 1.2757 - \dfrac{[4\sin(1.2757) - (3 \cdot 1.2757)]1.2757}{4(1.2757\cos(1.2757) - \sin(1.2757))} = 1.2757$