Electrical Engineering Department
Engineering mathematics Diploma
2023-2024

Chapter One

Laplace Transforms

Contents of Chapter One

- Laplace Transform of basic functions using the definition
- Transform of derivatives and integrals
- Properties of Laplace Transform
- Inverse Laplace Transform
- Solution of linear differential equations using Laplace Transform
- Circuit Applications

Laplace Transforms

If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}\{f\}$; thus

$$F(s) = \mathcal{L}\{f\} = \int_0^{\infty} f(t)e^{-st}dt$$

The operation $\mathcal{L}\{\ \}$ transforms $f(t)$, which is in the time domain, into $F(s)$, which is in the complex frequency domain, or simply (s-domain) where s is the complex variable $(\sigma + j\omega)$

Laplace Transforms

Evaluating Laplace transform using the definition

1. $f(t) = k$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} ke^{-st}dt$$

$$F(s) = -\frac{k}{s}[e^{-st}]_{t=0}^{t=\infty} = -\frac{k}{s}[e^{-s\infty} - e^{-s0}] = \frac{k}{s}$$

$$\mathcal{L}\{k\} = \frac{k}{s}$$

For $f(t) = 5$

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} 5e^{-st}dt = -\frac{5}{s}e^{-st}\Big|_0^{\infty} = \left[-\frac{5}{s}e^{-s\infty}\right] - \left[-\frac{5e^{-s0}}{s}\right] = \frac{5}{s}$$

Laplace Transforms

2. $f(t) = t$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty te^{-st}dt$$

$$\int udv = uv - \int vdu$$

By letting $u = t$ and $dv = e^{-st}dt$ we find

$$\int te^{-st}dt = -\frac{1}{s}te^{-st} + \frac{1}{s}\int e^{-st}dt = -\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}$$

$$F(s) = \left[-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right]_{t=0}^{t=\infty} = \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

In general,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Laplace Transforms

3. $f(t) = e^{-at}$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-at}e^{-st}dt$$

$$F(s) = \int_0^\infty e^{-(a+s)t}dt = \left[\frac{-e^{-(a+s)t}}{a+s}\right]_{t=0}^{t=\infty}$$

$$F(s) = \left[\frac{-e^{-(a+s)\infty}}{a+s} + \frac{e^{-(a+s)0}}{a+s}\right] = \frac{1}{a+s}$$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

Laplace Transforms

4. $f(t) = e^{at}$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{at}e^{-st}dt$$

$$F(s) = \int_0^{\infty} e^{-(s-a)t}dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_{t=0}^{t=\infty}$$

$$F(s) = \left[\frac{-e^{-(s-a)\infty}}{s-a} + \frac{e^{-(s-a)0}}{s-a}\right] = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Laplace Transforms

5. $f(t) = \cos \omega t$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} \cos \omega t \, e^{-st}dt = \int_0^{\infty} \left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] e^{-st}dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{j\omega t}e^{-st}dt + \int_0^{\infty} e^{-j\omega t}e^{-st}dt\right]$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{j\omega t}\} + \mathcal{L}\{e^{-j\omega t}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$$

Laplace Transforms

6. $f(t) = e^{-at} \sin \omega t$

$$\int_0^\infty e^{-at} * \sin(wt) * e^{-st} * dt = \int_0^\infty e^{-at} * \frac{e^{jwt} - e^{-jwt}}{2j} * e^{-st} * dt$$

$$= \frac{1}{2j} \int_0^\infty [e^{-at+jwt} e^{-st} - e^{-at-jwt} e^{-st}] * dt$$

$$= \frac{1}{2j} \left[\frac{1}{(s+a) - jw} - \frac{1}{(s+a) + jw} \right]$$

$$\mathcal{L}\{e^{-at} \sin \omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

Laplace Transforms

7. $\mathcal{L}\left\{\frac{df(t)}{dt}\right\}$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt$$

By using $\int u dv = uv - \int v du$

$$u = e^{-st} \quad \Longrightarrow \quad du = -se^{-st} dt$$

$$dv = \frac{df(t)}{dt} dt \quad \Longrightarrow \quad v = f(t)$$

$$\int_0^\infty \frac{df(t)}{dt} e^{-st} dt = [e^{-st} f(t)\,]_{t=0}^{t=\infty} + s \int_0^\infty f(t) e^{-st} dt$$

$$= 0 - f(0) + s \int_0^\infty f(t) e^{-st} dt$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

Laplace Transforms

We can extend the previous to show

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

In general

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

Laplace Transforms

8. $\mathcal{L}\{\int f(t)dt\}$

$$\mathcal{L}\left\{\int f(t)dt\right\} = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)dt\right]_{t=0}$$

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$$

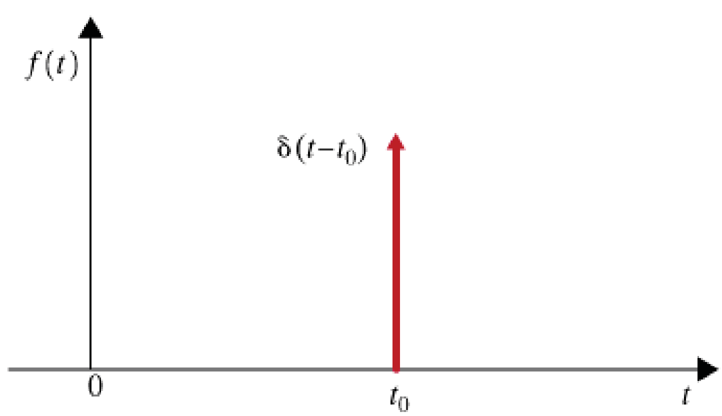
Laplace Transform Table

Function, $f(t)$	Laplace transform, $F(s)$	Function, $f(t)$	Laplace transform, $F(s)$
1	$\frac{1}{s}$	$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$
t	$\frac{1}{s^2}$	$\sinh bt$	$\frac{b}{s^2-b^2}$
t^2	$\frac{2}{s^3}$	$\cosh bt$	$\frac{s}{s^2-b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{-at}\sinh bt$	$\frac{b}{(s+a)^2-b^2}$
e^{at}	$\frac{1}{s-a}$	$e^{-at}\cosh bt$	$\frac{s+a}{(s+a)^2-b^2}$
e^{-at}	$\frac{1}{s+a}$	$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$	$u(t)$ unit step	$\frac{1}{s}$
$\cos bt$	$\frac{s}{s^2+b^2}$	$u(t-d)$	$\frac{e^{-sd}}{s}$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$	$\delta(t)$	1
		$\delta(t-d)$	e^{-sd}

Laplace Transforms

Example:

Find the Laplace transform of following impulse function



$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty \delta(t-t_0)e^{-st}dt = e^{-st_0}$$

Laplace Transforms

Other Examples

$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
(a) t^3	$\frac{6}{s^4}$	(e) $\cos(t/2)$	$\frac{s}{s^2+0.25}$	(h) $t\sin 4t$	$\frac{8s}{(s^2+16)^2}$
(b) t^7	$\frac{7!}{s^8}$	(f) $\sinh 3t$	$\frac{3}{s^2-9}$	(i) $e^{-t}\sin 2t$	$\frac{2}{(s+1)^2+4}$
(c) $\sin 4t$	$\frac{4}{s^2+16}$	(g) $\cosh 5t$	$\frac{s}{s^2-25}$	(j) $e^{3t}\cos t$	$\frac{s-3}{(s-3)^2+1}$
(d) e^{-2t}	$\frac{1}{s+2}$				

Properties of Laplace Transforms

- Linearity

$$\mathcal{L}\{af(t) \pm bg(t)\} = a\mathcal{L}\{f(t)\} \pm b\mathcal{L}\{g(t)\}$$

- First shift theorem (Frequency shift theorem)

$$\mathcal{L}\{e^{\pm at}f(t)\} = F(s \mp a)$$

- Second shift theorem (Time shift theorem)

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

Properties of Laplace Transforms

- Time scaling

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

- Multiplication by time

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

Properties of Laplace Transforms

Examples:

Determine the Laplace transform of

a) $f(t) = t^3 - 3e^{-4t} + \sin 2t$

b) $f(t) = (1 - e^{-3t}) \cos t - \frac{t^4}{4}$

c) $f(t) = e^{-5t} t^3$

d) $f(t) = (t-2)^2 u(t-2)$

e) $f(t) = t g'(t)$

Properties of Laplace Transforms

a)

$$\mathcal{L}\{t^3 - 3e^{-4t} + \sin 2t\} = \mathcal{L}\{t^3\} - 3\mathcal{L}\{e^{-4t}\} + \mathcal{L}\{\sin 2t\}$$

$$= \frac{6}{s^4} - \frac{3}{s+4} + \frac{2}{s^2+4}$$

b)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\cos t - e^{-3t}\cos t - \frac{t^4}{4}\right\}$$

$$= \mathcal{L}\{\cos t\} - \mathcal{L}\{e^{-3t}\cos t\} - \frac{1}{4}\mathcal{L}\{t^4\}$$

$$= \frac{s}{s^2+1} - \frac{s+3}{(s+3)^2+1} - \frac{6}{s^5}$$

Properties of Laplace Transforms

c) Let $g(t) = t^3$ then $G(s) = \frac{6}{s^4}$

Therefor

$$F(s) = \mathcal{L}\{e^{-5t}t^3\} = \mathcal{L}\{e^{-5t}g(t)\}$$

$$= G(s+5) = \frac{6}{(s+5)^4}$$

d) Let $g(t) = t^2$ then $G(s) = \frac{2}{s^3}$

and also $g(t-2) = (t-2)^2$

Therefor

$$F(s) = \mathcal{L}\{(t-2)^2 u(t-2)\} = \mathcal{L}\{g(t-2)u(t-2)\}$$

$$= e^{-2s}G(s) = \frac{2e^{-2s}}{s^3}$$

Properties of Laplace Transforms

e)

$$\mathcal{L}\{tg'(t)\} = -\frac{d}{ds}\mathcal{L}\{g'(t)\}$$

$$= -\frac{d}{ds}[sG(s) - g(0)]$$

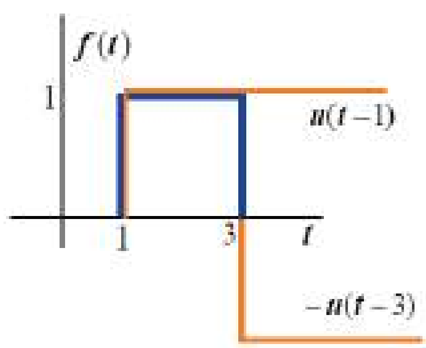
$$= -[G(s) + sG'(s) - 0]$$

$$= -G(s) - sG'(s)$$

Properties of Laplace Transforms

Examples:

$$f(t) = \begin{cases} 1 & 1 \le t \le 3 \\ 0 & \text{elsewhere} \end{cases}$$



$$f(t) = u(t-1) - u(t-3)$$

$$F(s) = e^{-s}\frac{1}{s} - e^{-3s}\frac{1}{s} = \frac{1}{s}\left(e^{-s} - e^{-3s}\right)$$

Properties of Laplace Transforms

Examples:

Let $u(t)$ be the unite step function. Find the Laplace transform of the ramp function

$$r(t) = tu(t)$$

Solution:

$$u(t) \leftrightarrow U(s) = \frac{1}{s}$$

$$tu(t) \leftrightarrow -\frac{d}{dt}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$t^2u(t) \leftrightarrow -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

By successive application of the property, one can show that

$$t^n(u(t)) \leftrightarrow \frac{n!}{s^{n+1}}$$

This result, plus linearity, allows computation of the transform of any polynomial

Properties of Laplace Transforms

Examples: Find the Laplace transform of

$$f(t) = te^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$$

Solution:

One can apply the time shifting property if the time variable always appears as it appears in the argument of the step. In this case as $t - 1$

$$f(t) = (t-1+1)e^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$$

$$f(t) = (t-1)e^{-(t-1)}u(t-1) + e^{-(t-1)}u(t-1) - e^{-(t-1)}u(t-1)$$

$$= (t-1)e^{-(t-1)}u(t-1)$$

$$tu(t) \leftrightarrow \frac{1}{s^2}$$

$$te^{-t}u(t) \leftrightarrow \frac{1}{(s+1)^2}$$

$$\therefore (t-1)e^{-(t-1)}u(t-1) \leftrightarrow \frac{e^{-s}}{(s+1)^2}$$

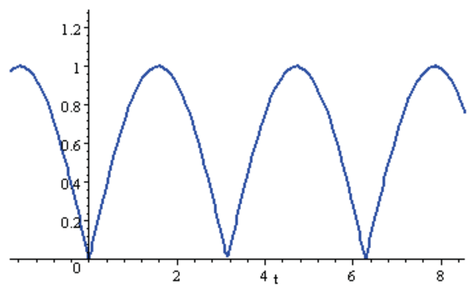
Laplace Transforms

Laplace Transform of a Periodic Function $f(t)$

The Laplace Transform of the periodic function, $f(t)$ with period p, equals the Laplace Transform of one cycle of the function, divided by $(1 - e^{-sp})$.

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_1(t)\}}{1 - e^{-sp}}$$

Example: Full-wave rectifier of $\sin t$ is



Solution: We have

$$f_1(t) = \sin t \times [u(t) - u(t - \pi)]$$

And the period $p = \pi$.

$$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{\sin t \times [u(t) - u(t - \pi)]\}$$

$$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{\sin t \times u(t)\} - \mathcal{L}\{\sin(t) \times u(t - \pi)\}$$

But since $\sin t = -\sin(t - \pi)$

$$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{\sin t \times u(t)\} + \mathcal{L}\{\sin(t - \pi) \times u(t - \pi)\}$$

$$= \frac{1}{s^2 + 1} + \frac{e^{-\pi s}}{s^2 + 1}$$

So the Laplace Transform of the periodic function is given by:

$$\mathcal{L}\{f(t)\} = \frac{1 + e^{-\pi s}}{(s^2 + 1)(1 - e^{-\pi s})}$$

Inverse Laplace Transform

The inverse Laplace transform of $F(s)$ is $f(t)$, i.e.

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{ts}ds$$

Where $\mathcal{L}^{-1}$ is inverse Laplace transform operator.

Examples: Find the inverse Laplace transform of

a) $\frac{2}{s^3}$ b) $\frac{2}{s^4}$ c) $\frac{1}{s^2+25}$

d) $\frac{5s-6}{s^2+9}$ e) $\frac{s+1}{(s+1)^2+4}$ f) $\frac{s}{(s+1)^2+4}$

Inverse Laplace Transform

Solutions:

From the table of Laplace Transform

(a) $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = t^2$

(b) $\mathcal{L}^{-1}\left\{\frac{2}{s^4}\right\} = \frac{2}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3}t^3$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+5^2}\right\} = \frac{1}{5}\sin 5t$

Inverse Laplace Transform

(d) Write $\frac{5s-6}{s^2+9} = 5\frac{s}{s^2+3^2} - 2\frac{3}{s^2+3^2}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{5s-6}{s^2+9}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= 5\cos 3t - 2\sin 3t$$

(e) $\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} = e^{-t}\cos 2t$

Inverse Laplace Transform

(f) Since the ILT of the terms cannot be found directly from the table, we need to rewrite it as the following

$$\frac{s}{(s+1)^2+4} = \frac{(s+1)-1}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

Inverse Laplace Transform

Most of the Laplace transforms that we encounter are proper rational functions of the form

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

Zeros: roots of numerator
Poles: roots of denominator

Partial Fraction Expansion:

If $m < n$ and the poles are distinct

$$F(s) = \frac{P(s)}{Q(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \frac{K_3}{s - p_3} + \cdots + \frac{K_n}{s - p_n}$$

Inverse Laplace Transform

If $m < n$ and the poles are duplicated

$$\frac{P(s)}{(s - p_1)^r} = \frac{K_1}{(s - p_1)^r} + \frac{K_2}{(s - p_1)^{r-1}} + \frac{K_3}{(s - p_1)^{r-2}} + \cdots + \frac{K_r}{s - p_1}$$

The Coefficients K_1, K_2,…K_r can be found as follow

$$K_n = \frac{1}{(n-1)!} \times \left[\frac{d^{n-1}}{ds^{n-1}} \left[(s - p_1)^r F(s) \right] \right]_{s=p_1}$$

Where $n = 1,2,3, \ldots, r$

Inverse Laplace Transform

Examples: Find the inverse Laplace transform of

(a) $\frac{s-8}{s(s-2)}$

(b) $\frac{9}{2s^2+7s-4}$

(c) $\frac{4s+1}{s^3+2s^2+s}$

(d) $\frac{7s-20}{s(s^2-4s+20)}$

(e) $\frac{s^2}{s^2+5s+6}$

(f) $F(s)=\frac{s^2+2s+3}{(s+1)^3}$

Inverse Laplace Transform

Solutions: We use the partial fraction technique

(a) $F(s) = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} = \frac{4}{s} - \frac{3}{s-2}$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{4}{s} - \frac{3}{s-2}\right] = 4 - 3e^{2t}$$

(b) $F(s) = \frac{9}{2s^2+7s-4} = \frac{2}{2s-1} - \frac{1}{s+4} = \frac{1}{s-1/2} - \frac{1}{s+4}$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s-1/2} - \frac{1}{s+4}\right] = e^{t/2} - e^{-4t}$$

Inverse Laplace Transform

(c) $$\mathcal{L}^{-1}\left\{\frac{4s+1}{s^3+2s^2+s}\right\}=\mathcal{L}^{-1}\left\{\frac{4s+1}{s(s+1)^2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{3}{(s+1)^2}-\frac{1}{s+1}\right\}$$

$$=1+3e^{-t}t-e^{-t}$$

where, if we let $F(s)=\frac{1}{s^2}$, then $f(t)=t$. Hence,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}=\mathcal{L}^{-1}\{F(s+1)\}=e^{-t}f(t)=e^{-t}t$$

Inverse Laplace Transform

(d) $$\mathcal{L}^{-1}\left\{\frac{7s-20}{s(s^2-4s+20)}\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{s+3}{s^2-4s+20}\right\}$$

$$=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{s+3}{(s-2)^2+16}\right\}$$

$$=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{(s-2)+5}{(s-2)^2+16}\right\}$$

$$=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{s-2}{(s-2)^2+16}+\frac{5}{(s-2)^2+16}\right\}$$

$$=-1+e^{2t}\cos 4t+\tfrac{5}{4}e^{2t}\sin 4t$$

Inverse Laplace Transform

(e)

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+5s+6}\right\} = \mathcal{L}^{-1}\left\{1-\frac{5s+6}{s^2+5s+6}\right\}$$

$$= \mathcal{L}^{-1}\left\{1-\frac{5s+6}{(s+2)(s+3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{1+\frac{4}{s+2}-\frac{9}{s+3}\right\}$$

$$= \delta(t)+4e^{-2t}-9e^{-3t}$$

(f)

$$F(s) = \frac{s^2+2s+3}{(s+1)^3} = \frac{K_1}{(s+1)^3}+\frac{K_2}{(s+1)^2}+\frac{K_3}{s+1}$$

$$(s+1)^3\frac{s^2+2s+3}{(s+1)^3} = (s+1)^3\left[\frac{K_1}{(s+1)^3}+\frac{K_2}{(s+1)^2}+\frac{K_3}{s+1}\right]$$

$$s^2+2s+3 = K_1+(s+1)K_2+(s+1)^2K_3$$

$$[s^2+2s+3]_{s=-1} = [K_1+(s+1)K_2+(s+1)^2K_3]_{s=-1} \;\Rightarrow\; K_1 = 2$$

$$[2s+2]_{s=-1} = [K_2+2(s+1)K_3]_{s=-1} \;\Rightarrow\; K_2 = 0$$

$$[2]_{s=-1} = [2K_3]_{s=-1} \;\Rightarrow\; K_3 = 1$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}+\frac{1}{s+1}\right]$$

$$f(t) = t^2e^{-t}+e^{-t}$$

Inverse Laplace Transform

Matlab command for partial fraction Expansion:

```
num = [2 5 3 6]
den = [1 6 11 6]
```

$$\frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

The command

```
[r,p,k] = residue(num,den)
```

Gives the following result

```
[r,p,k] = residue(num,den)

r =
        -6.0000
        -4.0000
         3.0000

p =
        -3.0000
        -2.0000
        -1.0000

k =
         2
```

$$2 + \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1}$$

Inverse Laplace Transform

The Convolution Theorem:

$f\,(t) * g(t)$ is called as the convolution of $f(t)$ and $g(t)$,

And it is defined by

$$f(t) * g(t) = \int_0^t f(t-v)g(v)dv$$

Convolution property: $f\,(t) * g(t) = g(t) * f\,(t)$

Therefore,

$$f\,(t) * g(t) = \int_0^t f(t-v)g(v)dv = \int_0^t f(v)g(t-v)dv = g(t) * f\,(t)$$

Sometime, $f\,(t) * g(t)$ is denoted as $(f * g)(t)$ or simply $f * g$.

In Laplace transform

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$$

Inverse Laplace Transform

Examples: Use the convolution theorem to find the inverse Laplace transforms of the following:

(a) $\dfrac{1}{(s-1)(s+2)}$

(b) $\dfrac{12}{s(s^2+9)}$

(c) $\dfrac{7}{s^2(s+5)}$

Inverse Laplace Transform

Solution

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\}$

$$= e^t * e^{-2t}$$

$$= \int_0^t e^{t-\upsilon} e^{-2\upsilon}\, d\upsilon = \int_0^t e^{t-3\upsilon}\, d\upsilon = \left[\frac{e^{t-3\upsilon}}{-3}\right]_0^t$$

$$= \frac{e^{-2t} - e^t}{-3} = \frac{e^t - e^{-2t}}{3}$$

Inverse Laplace Transform

Solution

(b) $\mathcal{L}^{-1}\left\{\frac{12}{s(s^2+9)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{3}{s^2+9}\right\}$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= 4(1 * \sin 3t)$$

$$= 4\int_0^t 1 \sin 3\upsilon \, d\upsilon$$

$$= 4\left[\frac{-\cos 3\upsilon}{3}\right]_0^t = \frac{4}{3}(1-\cos 3t)$$

Inverse Laplace Transform

(c) $\mathcal{L}^{-1}\left\{\frac{7}{s^2(s+5)}\right\} = 7\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+5}\right\}$

$$= 7\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= 7t * e^{-5t} \quad = 7\int_0^t \upsilon e^{-5(t-\upsilon)} d\upsilon = 7\int_0^t \upsilon e^{5(\upsilon-t)} d\upsilon$$

$$= 7\left[\frac{\upsilon e^{5(\upsilon-t)}}{5}\right]_0^t - 7\int_0^t \frac{e^{5(\upsilon-t)}}{5} d\upsilon$$

$$= 7\left(\frac{te^0 - 0}{5}\right) - 7\left[\frac{e^{5(\upsilon-t)}}{25}\right]_0^t = \frac{7t}{5} - \frac{7(1-e^{-5t})}{25}$$

$$= \frac{7}{25}(5t + e^{-5t} - 1)$$

Solution of LDEs Using Laplace Transform

Example: Solve the following Linear DE

$$y'' + 5y' + 6y = 0,\ y(0) = 2,\ \ y'(0) = 3$$

Solution: taking the Laplace transform of Linear DE

$$\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{0\} = 0$$

Now find the Laplace transform of derivatives

$$[s^s Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] + 6Y(s) = 0$$

Rearranging the equation

$$(s^2 + 5s + 6)Y(s) - (s + 5)y(0) - y'(0) = 0$$

Substituting in the initial conditions, we obtain

$$(s^2 + 5s + 6)Y(s) - 2(s + 5) - 3 = 0$$

$$Y(s) = \frac{2s + 13}{(s + 3)(s + 2)}$$

Solution of LDEs Using Laplace Transform

Using partial fraction decomposition, $Y(s)$ can be rewritten:

$$\frac{2s+13}{(s+3)(s+2)} = \frac{A}{(s+3)} + \frac{B}{(s+2)}$$

$$2s+13 = A(s+2) + B(s+3)$$

$$2s+13 = (A+B)s + (2A+3B)$$

$$A+B = 2,\ \ 2A+3B = 13$$

$$A = -7,\ B = 9$$

Thus

$$Y(s) = -\frac{7}{(s + 3)} + \frac{9}{(s + 2)}$$

Now we can find the inverse Laplace transform of $Y(s)$ to get $y(t)$

$$y(t) = -7e^{-3t} + 9e^{-2t}$$

Solution of LDEs Using Laplace Transform

Example: Solve the following Linear DE

$$y'' + y = \sin 2t, \; y(0) = 2, \;\; y'(0) = 1$$

Solution: taking the Laplace transform of Linear DE

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}$$

Now find the Laplace transform of derivatives

$$[s^s Y(s) - sy(0) - y'(0)] + Y(s) = \frac{2}{s^2 + 4}$$

Rearranging the equation

$$(s^2 + 1)Y(s) - sy(0) - y'(0) = \frac{2}{s^2 + 4}$$

Substituting in the initial conditions, we obtain

$$(s^2 + 1)Y(s) - 2s - 1 = \frac{2}{s^2 + 4}$$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}$$

Solution of LDEs Using Laplace Transform

Using partial fraction, $Y(s)$ can be rewritten:

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}$$

Then

$$\begin{aligned} 2s^3 + s^2 + 8s + 6 &= (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1) \\ &= (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D) \end{aligned}$$

Solving, we obtain $A = 2$, $B = 5/3$, $C = 0$, and $D = -2/3$. Thus

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}$$

Now we can find the inverse Laplace transform of $Y(s)$ to get $y(t)$

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Solution of LDEs Using Laplace Transform

Example: Solve the following Linear DE

$$y'' + 2y' + 5y = 3, \; y(0) = 0, \;\; y'(0) = 0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{3\}$$

$$[s^s Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{3}{s}$$

$$[s^2 + 2s + 5]Y(s) = \frac{3}{s}$$

$$Y(s) = \frac{3}{s(s+1-2j)(s+1+2j)} = \frac{A}{s} + \frac{B_1}{(s+1-2j)} + \frac{B_2}{(s+1+2j)}$$

$$y(t) = A + B_1 e^{-(1-2j)t} + B_1 e^{-(1+2j)t}$$

Where $A = 0.6$, $B_1 = -0.3 + 0.15j = 0.33e^{2.6779j}$, $B_2 = -0.3 - 0.15j = 0.33e^{-2.6779j}$

Initial and Final Value theorem

INITIAL VALUE THEOREM

Assume that $f(t)$ has Laplace transform.

Then,

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

FINAL VALUE THEOREM

Assume that $f(t)$ has Laplace transform and that $\lim_{n\to\infty} f(t)$ exist.

Then,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Note: $\lim_{n\to\infty} f(t)$ will exist if $F(s)$ has poles with negative real part and at most a single pole at $s = 0$.

Initial and Final Value theorem

Example: Given

$$F(s) = \frac{10(s+1)}{s(s^2+2s+2)}$$

Determine the initial and final values for $f(t)$.

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{10(s+1)}{s^2+2s+2} = 0$$

$F(s)$ has one pole at s=0 and the others have negative real part. The final value theorem can be applied.

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{10(s+1)}{s^2+2s+2} = 5$$

Note: Computing the inverse one can get

$$f(t) = 5 + 5\sqrt{2}e^{-t} \cos\left(t - \frac{3\pi}{4}\right)$$

Initial and Final Value theorem

Example: Investigate the application of initial and final value theorem to the Laplace transform function

$$F(s) = \frac{1}{(s+2)(s-3)}$$

Solution:

For the initial value theorem:

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{s}{(s+2)(s-3)} = 0$$

$$f(t) = \frac{1}{5}(e^{3t} - e^{-2t})$$

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{1}{5}(e^{0} - e^{0}) = 0$$

For the final value theorem:

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{s}{(s+2)(s-3)} = 0$$

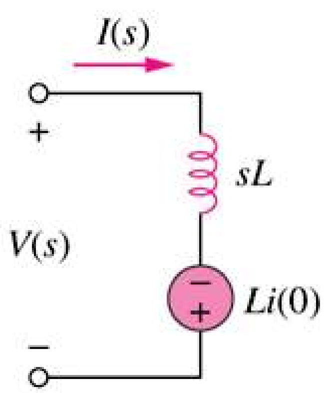
$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \frac{1}{5}(e^{\infty} - e^{-\infty}) = \infty$$

∴ the system is not stable (is not steady-state gain)

Circuit Application

1. RLC circuit with initial condition
2. Transfer functions
3. Block Diagram

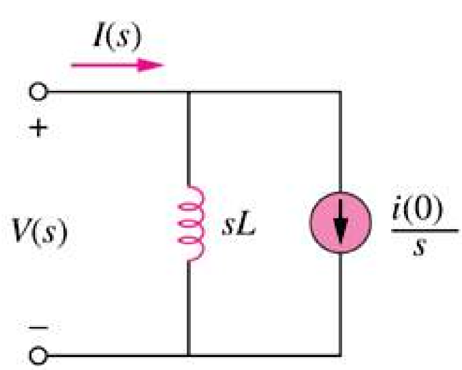
RLC circuit



$$v_L(t) = L\frac{di_L(t)}{dt}$$

Taking the Laplace transform

$$V_L(s) = (sL)I_L(s) - Li_L(0)$$

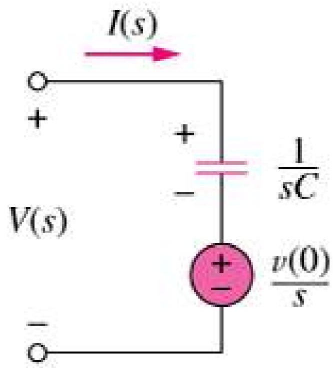


$$i_L(t) = \frac{1}{L}\int_0^t v_L(t)dt + i_L(0)$$

Taking the Laplace transform

$$I_L(s) = \frac{V_L(s)}{sL} + \frac{i_L(0)}{s}$$

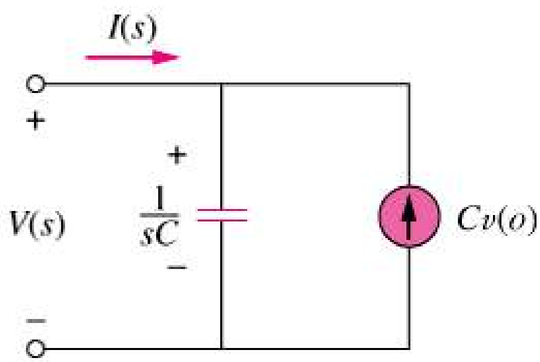
RLC circuit



$$v_c(t) = \frac{1}{C}\int_0^t i_c(t)dt + v_c(0)$$

Taking the Laplace transform

$$V_c(s) = \frac{1}{sC}I_c(s) + \frac{v_c(0)}{s}$$



$$i_c(t) = C\frac{dv_c(t)}{dt}$$

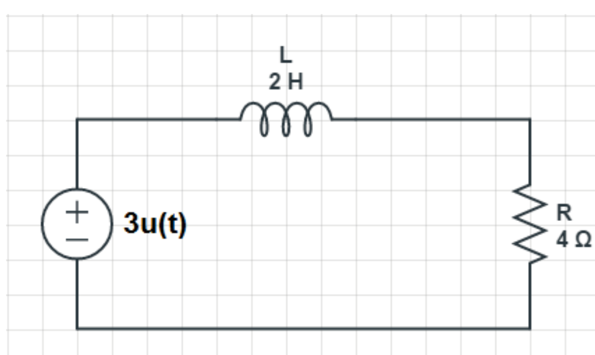
Taking the Laplace transform

$$I_c(s) = \frac{V_c(s)}{1/sC} - Cv_c(0)$$

RLC circuit

Example:

In the RL-Series circuit given that $i(0) = 5$ Amp, find $i(t)$



Using KVL

$$L\frac{di}{dt} + Ri = 3u(t)$$

Taking Laplace transform

$$L[sI(s) - i(0)] + RI(s) = \frac{3}{s}$$

RLC circuit

$$2[sI(s) - 5] + 4I(s) = \frac{3}{s}$$

$$I(s)[2s + 4] = \frac{3}{s} + 10$$

$$I(s)[s + 2] = \frac{1.5}{s} + 5 = \frac{5s + 3/2}{s}$$

$$I(s) = \frac{5s + 3/2}{s(s + 2)} = \frac{A}{s} + \frac{B}{s + 2} = \frac{A(s + 2) + Bs}{s(s + 2)}$$

Equating coefficients

$$\frac{3}{2} = 2A \quad \Longrightarrow \quad A = \frac{3}{4}$$

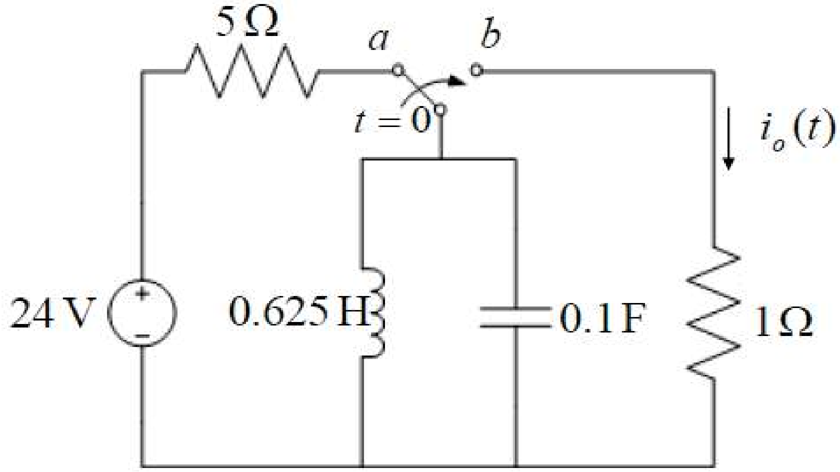
$$5 = A + B \quad \Longrightarrow \quad B = \frac{17}{4}$$

$$I(s) = \frac{5s + 3/2}{s(s + 2)} = \frac{3}{4s} + \frac{17}{4(s + 2)} \quad \Longrightarrow \quad \mathcal{L}^{-1}\{I(s)\} = i(t) = \frac{3}{4}u(t) + \frac{17}{4}e^{-2t}u(t)$$

RLC circuit

Example:

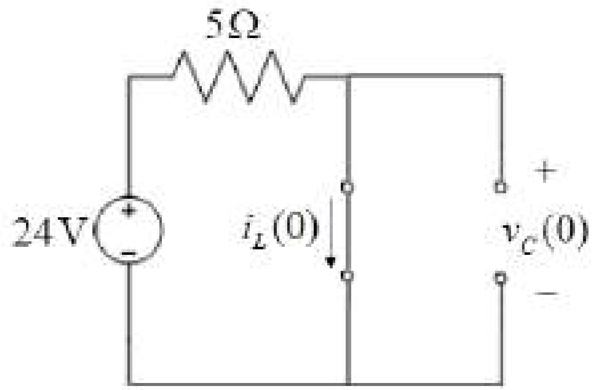
The switch in the following circuit moves from position a to position b at $t = 0$ second. Compute $i_o(t)$ for $t > 0$.



RLC circuit

Solution:

The i.c. (initial condition) are not given directly. Hence, at first we need to find the i.c. by analyzing the circuit when $t \leq 0$:

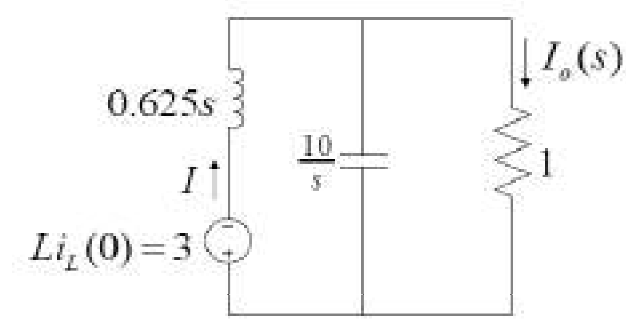


$$i_L(0) = \frac{24}{5} = 4.8A$$

$$v_L(0) = 0V$$

RLC circuit

Then, we can analyze the circuit for $t > 0$ by considering the i.c.



$$I = \frac{-3}{0.625s + \left(\frac{10}{s} \| 1\right)} = \frac{-3}{0.625s + \left(\frac{\frac{10}{s}}{\frac{10}{s} + 1}\right)} = \frac{-3}{0.625s + \left(\frac{10}{s + 10}\right)} = \frac{-3}{\frac{0.625s^2 + 6.25s + 10}{s + 10}}$$

$$I = \frac{-3(s + 10)}{0.625s^2 + 6.25s + 10}$$

RLC circuit

Using current divider rule, we find that

$$I_0 = \frac{\frac{10}{s}}{\frac{10}{s}+1} I = \frac{10}{(10+s)} \frac{-3(s+10)}{0.625s^2+6.25s+10} = \frac{-30}{0.625s^2+6.25s+10}$$

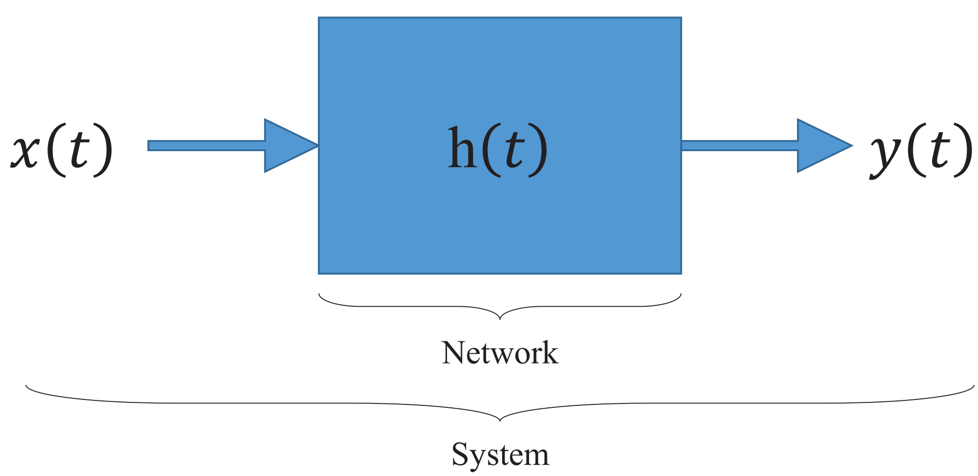
$$I_0 = \frac{-30}{0.625s^2+6.25s+10} = \frac{-30}{0.625(s^2+10s+16)} = \frac{-48}{s^2+10s+16}$$

Using partial fraction, we have

$$I_0(s) = \frac{-48}{(s+8)(s+2)} = \frac{8}{s+8} - \frac{8}{s+2}$$

$$i_0(t) = 8(e^{-8t} - e^{-2t})u(t) \quad \text{A}$$

Transfer Function



In time domain, $y(t) = h(t) * x(t)$

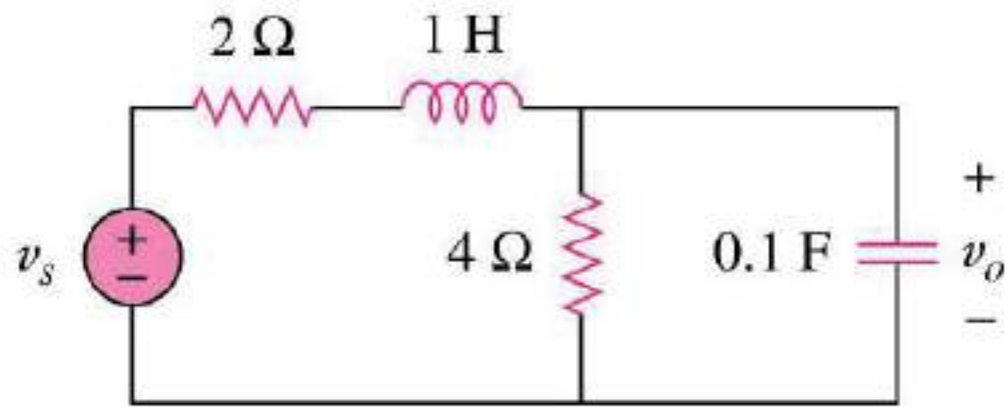
In s-domain, $Y(s) = H(s)X(s)$

$\therefore$Transfer Function, $H(S) = \dfrac{Y(s)}{X(s)}$

Transfer Function

Example:

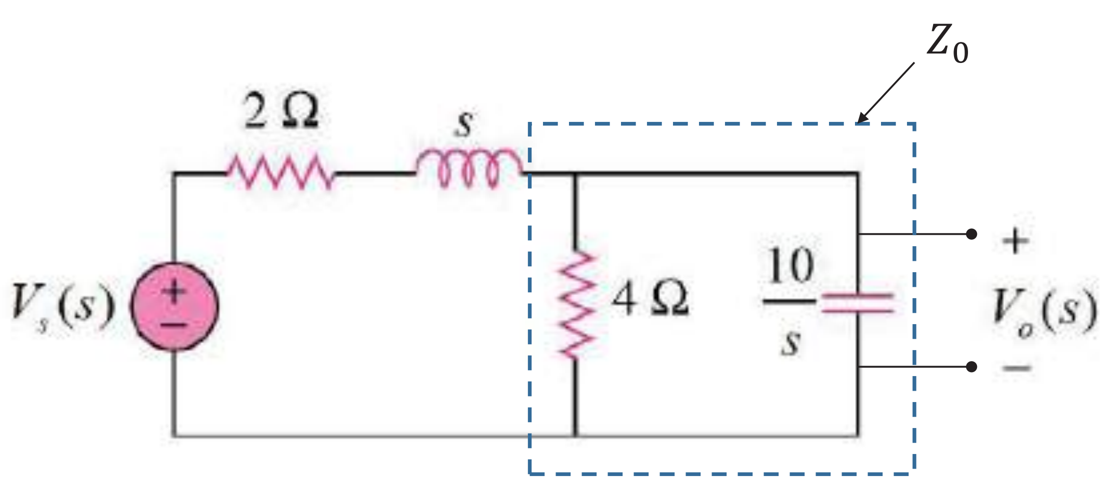
For the following circuit, find $H(s) = V_o(s)/V_i(s)$. Assume zero initial conditions.



Transfer Function

Solution:

Transform the circuit into s-domain with zero i.c.:



Transfer Function

$$Z_0 = 4 \left\| \frac{10}{s} = \frac{\frac{40}{s}}{4 + \frac{10}{s}} = \frac{40}{4s + 10} = \frac{20}{2s + 5}\right.$$

Using voltage divider

$$V_0 = \frac{\frac{20}{2s + 5}}{\frac{20}{2s + 5} + s + 2} V_s = \frac{20}{20 + (2s + 5)(s + 2)} V_s$$

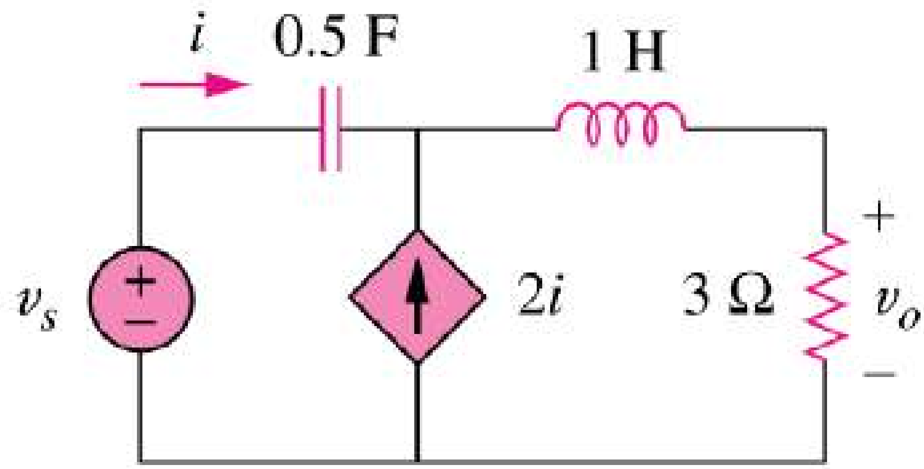
$$V_0 = \frac{20}{2S^2 + 9s + 30} V_s$$

$$H(S) = \frac{V_o(s)}{V_s(s)} = \frac{20}{2S^2 + 9s + 30}$$

Transfer Function

Example:

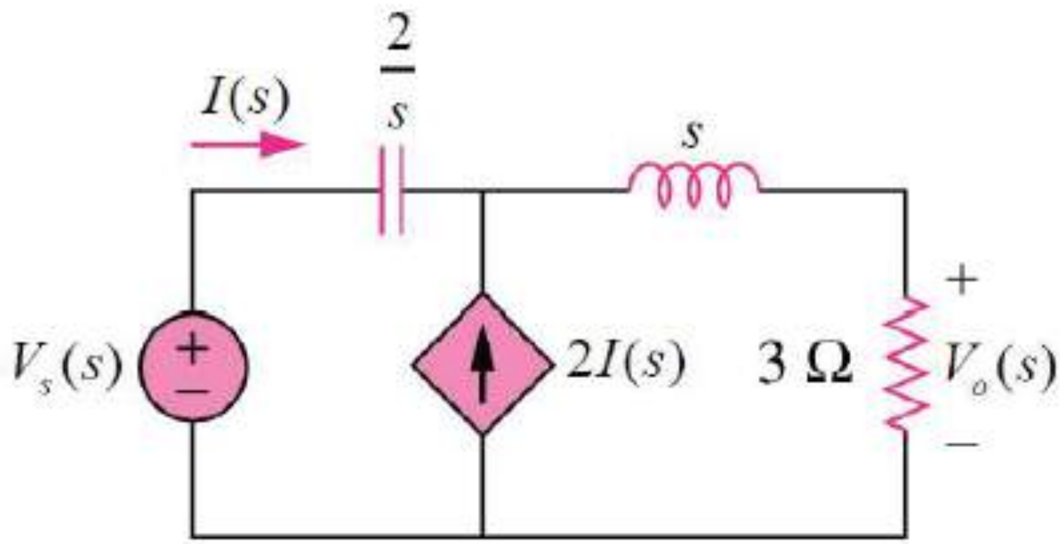
Obtain the transfer function $H(s) = V_0(s)/V_i(s)$, for the following circuit.



Transfer Function

Solution:

Transform the circuit into s-domain (We can assume zero i.c. unless stated in the question)



Transfer Function

We found that

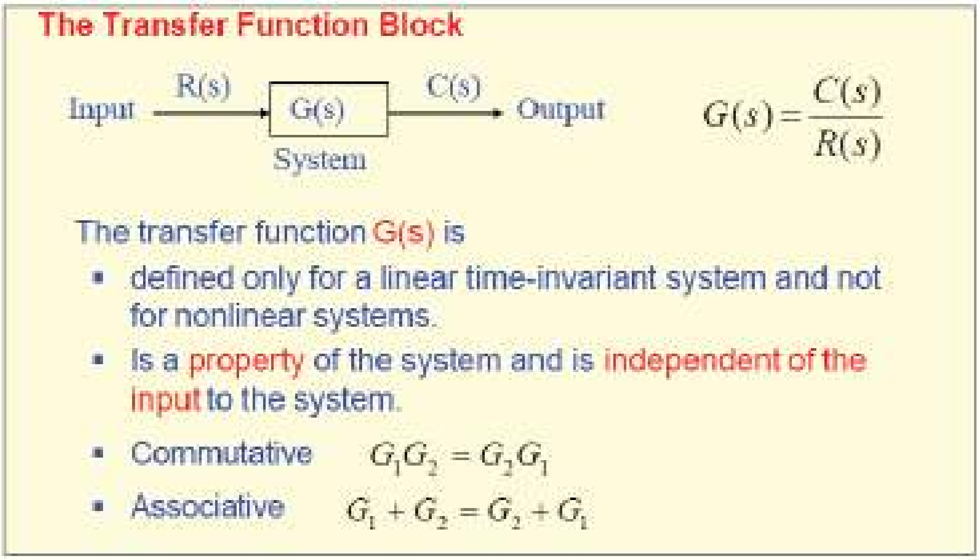
$$V_o = 3(I + 2I) = 9I$$

$$V_s = \frac{2}{s}I + (s+3)3I = \left(\frac{2}{s} + 3s + 9\right)I$$

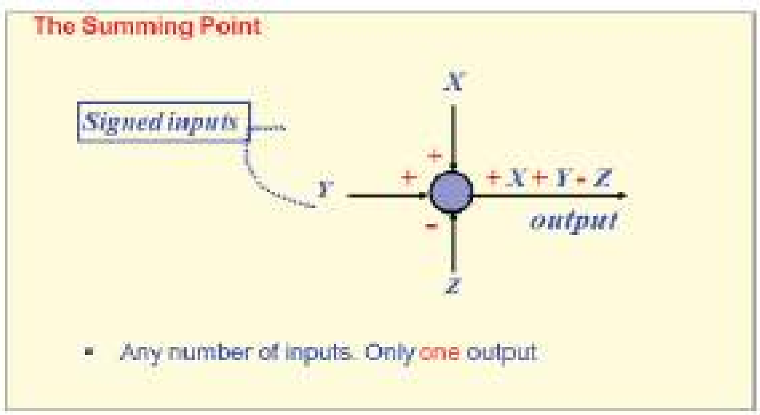
$$\therefore \quad H(s) = \frac{V_o(s)}{V_s(s)} = \frac{9}{\frac{2}{s} + 3s + 9} = \frac{9s}{3s^2 + 9s + 2}$$

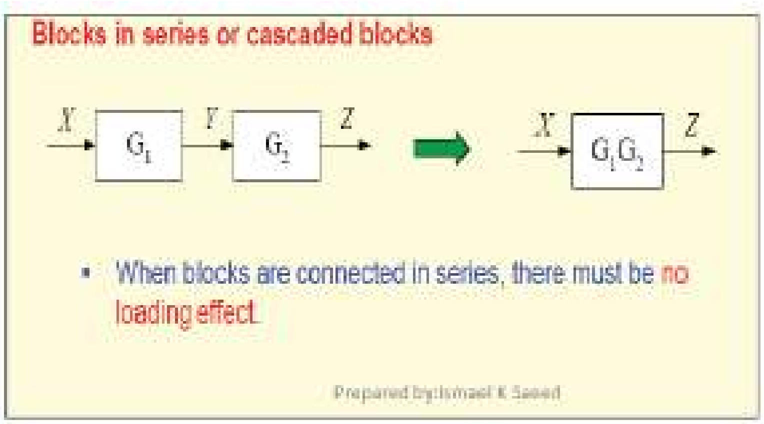
Block Diagram

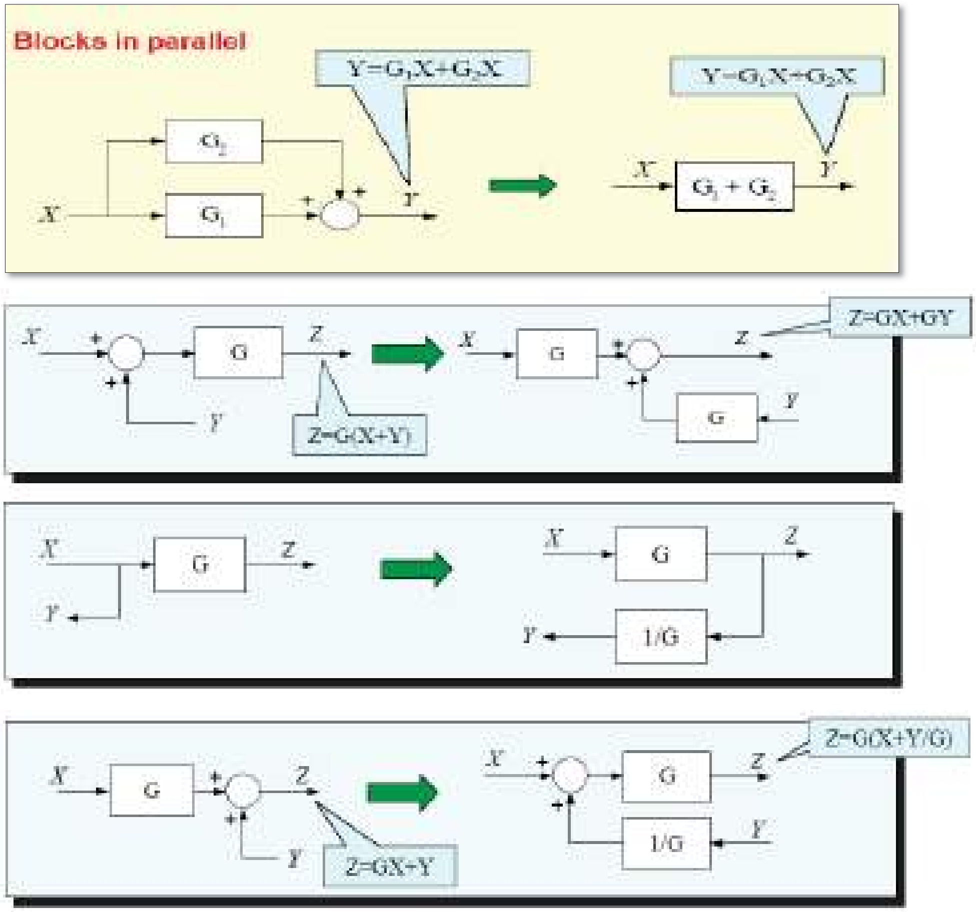
A block diagram is a graphical tool that can help us to visualize the model of a system and evaluate the mathematical relationships between their elements, using their transfer functions.



Block Diagram Elements

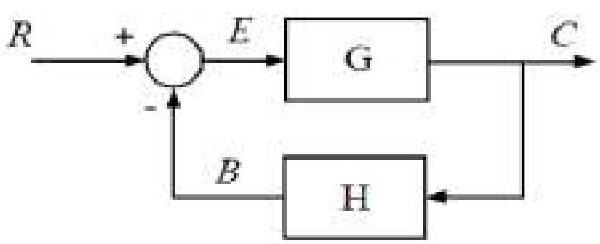






Block Diagram

Closed-loop Feedback System



R	is called the reference input
C	is the output or controlled variable
B	is the feedback
$E = (R - B)$	is the error
$G = \frac{C}{E}$	is called the feed-forward transfer function
$GH = \frac{B}{E}$	is called the open-loop transfer function

Block Diagram

Overall transfer function of closed-loop feedback system

$$E(s) = R(s) - B(s)$$

$$\frac{C(s)}{G(s)} = R(s) - B(s)$$

$$\frac{C(s)}{G(s)} = R(s) - C(s)H(s)$$

$$\frac{C(s)}{G(s)} + C(s)H(s) = R(s)$$

$$C(s)\left[\frac{1}{G(s)} + H(s)\right] = R(s)$$

$$C(s)\left[\frac{1 + G(s)H(s)}{G(s)}\right] = R(s)$$

$$\frac{C(s)}{R(s)} = \left[\frac{G(s)}{1 + G(s)H(s)}\right]$$

Block Diagram

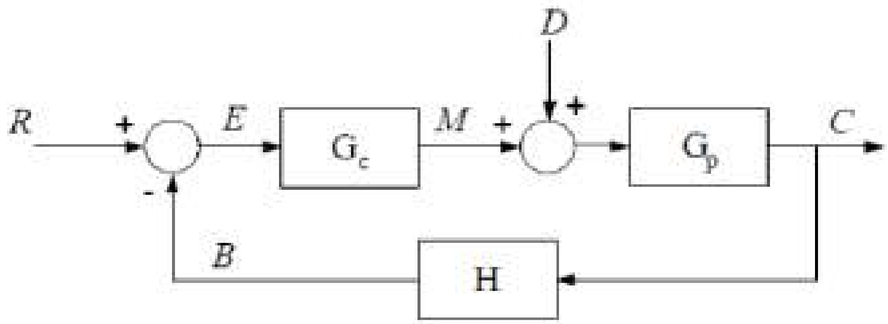
- Eliminating a negative feedback loop
  The overall transfer function for a negative feedback loop is given by

$$\frac{C(s)}{R(s)} = \left[\frac{G(s)}{1 + G(s)H(s)}\right]$$

- Eliminating a positive feedback loop
  The overall transfer function for a positive feedback loop is given by

$$\frac{C(s)}{R(s)} = \left[\frac{G(s)}{1 - G(s)H(s)}\right]$$

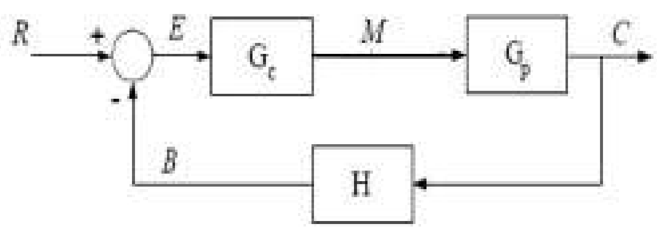
Block Diagram



G_e	is the controller transfer function
G_p	is the plant transfer function
M	is the manipulated variable
D	is the external disturbance
$G_e G_p = \dfrac{C}{E}$	is the feed-forward transfer function
$G_e G_p H = \dfrac{B}{E}$	is the open-loop transfer function

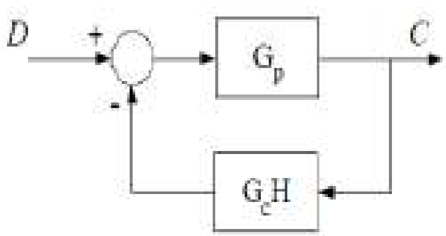
Block Diagram

Assuming $D = 0$, we can re-draw



$$\frac{C}{R} = \frac{G}{1+GH} = \frac{G_c G_p}{1+G_c G_p H}$$

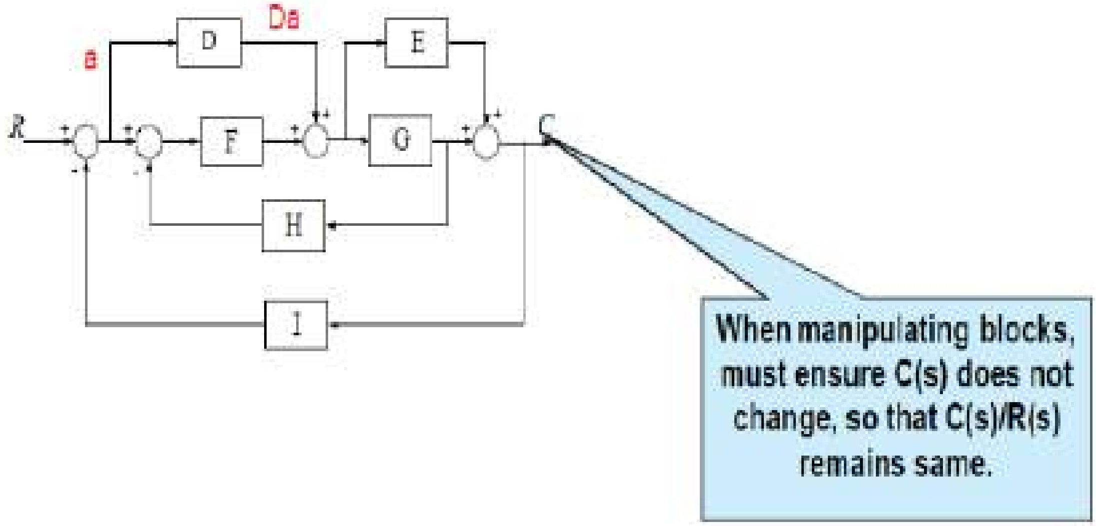
Assuming $R = 0$, we can re-draw



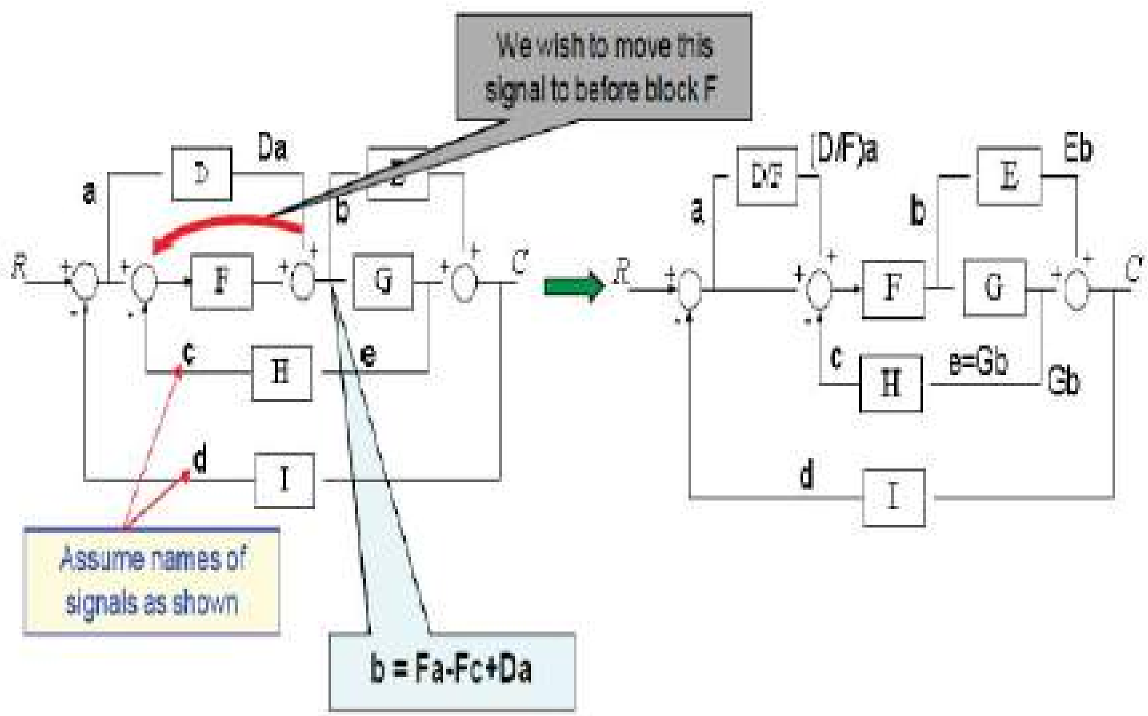
$$\frac{C}{D} = \frac{G}{1+GH} = \frac{G_p}{1+G_p G_c H}$$

Block Diagram

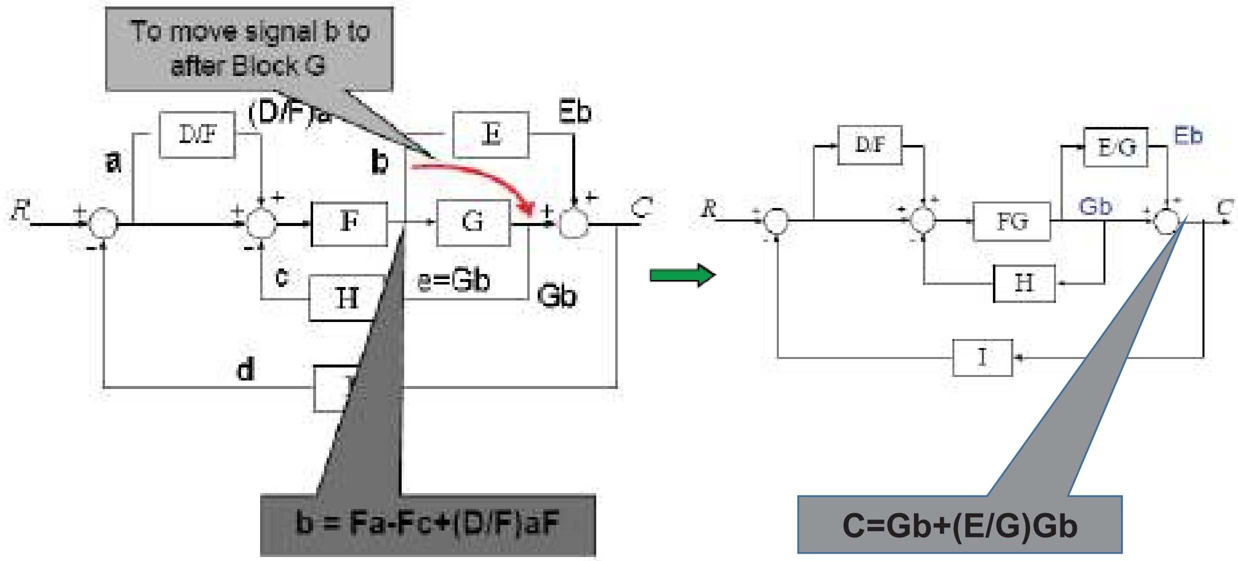
Example: Determine $C(s)/R(s)$



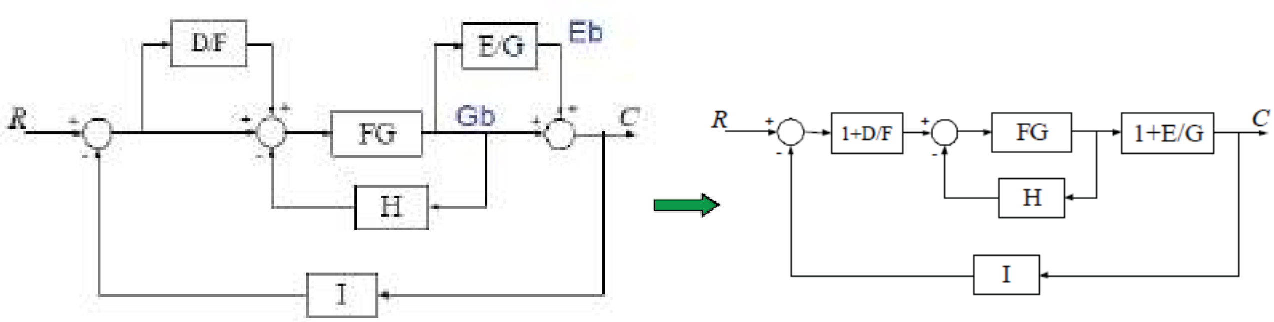
Block Diagram



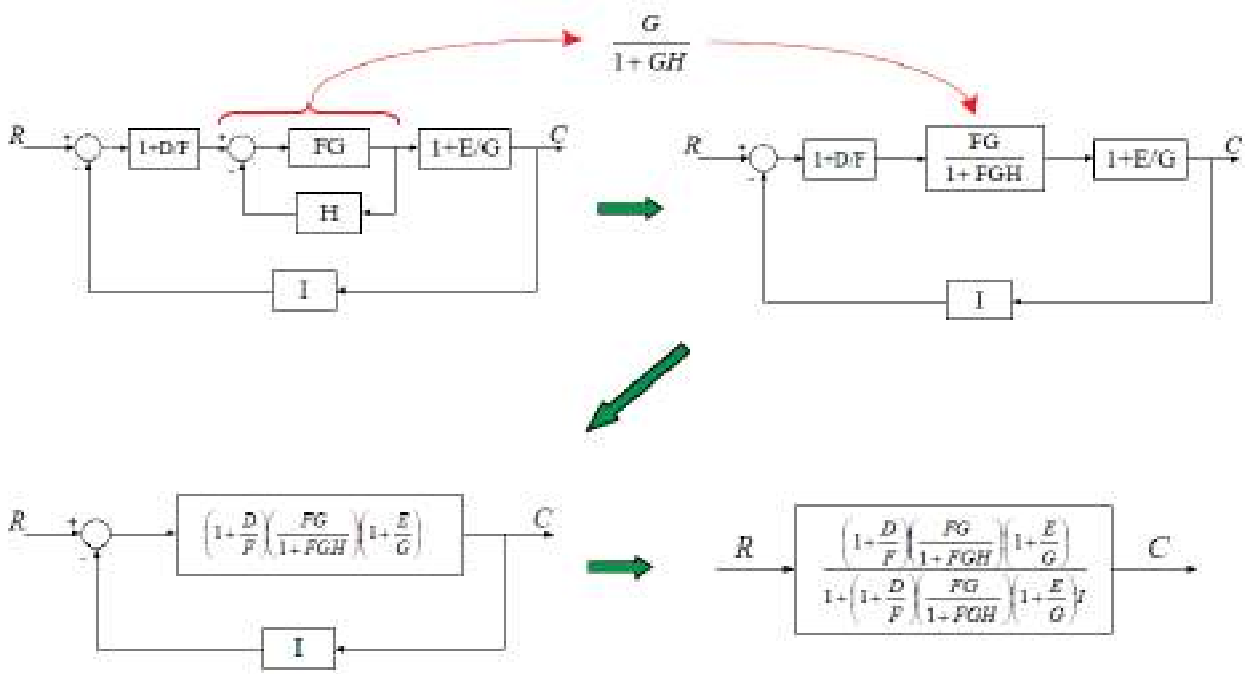
Block Diagram



Block Diagram

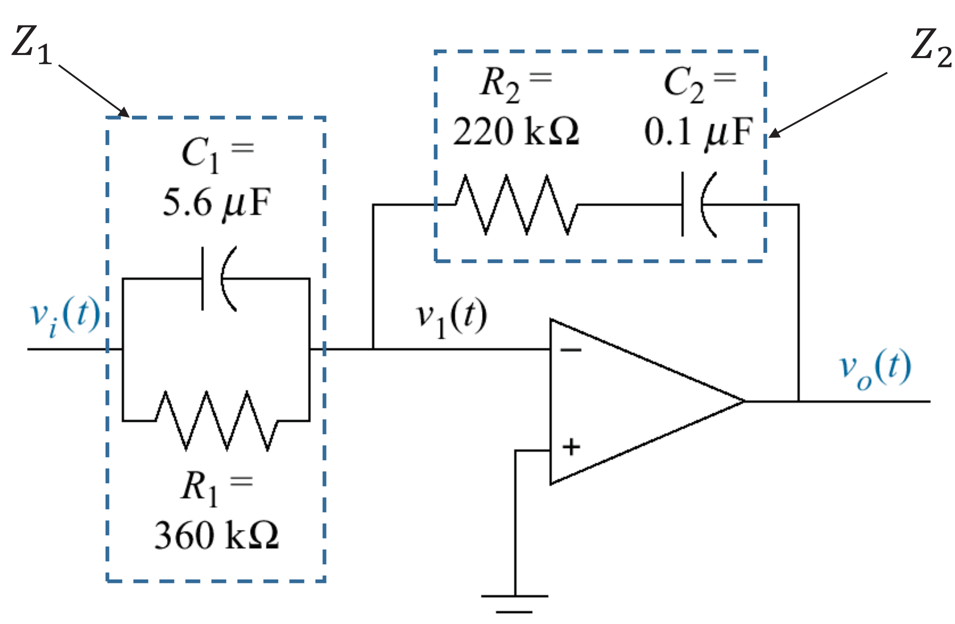


Block Diagram



Block Diagram

Inverting operational amplifier circuit:



Block Diagram

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6*10^{-6}s + \frac{1}{360*10^3}}$$

$$= \frac{360*10^3}{2.016s+1}$$

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220*10^3 + \frac{10^7}{s} = \frac{220*10^3 s + 10^7}{s}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{\frac{220*10^3 s + 10^7}{s}}{\frac{360*10^3}{2.016s+1}}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{(220*10^3 s + 10^7)(2.016s+1)}{360*10^3 s} = -1.232\,\frac{s^2 + 45.95s + 22.55}{s}$$