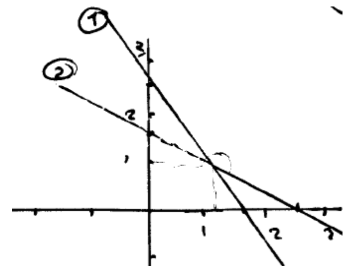
CHAPTER THREE

Solution of Linear Simultaneous Equations

4.1- Introduction: Linear systems of equations naturally occur in many places in engineering, such as structural analysis, ***dynamics and electric circuits***. Computers have made it possible to quickly and accurately solve larger and larger systems of equations.

In chapter (3) we saw how to find the root of the algebraic equations of the form [f(x) = 0].

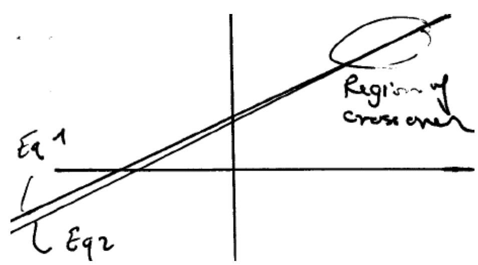
We may have several equations and several unknowns and must find those values of the unknowns which satisfy all the equations at the same time.



A simple case of simultaneous equations, consider the following : $f(x,y) = 3x + 2y - 5 = 0 \dots (1)$

$$g(x,y) = 2x + 3y - 5 \ = 0 \dots (2)$$

This can be solved graphically. The cross point between the two lines is the solution of the equations.



4.2- Ill conditioned Equations: If the two lines are almost parallel, then it is hard to tell just where the lines cross. This case can be detected when the characteristic determinant (det[A]) is near zero.

(Note: det[A] = $a_{11} * a_{22} - a_{12} * a_{21}$)

A system whose coefficient matrix is nearly singular is called ill-conditioned.

· When a system is ill-conditioned, the solution is very sensitive

• to small changes in the right-hand vector,

• to small changes in the coefficients.

4.2.1- Effect of errors: Consider the set of equations:

$$2x + 3y = -5$$
$$x - 4y = 14$$

The solution is (x=2, y= -3) with no error involved.

In practice, it is more likely to have equations:

$$2.37x + 3.06y = -5.63$$
$$0.93x - 3.72y = 14.78$$

Here, coefficients are correct to two decimal places, this means round off errors have been introduced and the accuracy of solution will be affected.

Consider also the two equations:

$$-10.0001x + 9y = -14$$
$$-9.999x \quad + 9y = -14$$

The two lines are nearly parallel and if the slope change very little, it will cause different point of intersection. Note that the determinant of the two above equations is almost zero.

4.3- **Methods of solutions**:

In chapter (3) we studied some methods to obtain a root of transcendental and algebraic equations. In this chapter we will study some methods to obtain solution of linear equations. In engineering and science applications we always come across these types of equations. Normally a system of linear equations with 'n' unknowns is represented as,

$$a_{11}\,x_1 + a_{12}\,x_2 + \cdots . + a_{1n}\,x_n = b_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + \cdots . + a_{2n}x_n = b_2$$

$$a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots . + a_{nn}x_n = b_n \quad .... (4.1)$$

Here $x_1, x_2, x_3, \ldots, x_n$, are the variables whose values are to be found. In this chapter we will study three major types of methods to obtain solution of linear equations. They are:

1) Direct Methods: There are two groups of methods:

First group that can be used with small number of equations (up to two or three equations) which used:

1. Graphical Method
2. Simple Direct Method
3. Cramer's Rule

Second group which can be used for large number of equations (more than three). These methods use elimination of variable. They transform the system of equations to triangular form. The important direct methods are:

1. Gauss elimination method,
2. Gauss Jordon elimination method.

2) **Matrix inversion method**.

3)Iterative Method: These methods use principles of successive approximation. The iterations are repeated till the required accuracy is obtained. The methods are:

- Jacobi's iteration method,
- Gauss seidel iteration method.

4.5- Gaussian elimination method:

(Carl Friedrich Gauss (1777 – 1855) – a German mathematician and scientist)

For more than two equations, this method can be used to reduce the system of equations in to 'triangular' form.

The procedure starts by:

1. Normalizing the first equation (the pivot row) by dividing each of its coefficients by $(a11)$.
2. The first equation is multiplied by the leading coefficient $(ai1)$ of each of the other equations and subtracted from each successive equation. The result will be the elimination of the first variable from all equations except the first equation.
3. Using the last $(n-1)$ equations the same procedure such that the second variable is eliminated from the last $(n-2) equations$.
4. The procedure is repeated until after (n)stage, the triangle form is complete.

We can change the upper part (or the lower part) of the matrix into zeros. After changing the matrix into triangular form it is possible to find the values of the variables by (substitution methods).

4.5.1- Substitution Methods

The methods we studied in the first two sections become complex when number of variables are more than three. Two substitution methods are discussed here. They are:

1- Backward substitution and
2- Forward substitution methods.

4.5.2- Backward Substitution Method:

Consider the following system of equations involving three unknowns,

$$a11\, x1 + a12\, \mathrm{x}_2 + a13\, x3 \;= b1$$
$$0 \;+ a22\, \mathrm{x}_2 + a23\, x3 \;= b2$$
$$0 \quad + \quad 0 \quad + a33\, x3 \;= b3$$

Let's form the matrix of these equations.

$$A \;= \begin{bmatrix} a11 & a12 & a13 \\ 0 & a22 & a23 \\ 0 & 0 & a33 \end{bmatrix}$$

In the matrix A, we observe that all the elements below diagonal are zero is called 'upper triangular form' of equations. To solve these equations we start from the last equation:

i.e., $a33\, \mathrm{x}_3 = b3 \quad or \quad \mathrm{x}_3 = \frac{b3}{a33}$

Then the second last equation:

$$a22\, \mathrm{x}_2 + a23\, \mathrm{x}_3 = b2$$

Putting the value of (x3) in this equation we get:

$$a22\, \mathrm{x}_2 + a23 * \frac{b3}{a33} \;= b2$$

Or $\quad \mathrm{x}_2 = \frac{1}{a22}\left(b2 - a23 * \frac{b3}{a33}\right)$

Similarly we can substitute the values of x_2 and x_3 to find x_1 from the first equation. Since we start from last equation , this is called (Backward Substitution Method)**.**

4.5.3-Forward Substitution Method

Now let's consider an alternate form of equations having three unknowns:

$$a11\, \mathrm{x}_1 \quad + \quad 0 \quad + \;0 \;= b1$$
$$a21\, \mathrm{x}_1 \quad + \quad a22\, \mathrm{x}_2 + \;0 \;= b2$$
$$a31\, \mathrm{x}_1 + a32\, \mathrm{x}_2 + a33\, \mathrm{x}_3 = b3$$

This can be written in matrix form as:

$$A \;= \begin{bmatrix} a11 & 0 & 0 \\ a21 & a22 & 0 \\ a31 & a32 & a33 \end{bmatrix}$$

Here we see that all the elements above the diagonal are zero. This is called 'lower triangular form' of equations. To solve these type of equations, we start from first equation:

$$a11\, \mathrm{x}_1 \;= b1, \quad or \;\; \mathrm{x}_1 = \frac{b1}{a11}$$

The second equation is: $\quad a21\, \mathrm{x}_1 + a22\, \mathrm{x}_2 \;= b2$

Substitution the value of x_1 in the above equation we get:

$$a21 * \frac{b1}{a11} + a22\, x_2 = b3$$

Or $$x_2 = \frac{1}{a22}\left(b2 - a21 * \frac{b1}{a11}\right)$$

Similarly we can substitute the values of x_1 and x_2 to find x_3 from the last equation. Since we start from the first equation, this method is called (Forward Substitution Method**).**

Example 1 Solve the following system of equations using Gauss- elimination method.

$$\begin{aligned} x + y + z &= 5 \\ 2x + 3y + 5z &= 8 \\ 4x + 5z &= 2 \end{aligned}$$

Solution: The augmented matrix of the equations is:

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix}$$

We will perform row operations.

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix} \xrightarrow{R2\ -\ 2R1\ ,} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{bmatrix}$$

$$\xrightarrow{R3 - 4R1} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{bmatrix}$$

13z=-26 so z=-2

y+3z=-2 , y+3*-2=-2, y-6=-2 , y=4

x+y+z=5, x+4-2=5 , x=3

The solution is :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

Example 2 Solve the following system of equations using Gauss- elimination method.

$$-3x + 2y - 6z = 6$$

$$5x + 7y - 5z = 6$$

$$x + 4y - 2z = 8$$

Solution: The augmented matrix of the equations is:

$$\begin{bmatrix} -3 & 2 & -6 & 6 \\ 5 & 7 & -5 & 6 \\ 1 & 4 & -2 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -3 & 2 & -6 & 6 \\ 5 & 7 & -5 & 6 \\ 1 & 4 & -2 & 8 \end{array}\right] \begin{array}{c} 3R_2 + 5R1 \\ 3R_3+R_1 \end{array} \left[\begin{array}{ccc|c} -3 & 2 & -6 & 6 \\ 0 & 31 & -45 & 48 \\ 0 & 14 & -12 & 30 \end{array}\right]$$

$$\begin{bmatrix} -3 & 2 & -6 & 6 \\ 0 & 31 & -45 & 48 \\ 0 & 14 & -12 & 30 \end{bmatrix} \quad 31R_3 - 14R_2 \quad \left[\begin{array}{ccc|c} -3 & 2 & -6 & 6 \\ 0 & 31 & -45 & 48 \\ 0 & 0 & 258 & 258 \end{array}\right]$$

$$z = \frac{258}{258=} = 1 \qquad y = \frac{48+45}{31} = \frac{93}{31} = 3$$

$$and \quad x = \frac{6 - 2 * 3 + 6 * 1}{-3} = \frac{6}{-3} = -2$$

The solution is :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

4.5.4-Gauss-Jordan Method: In the previous method , only the equation below the pivot row were manipulated, where as in the Gauss-Jordan method, the manipulation takes place both above and below the pivot row. Thus the coefficients of the matrix is reduced to a unit matrix (a matrix that has ones in the diagonal and zeros elsewhere).

Example 4.4: Solve the following system of equations using Gauss-Jordan elimination method.

$$\begin{aligned} x + y + z &= 5 \\ 2x + 3y + 5z &= 8 \\ 4x + 5z &= 2 \end{aligned}$$

Solution: The augmented matrix of the equations is:

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix}$$

We will perform row operations until obtain the unit matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix} \xrightarrow{R2\ -\ 2R1\ ,} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{bmatrix}$$

$$\xrightarrow{R3 - 4R1} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{bmatrix}$$

$$R3\ +\ 4R2 \quad \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{bmatrix}$$

$$(1/13)R3 \quad \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$R2\text{-}3R3 \quad \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$R1\ -\ R3 \quad \begin{bmatrix} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

R1 - R2 $$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

From the final matrix, we can read the solution:

x=3, y= 4 and z=-2

4.6-Matrix Inversion Method

Finding the inverse matrix by using Gauss-Jordan Method

Example :Given matrix *A*, the Gauss-Jordan method can be used to find its inverse.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Augment *A* with the identity matrix and then reduce:

$R_2 - 3R_1$
$R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \Leftrightarrow R_3 \quad \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & -5 & -3 & 1 & 0 \end{array}\right]$$

$$-3 \times R_2 + R_3 \quad \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{array}\right]$$

$$R_1 + \tfrac{2}{5}R_3, \quad -\tfrac{1}{5} \times R_3 \quad \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & \tfrac{2}{5} & -\tfrac{6}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\tfrac{1}{5} & \tfrac{3}{5} \end{array}\right]$$

$R_1 + R_2$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 0 & \frac{2}{5} & -\frac{1}{5} \\ -1 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

If we have the inverse of a matrix, we can use it to solve a set of equations, $Ax = b$:

$$A^{-1}Ax = A^{-1}b$$

$$\Rightarrow \quad x = A^{-1}b$$

See Table 1 for the comparison of *singular* and *nonsingular* matrices.

Table: A comparison of singular and nonsingular matrices

For Singular matrix A	For Nonsingular Matrix A
It has no inverse, A^{-1}	It has an inverse, A^{-1}
Its determinant is zero	The determinant is nonzero
There is no unique solution to the system $Ax = b$	There is a unique solution to the system $Ax = b$
Gaussian elimination cannot avoid a zero on the diagonal	Gaussian elimination does not encounter a zero on the diagonal
Rows are linearly dependent	Rows are linearly independent
Columns are linearly dependent	Columns are linearly independent

4.7-Iterative Methods:

Iterative Methods: Describes how a linear system can be solved in an entirely different way, by beginning with an initial estimate of the solution and performing computations that eventually arrive at the correct solution.

An iterative method is particularly important in solving systems that have few nonzero coefficients.

4.7.1-Jacobi method:(Gauss Jacobi method is the first iterative method used to solve linear system of equations.

This is the method of simultaneous displacements:

$$x(i)^{p+1} = \frac{1}{a(ii)}\left[b(i) - \sum_{\substack{j=1 \\ j\neq i}}^{n} a(ij)x_j^p\right]$$

The procedure is to assume a set of starting values (initial guess), for x_i(i= 1,2,3…..) and then to calculate new values by substitution for (i=1,2,…,n). This is repeated until some criterion met.

NOTE: Before starting the solution the diagonal coefficient values must be the largest, which might need rearranging the equations this is called (**partial pivoting**).

Definition of Strictly Diagonally Dominant Matrix

An $n \times n$ matrix A is *diagonally dominant* if and only if;

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, i = 1, 2, \ldots\ldots, n$$

An $n \times n$ matrix A is **strictly diagonally dominant** if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries

A system is called ***diagonally dominant*** if the system of equations can be ordered so that each diagonal entry of the coefficient matrix is larger in magnitude than the sum of the magnitudes of the order coefficients in that row. For such a system, the iteration will converge for any starting values.

in the same row. That is,

$$|a_{11}| > |a_{12}| + |a_{13}| + \cdots \ldots \ldots |a_{1n}|$$
$$|a_{22}| > |a_{21}| + |a_{23}| + \cdots \ldots \ldots |a_{2n}|$$
$$|a_{nn}| > |a_{n1}| + |a_{n2}| + \cdots \ldots \ldots |a_{n,n-1}|$$

Although this may seem like a very restrictive condition, it turns out that there are very many applied problems that have this property.

• i.e.,

$$6x1 - 2x2 + x3 = 11$$
$$x1 + 2x2 - 5x3 = -1$$
$$-2x1 + 7x2 + 2x3 = 5$$

• The solution is $x_1 = 2$, $x_2 = 1$, $x_3 = 1$.

However, before we begin our iterative scheme we must first reorder the equations so that the coefficient matrix is diagonally dominant.

After reordering;

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

Is the solution same? Check it out as an exercise.

Consider the set of equations:

$$a11\, x_1 + a12\, x_2 + a13\, x_3 = b1$$
$$a21\, x_1 + a22\, x_2 + a23\, x_3 = b2$$
$$a31\, x_1 + a32\, x_2 + a33\, x_3 = b3$$

Using Jacobi-formula, the above equations can be rewritten as:

$$x_1^{(1)} = \frac{1}{a11}\left(b1 - a12 * x2^{(0)} - a13 * x_3^{(0)}\right)$$
$$x_2^{(1)} = \frac{1}{a22}\left[b2 - a21 * x_1^{(0)} - a23 * x_3^{(0)}\right]$$
$$x_3^{(1)} = \frac{1}{a33}\left[b3 - a31 * x_1^{(0)} - a32 * x_2^{(0)}\right]$$

Example 4.6: Solve the following set of equations using Jacobi- method.

$$x_1 + 2x_2 + x_3 = 2$$
$$3x_1 + 3x_2 + 4x_3 = 1$$
$$3x_1 + x_2 + 4x_3 = 3$$

Solution:

Step#1: Applying the above note, the equations must be rearranged in the following manner to have largest values on the diagonal

$$3x_1 + 3x2 + 4x3 = 1$$
$$x_1 + 2x2 + x3 = 2$$
$$3x_1 + x2 + 4x3 = 3$$

Step#2: Rewrite the equations according to the rule above:

$$x_1^{(1)} = \frac{1}{3}\left(1 - 3 * x_2^{(0)} - 4 * x_3^{(0)}\right)$$

$$x_2^{(1)} = \frac{1}{2}\left[2 - x_1^{(0)} - x_3^{(0)}\right]$$

$$x_3^{(1)} = \frac{1}{4}\left[3 - 3 * x_1^{(0)} - x_2^{(0)}\right]$$

Step#3: Starting with initial guess , $x^0 = [0 \;\; 0 \;\; 0\,]^T$

First iteration:

$$x_1^{(1)} = \frac{1}{3}(1 - 3 * (0) - 4 * (0)) = 1/3$$

$$x_2^{(1)} = \frac{1}{2}[2 - 1 * (0) - 1 * (0)] = 1$$

$$x_3^{(1)} = \frac{1}{4}[3 - 3 * (0) - 1 * (0)] = 3/4$$

Thus, $x^{(1)} = [\ 1/3 \quad 1 \quad 3/4]^T$

Second iteration:

$$x_1^{(2)} = \frac{1}{3}\left(1 - 3 * 1 - 4 * (\frac{3}{4})\right) = -5/3$$

$$x_2^{(2)} = \frac{1}{2}\left[2 - 1 * \left(\frac{1}{3}\right) - 1 * (\frac{3}{4})\right] = 0.2916$$

$$x_3^{(2)} = \frac{1}{4}\left[3 - 3 * \left(\frac{1}{3}\right) - 1 * 1\right] = 1/4$$

Thus, $x^{(2)} = [\ -1.666 \quad 0.2916 \quad 0.25]$

And so on until the difference between the new values and the previous values becomes less than the set error.

4.7.2-Gauss-Seidl Method: The method is similar to the previous one (Jacobi method) but with a small difference. *It uses any new value available from calculation in the next equation.* Suppose we have the set of equations:

$$a11\, x_1 + a12\, x_2 + a13\, x_3 = b1$$
$$a21\, x_1 + a22\, x_2 + a23\, x_3 = b2$$
$$a31\, x_1 + a32\, x_2 + a33\, x_3 = b3$$

It can be rewritten as in formula:

$$x_1^{(1)} = \frac{1}{a11}\left(b1 - a12 * x_2^{(0)} - a13 * x_3^{(0)}\right)$$
$$x_2^{(1)} = \frac{1}{a22}\left[b22 - a21 * x_1^{(1)} - a23 * x_3^{(0)}\right]$$
$$x_3^{(1)} = \frac{1}{a33}\left[b3 - a31 * x_1^{(1)} - a32 * x_2^{(1)}\right]$$

Example 4.7: Consider the equations:

$$x_1 + 6x_2 + 2x_3 = 9$$
$$-x_1 - 2x_2 + 5x_3 = 2$$
$$4x_1 - x_2 + x_3 = 4$$

Solution:

Step#1: Since the diagonal coefficients are not the largest it must be rewritten as:

$$4x_1 - x_2 + x_3 = 4$$
$$x_1 + 6x_2 + 2x_3 = 9$$
$$-x_1 - 2x_2 + 5x_3 = 2$$

Step#2: Let $x^{(0)} = [\,0 \quad 0 \quad 0]^T$, then the first iteration will be:

$$x_1^{(1)} = \frac{1}{4}\left[\,4 + x2^{(0)} - x3^{(0)}\right] = \frac{1}{4}[4 + 0 - 0] = 1$$
$$x2^{(1)} = \tfrac{1}{6}[9 - x_1^{(1)} - 2 * x3^{(0)}] = \tfrac{1}{6}[\,9 - 1 - 0\,] = 4/3=1.3333$$
$$x3^{(1)} = \tfrac{1}{5}\left[\,2 + x_1^{(1)} + 2 * x2^{(1)}\right] = \tfrac{1}{5}\left[2 + 1 + \tfrac{8}{3}\right] = 17/15=1.1333$$

$$x_1^{(2)} = \frac{1}{4}\left[\,4 + x2^{(1)} - x3^{(1)}\right] = \frac{1}{4}[4 + 1.3333 - 1.1333] = 1.05$$

$$x2^{(2)} = \tfrac{1}{6}[9 - x_1^{(2)} - 2 * x3^{(1)}] = \tfrac{1}{6}[\,9 - 1.05 - 2 * 1.1333\,] = 0.9472$$
$$x3^{(2)} = \frac{1}{5}\left[\,2 + x_1^{(2)} + 2 * x2^{(2)}\right] = \frac{1}{5}[2 + 1.05 + 2 * 0.9472] = 0.9889$$

The following table gives the results of five iteration and the final result.

Iteration	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$
0	0	0	0
1	1.0	1.3333	1.1333
2	1.05	0.9472	0.9889
3	0.9896	1.005	0.999
4	1.001	0.9999	1.0
5	1.0	1.0	1.0

Convergence of Gauss – Seidel Method

The sufficient conditions for the convergence of two nonlinear equations u(x, y) and v(x, y) are,

$$\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial v}{\partial x}\right| < 1 \quad \ldots\ldots\ldots\ldots.(1)$$

And

$$\left|\frac{\partial u}{\partial y}\right| + \left|\frac{\partial v}{\partial y}\right| < 1 \quad \ldots\ldots\ldots\ldots..(2)$$

The criteria is also applicable to linear equations. For two simultaneous linear equations, the gauss – seidel algorithm is given as,

$$u(x_1, x_2) = \frac{1}{a_{11}}[b_1 - a_{12}x_2] \quad \ldots\ldots\ldots\ldots\ldots(3)$$

And

$$v(x_1, x_2) = \frac{1}{a_{22}}[b_2 - a_{21}x_1] \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Hence, the partial derivatives of above equations are,

$$\frac{\partial u}{\partial x_1} = 0, \quad \frac{\partial u}{\partial x_2} = -\frac{a_{12}}{a_{11}} \quad \text{(from equation 3)}$$

$$and \quad \frac{\partial v}{\partial x_2} = 0, \quad \frac{\partial v}{\partial x_1} = -\frac{a_{21}}{a_{22}} \quad \text{(from equation 4)}$$

Putting these partial derivatives in equation 1 we get,

$$\left|\frac{a_{21}}{a_{22}}\right| < 1$$

$$and \left|\frac{a_{12}}{a_{11}}\right| < 1$$

Above equations can also be written as,

$$|a_{22}| > |a_{21}|$$

$$and\ |a_{11}| > |a_{12}|$$

That is the diagonal elements must be greater than the off diagonal elements in each row. The above criteria can be extended for multiple equations, and it will be,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \dots\dots\dots\dots\dots\dots..(5)$$

This is the condition for convergence of the gauss seidel method

Problems:

1. Solve using: a)Gauss elimination b) Gauss-Jordan method

$$x_1 + x_2 + x_3 - x_4 = 2$$
$$x_1 - x_2 - x_3 + x_4 = 0$$
$$2x_1 + x_2 - x_3 + 2x_4 = 9$$
$$3x_1 + x_2 + 2x_3 - x_4 = 7$$

2. Solve the following equation by:
   a) Gauss-Seidle method b) Jacobi method.

$$x - y + z = -4$$
$$5x - 4y\ 3z = -12$$
$$2x + y + z = 11$$

3. For the set of equations:

$$\begin{cases} 3.1x_1 & + & 1.5x_2 & + & 1.0x_3 & = & 10.83; \\ 1.5x_1 & + & 2.5x_2 & + & 0.5x_3 & = & 9.20; \\ 1.0x_1 & + & 0.5x_2 & + & 4.2x_3 & = & 17.10. \end{cases}$$

a) Compute the determinant using *Gaussian elimination with back substitution.*

b) Substitute your results back into the original equations to verify your solution.

4. Determine the inverse matrix using *Gauss- Gordan method*

$$A = \begin{bmatrix} 2.1 & -4.5 & -2.0 \\ 3.0 & 2.5 & 4.3 \\ -6.0 & 3.5 & 2.5 \end{bmatrix}.$$

Multiply the inverse by the original coefficient matrix and assess whether the result is close to the identity matrix.

5. Given the linear system of equations:

$$\begin{cases} 8.714x_1 & + & 2.180x_2 & + & 5.684x_3 & = & 49.91; \\ -1.351x_1 & + & 10.724x_2 & + & 5.224x_3 & = & 50.17; \\ 2.489x_1 & - & 0.459x_2 & + & 6.799x_3 & = & 32.68. \end{cases}$$

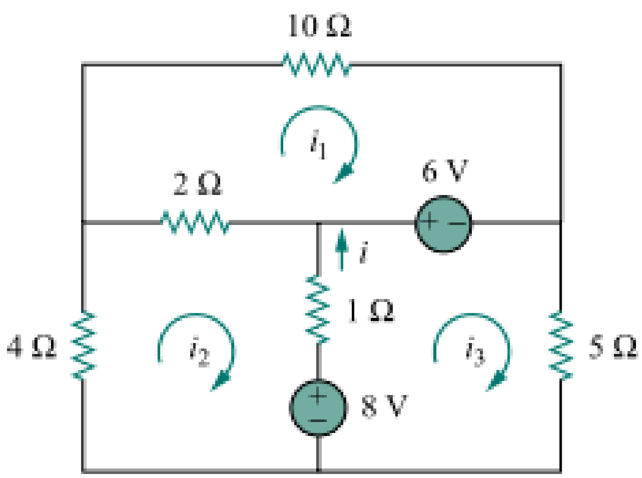
Solve it with: **a)** *Gauss-Jordan method*, **b)** *Jacobi iteration***,** **c)** *Gauss - Seidel iteration*. For **b)** and **c)** provide a means to estimate the error.

4.8 Engineering Applications

In this chapter we studied various methods to solve the system of linear equations. Now we will apply these methods to solve the circuit analysis problems.

Ex. 4.8.1

Fig. 1 shows an electrical circuit. Write the loop equations for this circuit and obtain the loop current using Gauss – seidel iterative method.



For loop 1 : $6 = 12i_1 - 2i_2 \longrightarrow 3 = 6i_1 - i_2$ (1)

For loop 2 : $-8 = 7i_2 - 2i_1 - i_3 \longrightarrow 8 = 2i_1 - 7i_2 + i_3$ (2)

For loop 3 : $-8 + 6 + 6i_3 - i_2 = 0 \longrightarrow 2 = 6i_3 - i_2$ (3)

Putting (1), (2), and (3) in matrix form,

$$\begin{bmatrix} 6 & -1 & 0 \\ 2 & -7 & 1 \\ 0 & -1 & 6 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 6 & -1 & 0 \\ 2 & -7 & 1 \\ 0 & -1 & 6 \end{vmatrix} = -234 \qquad \Delta_2 = \begin{vmatrix} 6 & 3 & 0 \\ 2 & 8 & 1 \\ 0 & 2 & 6 \end{vmatrix} = 240 \qquad \Delta_3 = \begin{vmatrix} 6 & -1 & 3 \\ 2 & -7 & 8 \\ 0 & -1 & 2 \end{vmatrix} = -38$$

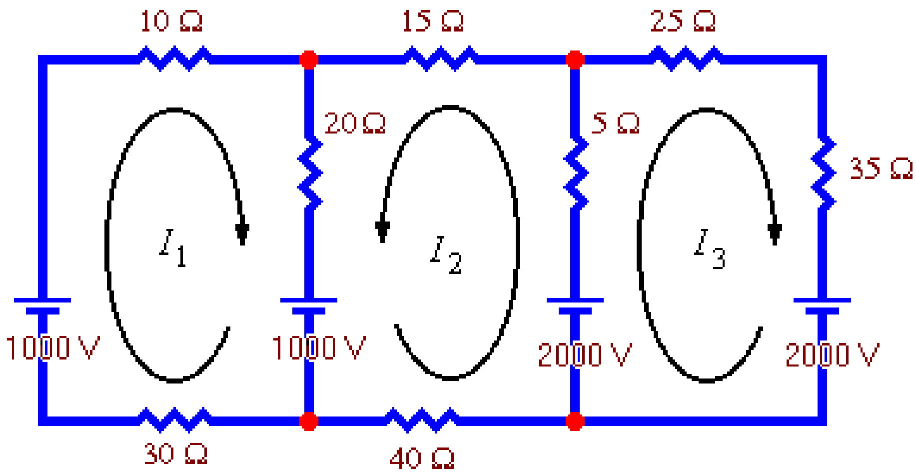
Clearly,

$$i = i_3 - i_2 = \frac{\Delta_3 - \Delta_2}{\Delta} = \frac{-38 - 240}{-234}$$

$$i = \mathbf{1.188\ A}$$

<u>Home work :</u>

Find the current flowing in each branch of this circuit using **Gauss- Jordan elimination method**.



Then draw the circuit with currents in each branch.