

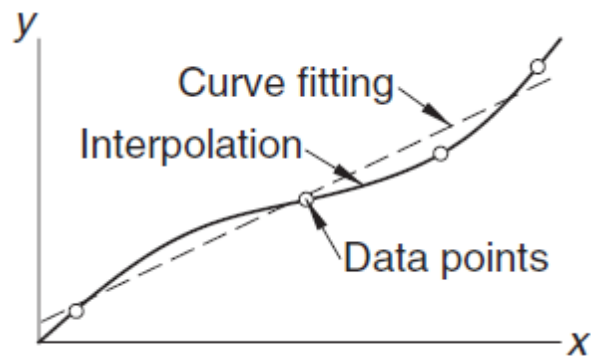
# CHAPTER FOUR

## Curve Fitting and Interpolation

**5.1- Introduction:** It is the technique to fit curves to data in order to obtain intermediate estimates. There are two general approaches:

**First :** where data exhibits a significant degree of error, the strategy is to derive a single curve that represents the general trend of the data. ( One approach (regression) is called least-squares).

**Second:** where data is very precise, the basic approach is to fit a curve or a series of curves that pass directly through each point. The estimation of values between well-known discrete points is called (interpolation).



**5.2-Least - Squares Approximation** (regression):

### 5.2.1- Linear regression:

The simplest example of a least squares approximation is fitting a straight line:

$$y = a_0 + a_1x + e$$

Where  $(a_0)$  and  $(a_1)$  are coefficients expressing the intercept and slope respectively, and  $(e)$  is the error or residual:

$$e = y - a_0 - a_1x$$

- Thus, the error, or residual is the discrepancy between the true value(**observed value**) of  $(y)$  and the approximate value (**expected value**) of  $(a_0 + a_1x)$  predicted by the linear equation.( **Residual:** Difference between measured and calculated Y-values

**5.2.2- Best Fit:** The sum of errors is:

$$\sum_{i=1}^n e_i = \sum_{i=1}^n [y_i - a_0 - a_1x_i] \dots (1)$$

To avoid cancelling of negative error value with the positive values we change all the signs of error values to positive by squaring. The summation of the square of the errors is:

$$Sr = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - a_0 - a_1 * x_i)^2 \dots (2)$$

**5.2.3- Least-Squares of a straight line:** To minimize the error we take the derivative of equation (2) with respect of one variable and equated to zero, then we repeat for the other variables the same procedure.

For a straight line we do this with respect to  $(a_0)$  then  $(a_1)$ :

$$\frac{\partial Sr}{\partial a_0} = -2 \sum_{i=1}^n [y_i - a_0 - a_1 * x_i] = 0 \quad [\text{w. r. t } a_0]$$

$$\frac{\partial Sr}{\partial a_1} = -2 \sum_{i=1}^n [y_i - a_0 - a_1 * x_i] * x_i = 0 \quad [\text{w. r. t } a_1]$$

Or 
$$\sum_{i=1}^n y_i - \sum_{i=1}^n a_0 - \sum_{i=1}^n a_1 * x_i = 0$$

And 
$$\sum_{i=1}^n y_i * x_i - \sum_{i=1}^n a_0 * x_i - \sum_{i=1}^n a_1 * x_i^2 = 0$$

Since  $\sum_{i=1}^n a_0 = n * a_0$ , then we shall have two equations:

$$n * a_0 + \sum_{i=1}^n a_1 * x_i = \sum_{i=1}^n y_i \dots \dots \dots (3)$$

$$\sum_{i=1}^n a_0 * x_i + \sum_{i=1}^n a_1 * x_i^2 = \sum_{i=1}^n x_i * y_i \dots (4)$$

Thus, 
$$a_1 = \frac{n * \sum_{i=1}^n [x_i y_i] - \sum_{i=1}^n x_i * \sum_{i=1}^n y_i}{n * \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i * \sum_{i=1}^n x_i}$$

And

$$a_0 = \frac{\sum_{i=1}^n x_i^2 * \sum_{i=1}^n y_i - \sum_{i=1}^n x_i * \sum_{i=1}^n [x_i * y_i]}{n * \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i * \sum_{i=1}^n x_i}$$

or: 
$$a_0 = \bar{y} - a_1 \bar{x}$$

### Definition

#### Arithmetic means:

The sum of individual data points( $y_i$ ) divided by the number of points ( $n$ ).

Or:  $\bar{y} = \frac{\sum y_i}{n}$  Where ( $n$ ) is the number of data points.

And  $\bar{x} = \frac{\sum x_i}{n}$

**Example 5.1:** Fit a straight line to the ( $x$ ) and ( $y$ ) values of the table below.

<b>X(i)</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>6</b>	<b>8</b>
<b>Y(i)</b>	2	3	4	4	5	6

**Solution:** We have six data points (i.e.  $n=6$ )

make a table of needed coefficients in equations

(i)	$x_i$	$y_i$	$x_i^2$	$x_i * y_i$
1	1	2	1	2
2	2	3	4	6
3	3	4	9	12
4	4	4	16	16
5	6	5	36	30
6	8	6	64	48
	$\sum x_i = 24$	$\sum y_i = 24$	$\sum x_i^2 = 130$	$\sum y_i * x_i = 114$

Then find

$$a_1 = \frac{n * \sum_{i=1}^n [x_i y_i] - \sum_{i=1}^n x_i * \sum_{i=1}^n y_i}{n * \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i * \sum_{i=1}^n x_i}$$

$$a_1 = \frac{6 * 114 - 24 * 24}{6 * 130 - 24 * 24} = \frac{684 - 576}{780 - 576} = \frac{108}{204} = 0.5294$$

$$a_0 = \bar{y} - a_1 \bar{x} = 4 - 0.5294(4) = 4 - 2.1176 = 1.8824$$

The straight line equation is:  $y = a_0 + a_1 * x$

Thus the required straight line of the equation is:

$$y = 1.8824 + 0.5294 * x$$

### 5.2.4-Polynomial Regression:

Some data cannot be represented by a straight line. For these cases, a curve would be better suited to fit the data.

Another alternative is to fit polynomials to data using polynomial regression.

The least-squares procedure can be readily extended to fit the data to an  $m^{\text{th}}$  - degree polynomial:

$$y = a_0 + a_1 * x + a_2 * x^2 + \dots + a_m * x^m + e$$

And the sum of the squares is:

$$Sr = \sum_{i=1}^n (y_i - a_0 - a_1 * x - a_2 * x^2 - \dots - a_m * x^m)^2$$

Following the same procedure as before (i.e. taking the derivative of the equation with respect to the unknowns,  $a_0 \dots a_m$ ), we get:

$$\frac{\partial Sr}{\partial a_0} = -2 \sum_{i=1}^n [y_i - a_0 - a_1 * x_i] = 0$$

$$\frac{\partial Sr}{\partial a_1} = -2 \sum_{i=1}^n [y_i - a_0 - a_1 * x_i] * x_i = 0$$

$$\dots$$
$$\frac{\partial Sr}{\partial a_m} = -2 \sum_{i=1}^n [y_i - a_0 - a_1 * x_i - \dots - a_m * x_i^m] * x_i^m = 0$$

By re-arranging the above equations we get the following set of equations:

$$a_0 n + a_1 \sum x_i + a_2 \sum x_i^2 + \dots + a_m \sum x_i^m = \sum y_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + \dots + a_m \sum x_i^{m+1} = \sum x_i y_i$$

$$a_0 \sum x_i^m + a_1 \sum x_i^{m+1} + \dots + a_m \sum x_i^{2m} = \sum x_i^m y_i$$

$$\begin{aligned} a_0 n + a_1 \sum x_i + a_2 \sum x_i^2 &= \sum y_i \\ a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 &= \sum x_i y_i \\ a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 &= \sum x_i^2 y_i \end{aligned}$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

**Example 5.2:**

Find the least squares polynomial of the second degree for the data in the table:

<b>X</b>	<b>1.2</b>	<b>1.5</b>	<b>1.8</b>	<b>2.6</b>	<b>3.1</b>	<b>4.3</b>	<b>4.9</b>	<b>5.3</b>
<b>Y</b>	4.5	5.1	5.8	6.7	7.0	7.3	7.6	7.4
<b>X</b>	5.7	6.4	7.1	7.6	8.6	9.2	9.8	
<b>Y</b>	7.2	6.9	6.6	5.1	4.5	3.4	2.7	

**Solution:** By making a table to find the coefficients of  $a_0$ ,  $a_1$  and  $a_2$

<b>I</b>	<b>x</b>	<b>y</b>	<b><math>x^2</math></b>	<b><math>x^3</math></b>	<b><math>x^4</math></b>	<b>x y</b>	<b><math>x^2 y</math></b>
1	1.2	4.5	1.44	1.73	2.07	5.40	6.48
2	1.5	5.1	2.25	3.38	5.06	7.65	11.48
3	1.8	5.8	3.24	5.83	10.50	10.44	18.79
4	2.6	6.7	6.76	17.58	45.70	17.42	45.29
5	3.1	7.0	9.61	29.79	92.35	21.70	67.27
6	4.3	7.3	18.49	79.51	341.88	31.39	134.98
7	4.9	7.6	24.01	117.65	576.48	37.24	182.48
8	5.3	7.4	28.09	148.88	789.05	39.22	207.87
9	5.7	7.2	32.49	185.19	1055.60	41.04	233.93
10	6.4	6.9	40.96	262.14	1677.72	44.16	282.62
11	7.1	6.6	50.41	357.91	2541.17	46.86	332.71
12	7.6	5.1	57.76	438.98	3336.22	38.76	294.58
13	8.6	4.5	73.96	636.06	5470.08	38.70	332.82
14	9.2	3.4	84.64	778.69	7163.93	31.28	287.78
15	9.8	2.7	96.04	941.19	9223.68	26.46	259.31
	<b><math>\Sigma x =</math></b>	<b><math>\Sigma y =</math></b>	<b><math>\Sigma x^2 =</math></b>	<b><math>\Sigma x^3 =</math></b>	<b><math>\Sigma x^4 =</math></b>	<b><math>\Sigma xy =</math></b>	<b><math>\Sigma x^2 y =</math></b>
	<b>79.1</b>	<b>87.8</b>	<b>530.15</b>	<b>4004.50</b>	<b>32331.49</b>	<b>437.72</b>	<b>2689.37</b>

then solve it we get:

$$15a_0 + 79.1a_1 + 530.15a_2 = 87.8$$

$$79.1a_0 + 530.15a_1 + 4004.3a_2 = 437.72$$

$$530.15a_0 + 4004.3a_1 + 32331.49a_2 = 2698.37$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

$$\begin{bmatrix} 15 & 79.1 & 530.15 \\ 79.15 & 30.15 & 4004.3 \\ 530.15 & 4004.3 & 32331.4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 87.8 \\ 437.72 \\ 2698.37 \end{bmatrix}$$

Using matrix inversion to find the unknowns we get:

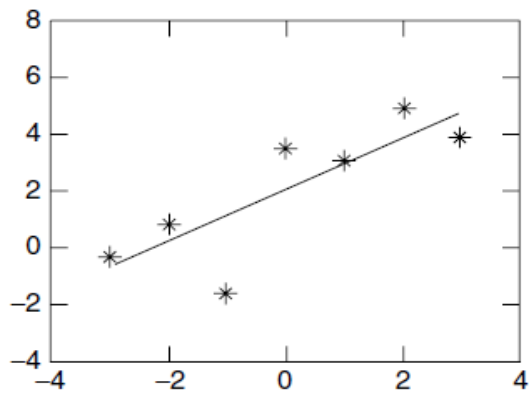
$$a_0 = 2.78, \quad a_1 = 1.94, \quad a_2 = -0.2$$

And the polynomial is:  $y = 2.78 + 1.94x - 0.2x^2$

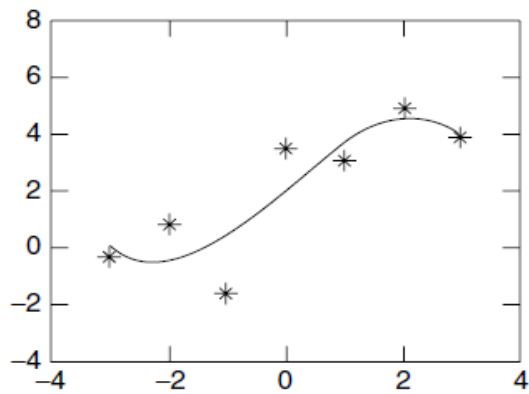
**Example 5.3:** Consider the data in the table. Use least squares curve fitting to find the polynomials of degrees 1, 3, 5 and 7.

<b>X</b>	<b>-3</b>	<b>-2</b>	<b>-1</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>
<b>Y</b>	-0.2774	0.8958	-1.5651	3.4565	3.0601	4.8568	3.8982

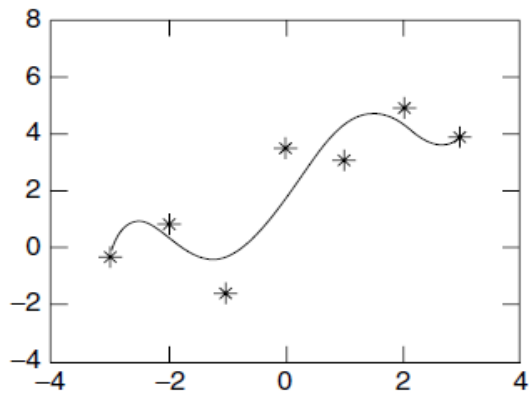
**Solution:** By using MATLAB (poly fit) we get:



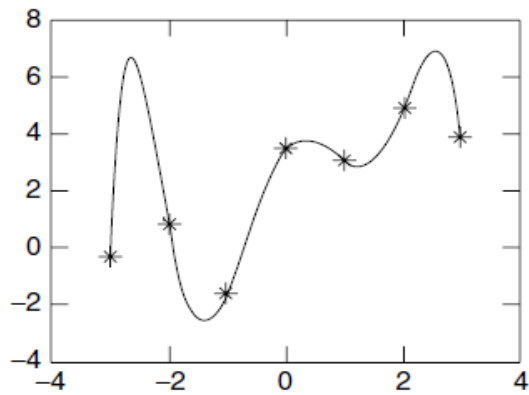
(a) Polynomial of degree 1



(b) Polynomial of degree 3



(c) Polynomial of degree 5



(d) Polynomial of degree 7



**Example 5.4:** Fit a second order polynomial to the data in the table:

X	0	1	2	3	4	5
y	2.1	7.7	13.6	27.2	40.9	61.1

**Solution:** By making a table to find the coefficients of the unknowns we get:

i	$x_i$	$y_i$	$x_i^2$	$x_i^3$	$x_i^4$	$x_i y_i$	$x_i^2 y_i$
1	0	2.1	0	0	0	0	0
2	1	7.7	1	1	1	7.7	7.7
3	2	13.6	4	8	16	27.2	54.4
4	3	27.2	9	27	81	81.6	244.8
5	4	40.9	16	64	256	163.6	654.4
6	5	61.1	25	125	625	305.5	1527.5
$\Sigma$	15	152.6	55	255	979	585.6	2488.8

Therefore, the three equation for the three unknowns ( $a_0$ ,  $a_1$ , and  $a_2$ ) are:

$$\begin{aligned} 6a_0 + 15a_1 + 55a_2 &= 152.6 \\ 15a_0 + 55a_1 + 255a_2 &= 585.6 \\ 55a_0 + 255a_1 + 979a_2 &= 2488.9 \end{aligned}$$

$$\begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 255 \\ 55 & 255 & 979 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 152.6 \\ 585.6 \\ 2488.9 \end{bmatrix}$$

Solving the above equations through matrix inversion or Gauss elimination gives:

$$a_0 = 2.47857, \quad a_1 = 2.35929, \quad a_2 = 1.86071$$

Therefore the least-squares quadratic equation is:

$$y = 2.47857 + 2.35929x + 1.86071x^2$$

### 5.3-Interpolation : (insertion)(توسيط)(It is the process of finding the value of a function for any value of argument (independent variable) within an interval

It is to estimate intermediate values between precise data points. The most common method used for this purpose is polynomial interpolation. The general formula for an nth order polynomial is:

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \dots (1)$$

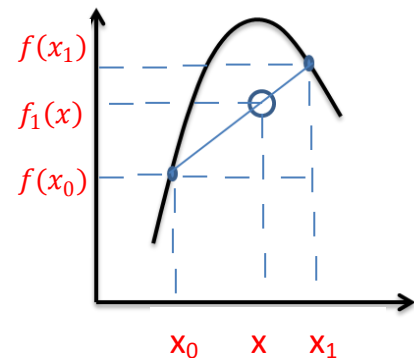
For (n+1) data points, there is one and only one polynomial of order (n) or less that passes through all points.

**5.3.1-Linear Interpolation:** The simplest form of interpolation is to connect two data points with a straight line. From the graph:

$$\frac{f_1(x) - f(x_0)}{(x - x_0)} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

Or

$$f_1(x) = f(x_0) + \frac{[f(x_1) - f(x_0)]}{(x_1 - x_0)} * [(x - x_0)] \dots (2)$$



Which is the linear interpolation formula and the notation ( $f_1(x)$ ) means, this is the first order interpolating polynomial.

The term  $[f(x_1) - f(x_0)] / (x_1 - x_0)$  is an approximation for the first derivative or the value of ( $a_1$ ).

The smaller the interval between the data points, the better the approximation.

The general formula of the first-order polynomial (i.e. straight line) is:

$$f(x) = a_0 + a_1x$$

Then equation(2) can be rewritten as:

$$f_1(x) = a_0 + a_1(x - x_0) \dots (3)$$

Where ,  $a_0 = f(x_0)$ , and  $a_1 = [f(x_1) - f(x_0)] / (x_1 - x_0)$ .

**Example 5.5:** Estimate the natural logarithm of (2) using linear interpolation.

a)First perform computation between the data:  $\ln(1) = 0$  and  $\ln(6) = 1.7917595$ .

b)Then repeat between:  $\ln(1)$  and  $\ln(4) = 1.3862944$ .

c) find  $\epsilon_r$  in each case if the exact value of  $\ln(2)=0.69314718$ .

### Solution:

Using general formula for points:  $\ln(1)$ , and  $\ln(6)$ :

a)

$$f_1(x) = f(x_0) + \frac{[f(x_1) - f(x_0)]}{(x_1 - x_0)} * [(x - x_0)]$$

$$\text{Then } f_1(x) = 0 + \frac{1.7917595 - 0}{(6-1)} * (2 - 1) = 0.3583519$$

$$\epsilon_r = \frac{0.69314718 - 0.3583519}{0.69314718} * 100 = 48\%$$

b) For the points  $\ln(1)$  and  $\ln(4)$ ,  $f_1(2)$  will be:

$$f_1(x) = 0 + \frac{1.3862944 - 0}{(4 - 1)} * (2 - 1) = 0.46209813$$

$$\epsilon_r = \frac{0.69314718 - 0.46209813}{0.69314718} * 100 = 33.3\%$$

33.3% (i.e. less error for smaller interval)

### 5.3.2-Quadratic Interpolation:

If three points are available, this can be used to have a second-order polynomial (Quadratic polynomial or parabola), of the form:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \dots (4)$$

Or

$$f_2(x) = b_0 + b_1x - b_1x_0 + b_2x^2 + b_2x_0x_1 - b_2xx_0 - b_2xx_1$$

Since the general function is:  $f_2(x) = a_0 + a_1x + a_2x^2$

Then

$$a_0 = b_0 - b_1x_0 + b_2x_0x_1$$

$$a_1 = b_1 - b_2x_0 - b_2x_1$$

$$a_2 = b_2$$

To find the constants, by assuming  $x=x_0$  in equation(4) we get:

$$b_0 = f(x_0) \dots \dots (5)$$

Substitute equation(5) in (4),

$$f_2(x) = f(x_0) + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

then assume  $x=x_1$

$$f_2(x_1) = f(x_0) + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f_2(x_1) = f(x_0) + b_1(x_1 - x_0)$$

we get:

$$b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \dots \dots (6)$$

Substitute (5) and (6) in (4),

$$f_2(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0) + b_2(x - x_0)(x - x_1)$$

then assume  $x=x_2$

$$f_2(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

we get:

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)}$$

$$f_2(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

Note that the first two terms ( $b_0$  and  $b_1$ ) are equivalent to linear interpolation from  $(x_0)$  to  $(x_1)$ . The last term (i.e.  $b_2 (x-x_0)(x-x_1)$ ) introduces the second-order curvature into the formula.

**Example 5.6:** Fit a parabola (i.e second-order polynomial) to the three points in the previous example to find  $\ln(2)$ .

**Solution:**

$$\begin{array}{ll} x_0 = 1 & f(x_0) = 0 \\ x_1 = 4 & f(x_1) = 1.3862944 \\ x_2 = 6 & f(x_2) = 1.7917595 \end{array}$$

$$b_0 = f(x_0) = 0$$

$$b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{1.3862944 - 0}{4 - 1} = 0.46209813$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)}$$

$$= \frac{\frac{1.7917595 - 1.3862944}{(6 - 4)} - \frac{1.3862944 - 0}{(4 - 1)}}{(6 - 1)} = -0.051873116$$

By substituting in quadratic formula we get:

$$f_2(x) = 0 + 0.46209813(x - 1) - 0.051873116(x - 1)(x - 4)$$

Which can be evaluated at  $x=2$  for:  $f_2(2) = 0.565844$  which represents a percent relative error  $\text{Er}=18.4\%$  which is a good improvement by the introduction of the quadratic term. if the exact value of  $\ln(2)=0.69314718$

$$\text{Er} = \left| \frac{0.69314718 - 0.565844}{0.69314718} \right| \times 100 = \left| \frac{0.1273031}{0.69314718} \right| \times 100 = 18.36\%$$

### 5.3.3-Newton's Interpolating polynomial (NDD):

For an  $n$ th order polynomial,  $(n+1)$  points are required  $(x_0, x_1, \dots, x_n)$ . Using these data points, the following equations are used to evaluate the coefficients:

$$b_0 = f(x_0)$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

:

$$b_n = f[x_n, \dots, x_1, x_0]$$

- Divided difference are calculated as follows:

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$f[x_n, x_{n-1}, \dots, x_2, x_1] = \frac{f[x_n, x_{n-1}, \dots, x_2] - f[x_{n-1}, x_{n-2}, \dots, x_1]}{x_n - x_1}$$

- Divided differences are calculated using divided difference of a smaller number of terms:

$x_i$	$f(x_i)$	First	Second	Third
$x_1$	$f(x_1)$	$f[x_2, x_1]$	$f[x_3, x_2, x_1]$	$f[x_4, x_3, x_2, x_1]$
$x_2$	$f(x_2)$	$f[x_3, x_2]$	$f[x_4, x_3, x_2]$	
$x_3$	$f(x_3)$	$f[x_4, x_3]$		
$x_4$	$f(x_4)$			

Where  $b_1, b_2, \dots, b_n$  are the first, second, ..., nth divided differences. [e.g. the 3<sup>rd</sup> order polynomial as below]

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) \\ + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

Which is called NDD interpolating polynomial (or formula)

**Example 5.7:** Use NDD to find the value of  $f(2)$  from the data in the table:

<b>X</b>	<b>-1.0</b>	<b>0.0</b>	<b>0.5</b>	<b>1.0</b>	<b>2.5</b>	<b>3.0</b>
<b>y=f(x)</b>	3.0	-2.0	-0.375	3.0	16.125	19.0

**Solution:**

Step#1: We must compute the first, second and third divided differences.

1st divided difference      2<sup>nd</sup> divided difference      3<sup>rd</sup> divided difference

$\frac{-2.000 - 3.000}{0 - (-1.0)} = -5.000$	$\frac{3.250 - (-5.000)}{0.5 - (-1.0)} = 5.500$	$\frac{3.500 - 5.500}{1.0 - (-1.0)} = -1.000$
$\frac{-0.375 - (-2.000)}{0.5 - 0.0} = 3.250$	$\frac{6.750 - 3.250}{1.0 - 0.0} = 3.500$	$\frac{1.000 - 3.500}{2.5 - 0.0} = -1.000$
$\frac{3.000 - (-0.375)}{1.0 - 0.5} = 6.750$	$\frac{8.750 - 6.750}{2.5 - 0.5} = 1.000$	$\frac{-1.500 - 1.000}{3.0 - 0.5} = -1.000$
$\frac{16.125 - 3.000}{2.5 - 1.0} = 8.750$	$\frac{5.750 - 8.750}{3.0 - 1.0} = -1.500$	
$\frac{19.000 - 16.125}{3.0 - 2.5} = 5.750$		

Or it can be prepared in a table:

		<b>1<sup>st</sup> divided</b>	<b>2<sup>nd</sup> divided</b>	<b>3<sup>rd</sup> divided</b>
		Difference	difference	Difference
<b>X</b>	<b>f(x)</b>	$f(x_0, x_1)$	$f(x_0, x_1, x_2)$	$f(x_0, x_1, x_2, x_3)$
<b>-1.00</b>	3.000			
		-5.000		
<b>0.00</b>	-2.000		5.500	
		3.250		-1.000
<b>0.50</b>	-0.375		3.500	
		6.750		-1.000
<b>1.00</b>	3.000		1.000	
		8.750		-1.000
<b>2.50</b>	16.125		-1.500	
		5.750		
<b>3.00</b>	19.000			

Using NDD formula above we get:

$$f(2) = 3 + (2 + 1) * (-5) + (2 + 1)(2 - 0) * (5.5) + (2 + 1)(2 - 0)(2 - 0.5) * (-1) = 12$$

**Example 5.8:** Use the data table below to find  $f(2)$  using NDD formula.

X	1	4	5	6
Y	0	1.3862944	1.6094379	1.7917595

If the exact value of  $f(2) = 0.69314718$ , find  $\epsilon_r\%$ .

**Solution:** By making a table to find the 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> differences:

n	$X_i$	$Y_i$	$F(x_1, x_0)$	$F(x_2, x_1, x_0)$	$F(x_3, \dots, x_0)$
1	1	0			
			0.46209813		
2	4	1.3862944		-0.05973865	
			0.2231435		0.00786554
3	5	1.6094379		-0.02041095	
			0.18232160		
4	6	1.7917595			

The results represent the coefficients ( $b_1$ ,  $b_2$  and  $b_3$ ). Thus:

$$f_3(x) = 0 + 0.46209813(x - 1) - 0.05973865(x - 1)(x - 4) + 0.00786554(x - 1)(x - 4)(x - 5)$$

$$\begin{aligned} \text{Thus } f_3(2) &= 0.46209813(2-1) - 0.05973865(2-1)(2-4) + 0.00786554(2-1)(2-4)(2-5) \\ &= 0.46209813 + 0.1194773 + 0.04719324 \\ &= 0.62876867 \end{aligned}$$

$$\epsilon_r\% = \left| \frac{0.69314718 - 0.62876867}{0.69314718} \right| \times 100 = \left| \frac{0.0643788}{0.69314718} \right| \times 100 = 9.28\%$$



### 5.3.4- Lagrange Interpolating Polynomial:

It is a reformation of Newton difference polynomial. It can be represented as:

$$f_n(x) = \sum_{i=0}^n L_i(x) * f(x_i) \dots \dots (9)$$

$$\text{Where } L_i(x) = \prod_{j=0, j \neq i}^n \frac{(x-x_j)}{(x_i-x_j)} \dots \dots \dots (10)$$

Where  $(\prod)$  means: the product of.

For example the linear version (i.e.  $n=1$ ) which needs two points is:

$$f_1(x) = \frac{(x-x_1)}{(x_0-x_1)} * f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} * f(x_1) \dots (11)$$

And the 2<sup>nd</sup>-order version (i.e.  $n=2$ ) which needs three points is:

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} * f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} * f(x_1) \\ + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} * f(x_2) \dots \dots (12)$$

For  $(n+1)$  points the following general formula is used:

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) \\ + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\dots(x_n-x_{n-1})} f(x_n)$$

And, like NDD method, has the advantage that the values  $x_0, x_1, \dots, x_n$  need not be equally spaced or taken in consecutive order.

**Example 5.9:** Use Lagrange interpolating polynomial of the 1<sup>st</sup> and 2<sup>nd</sup> order to evaluate (ln(2)) using the following data:

<b>X</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>6</b>
<b>Y=ln(x)</b>	0	-----	1.3862944	1.7917595

**Solution:**

1. The 1<sup>st</sup>-order polynomial (equation(11)) is:

$$f_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} * f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} * f(x_1)$$

And therefore, the estimate at x=2 is:

$$f_1(2) = \frac{(2 - 4)}{(1 - 4)} * (0) + \frac{(2 - 1)}{(4 - 1)} * (1.3862944) = 0.4620981$$

2. In a similar way, the 2<sup>nd</sup> - order polynomial using equation(12):

$$f_2(2) = \frac{(2 - 4)(2 - 6)}{(1 - 4)(1 - 6)} * (0) + \frac{(2 - 1)(2 - 6)}{(4 - 1)(4 - 6)} * (1.3862944) + \frac{(2 - 1)(2 - 4)}{(6 - 1)(6 - 4)} * (1.7917595) = 0.56584437$$

**Example 5.10:** Use Lagrange's interpolation formula to fit a polynomial to the data:

<b>X</b>	<b>-1</b>	<b>0</b>	<b>2</b>	<b>3</b>
<b>Y</b>	-8	3	1	12

And hence find y at x=1.

**Solution:** Here the given data contains four pairs of values.

By Lagrange's formula,

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} * f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} * f(x_1) + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} * f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} * f(x_3)$$

**Answer is: f(x)=2**

**Sol:**

$$f(x) = \frac{(1-0)(1-2)(1-3)}{(-1-0)(-1-2)(-1-3)} * -8 + \frac{(1-(-1))(1-2)(1-3)}{(0-(-1))(0-2)(0-3)} * 3 \\ + \frac{(1-(-1))(1-0)(1-3)}{(2-(-1))(2-0)(2-3)} * 1 + \frac{(1-(-1))(1-0)(1-2)}{(3-(-1))(3-0)(3-2)} * 12$$

$$f(x) = \frac{(1)(-1)(-2)}{(-1)(-3)(-4)} * -8 + \frac{(2)(-1)(-2)}{(1)(-2)(-3)} * 3 + \frac{(2)(1)(-2)}{(3)(2)(-1)} * 1 \\ + \frac{(2)(1)(-1)}{(4)(3)(1)} * 12$$

$$f(x) = \frac{2}{-12} * -8 + \frac{4}{6} * 3 + \frac{-4}{-6} * 1 + \frac{-2}{12} * 12$$

$$f(x) = \frac{4}{3} + 2 + \frac{2}{3} - 2 = \frac{6}{3} = 2$$

## Problems:

1. Given the data: 0.95, 1.42, 1.3, 1.55, 1.68, 1.32, 1.15, 1.47, 1.95, 1.66, 1.46, 1.47, 1.92, 1.35, 1.05, 1.85, 1.74, 1.65, 1.78, 1.71, 2.39, 1.82, 2.09, 2.14, 2.27 .

Determine:    a)The mean            b)The standard deviation            c)The variance  
d)The coefficient of variance.

2. Use least-squares regression to fit a straight line to:

<b>X</b>	<b>1</b>	<b>3</b>	<b>5</b>	<b>7</b>	<b>10</b>	<b>12</b>	<b>13</b>	<b>16</b>	<b>18</b>	<b>20</b>
<b>Y</b>	4	2	6	5	8	7	10	9	12	11

3. Fit a parabola (quadratic) to:

<b>X</b>	<b>0</b>	<b>2</b>	<b>4</b>	<b>6</b>	<b>9</b>	<b>11</b>	<b>13</b>	<b>15</b>	<b>17</b>	<b>19</b>	<b>23</b>	<b>25</b>	<b>28</b>
<b>Y</b>	1.2	0.6	0.4	-0.2	0	-0.6	-0.4	-0.2	-0.40	0.2	0.4	1.2	1.8

Plot the results.

4. Obtain the missing terms in the data given below using Lagrange's formula.

<b>X</b>	<b>10</b>	<b>15</b>	<b>20</b>	<b>25</b>	<b>30</b>	<b>35</b>
<b>Y</b>	43	-----	29	32	-----	78

5-

The density of air  $\rho$  varies with elevation  $h$  in the following manner:

$h$ (km)	0	3	6
$\rho$ (kg/m <sup>3</sup> )	1.225	0.905	0.652

Express  $\rho(h)$  as a quadratic function using Lagrange's method.

6-

The points

$x$	-2	1	4	-1	3	-4
$y$	-1	2	59	4	24	-53

lie on a polynomial. Use the divided difference table of Newton's method to determine the degree of the polynomial.