DIFFERENTIAL EQUATIONS

Introduction

An Equation containing differentials or derivatives, of dependent variable with respect to Independent variables is called a "Differential Equations" (D.E)

Examples:-

(i) $(2x-y)\,dy+(x+2y)\,dx=0$ (ii) $L\frac{di}{dt}+Ri=E$

(iii) $y=x\frac{dy}{dx}+c/\frac{dy}{dx}$ (iv) $\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=0$

(v) $\left(\frac{d^3y}{dx^3}\right)^2+2\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}+x^2\left(\frac{dy}{dx}\right)^3=0$ $\left(\frac{d^{\overset{\text{(order)}}{3}}y}{dx^3}\right)^{\underset{\text{(degree)}}{2}}$

(vi) $\left(1+\frac{d^2y}{dx^2}\right)^3=\frac{d^3y}{dx^2}$ (vii) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=nu$

(viii) $\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u}{\partial y^2}=x$

TYPES OF (D.E) — Ordinary D.E / Partial D.E

* The eqns $i \xrightarrow{\text{to}} vi$ are all ordinary (D.E)
* The eqns $vii \xrightarrow{\&} viii$ are partial (D.E)

ORDER:- The order of the highest ordered derivative occurring in the (D.E)

Thus eqns i, ii, iii, & vii $\longrightarrow$ are of the FIRST ORDER

iv , viii ⟶ are of SECOND ORDER
v , vi ⟶ are of THIRD ORDER

DEGREE :- The degree of the highest order derivative occurring is called degree.

i, ii, iv, vii, viii ⟶ 1ST degree.
iii , v ⟶ 2nd degree.
vi ⟶ 3rd degree.

1- FIRST ORDER D.E :-

1st order D.E can be solved by integration if it is possible to collect y-terms with dy, and all x-terms with dx «i.e separating variable so these eqns called «SEPAREBLE Eqns»

$$f(y)\,dy + g(x)\,dx \quad ----- (1)$$

then the general solution is

$$\int f(y)\,dy + \int g(x)\,dx = c \quad ------ (2)$$

c - arbitrary constant

Example (1):-

solve the following D.E

a- $\dfrac{dy}{dx} = \dfrac{2x+3}{4y-5}$

$(4y-5)\,dy = (2x+3)\,dx$ ⟶ Integrate Both sides

$$\therefore \int (4y-5)\,dy = \int (2x+3)\,dx + c$$

$$\frac{4y^2}{2} - 5y = \frac{2x^2}{2} + 3x + c$$

$$2y^2 - 5y = x^2 + 3x + c$$

Hence $x^2 - 2y^2 + 3x + 5y + c = 0$ is the sol.

(3)

b) $9yy' + 4x = 0$ $\rightsquigarrow y' = dy/dx$

$9y\frac{dy}{dx} + 4x = 0$ By separating Variables & Integrate.

$\therefore \int 9y\,dy = \int -4x\,dx$

$\frac{9}{2}y^2 = -2x^2 + c \implies \frac{9}{2}y^2 + 2x^2 = c$

$\frac{9y^2 + 4x^2}{2} = 2c \rightsquigarrow \times\frac{1}{9} \times \frac{1}{4}$

$$\therefore \frac{y^2}{4} + \frac{x^2}{9} = \bar{C} \qquad \bar{C} = \frac{c}{18}$$

family of paraboles

c) $x^2\frac{dy}{dx} = \cos^2 y$

$\implies x^2 dy = \cos^2 y \, dy$ dx separting variables

$\frac{dx}{x^2} = \frac{dy}{\cos^2 y} \implies x^{-2} dx = \sec^2 y \, dy$

Integrating Bath sides

$\int \sec^2 y \cdot dy = \int x^{-2} dx + c$

$\therefore \tan y = \frac{x^{-1}}{-1} + c \implies \tan y = \frac{-1}{x} + c \implies \boxed{\tan y + x^{-1} = c}$ is the sol.

d) $(x+1)\frac{dy}{dx} = x(y^2+1)$

$(x+1)dy = x(y^2+1)dx \xrightarrow[\text{Variables}]{\text{separting}} \frac{1}{y^2+1}dy = \frac{x}{x+1}dx$

Indagrate Bath sides

$\int \frac{1}{y^2+1} dy = \int (1 - \frac{1}{x+1}) dx + c$

$\tan^{-1} y = x - \ln(x+1) + c$ is the sol.

E) $a\left[x\dfrac{dy}{dx}+2y\right]=xy\dfrac{dy}{dx}$

sol:

$$ax\,dy+2ay\,dx=xy\,\overset{dy}{dx}\Rightarrow 2ay\,dx=xy\,dy-ax\,dy$$

$$2ay\,dx=x(y-a)\,dy \qquad \times\frac{1}{y}\cdot\frac{1}{x} \text{ (separating varible)}$$

$$\therefore \frac{2a}{x}\,dx=\frac{(y-a)}{y}\,dy \rightsquigarrow \text{Integrate Both sides}$$

$$\int\frac{2a}{x}\,dx=\int\left(1-\frac{a}{y}\right)dy+c_1$$

$$2a\ln x=y-a\ln y+c_1$$

$$2a\ln x+a\ln y=y+a\ln c$$

$$a\left[\ln x^2+\ln y-\ln c\right]=y$$

$$\therefore a\left[\ln x^2+\ln y/c\right]=y$$

$$a\left[\ln x^2\cdot\frac{y}{c}\right]=y$$

$$\therefore \ln x^2\cdot\frac{y}{c}=\frac{y}{a} \Rightarrow \boxed{\frac{x^2y}{c}=e^{y/a}}$$

is the sol

Hint: $c_1=a\ln c$

$2\ln x=\ln x^2$

$\ln x+\ln y=\ln xy$

$\ln x-\ln y=\ln\dfrac{x}{y}$

$e^{\ln A}=A$

$e^{m\ln A}=A^m$

F) The slope of the tangent to any curve at (x,y) is given by x^2y+x^2, IF the curve passes through $(0,2)$, find the eqn of the curve

sol:-

The slope of the tangent at point $(x,y)=\dfrac{dy}{dx}$

$$\therefore \frac{dy}{dx}=x^2y+x^2$$

$$dy=x^2(y+1)\,dx \longrightarrow \times\frac{1}{y+1} \quad \text{separating varible}$$

$$\frac{dy}{y+1}=x^2\,dx$$

Integrate both sides

$$\int\frac{1}{y+1}\,dy=\int x^2\,dx+c$$

$$\ln(y+1) = \frac{x^3}{3} + C \quad \text{is the sol.}$$

The curve passes through $(0, 2)$ i.e $x=0$, $y=2$

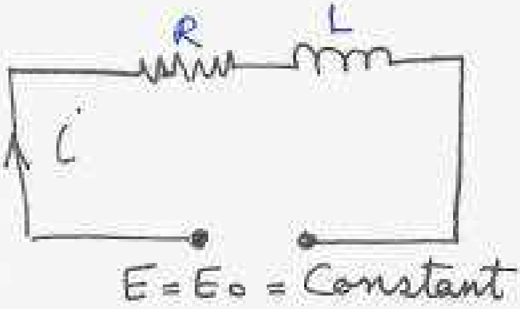
$\therefore \ln(2+1) = 0 + C \implies C = \ln 3$ put in eqn

$\therefore \ln(y+1) = \frac{x^3}{3} + \ln 3 \implies \ln(y+1) - \ln 3 = \frac{x^3}{3}$

$\ln \frac{y+1}{3} = \frac{x^3}{3} \implies \therefore \frac{y+1}{3} = e^{x^3/3}$

$\therefore y = 3e^{x^3/3} - 1$ is the curve eqn.

9) - a simple circuit R & L, Voltage E and the current i at any time is:-



$$L\frac{di}{dt} + Ri = E$$

IF E is constant and initially no current passes through the circuit, Find the relation connecting i & t

sol: D.E is $\quad L\frac{di}{dt} + Ri = E \implies L\frac{di}{dt} = E - Ri$

$L\,di = (E - Ri)\,dt \implies \therefore L \cdot \frac{1}{E - Ri} \cdot di = dt$

$L\int \frac{1}{E-Ri} \cdot di = \int dt + C$

$\therefore \frac{L}{-R} \ln[E - Ri] = t + C \quad \text{---} \ (1)$ is the sol

* Initially No current passes (i.e $i_0 = 0$ at $t = 0$

$\therefore \frac{L}{-R} \ln E = 0 + C \implies \therefore \frac{L}{-R} \ln E = C$

Put (C) in eqn (1)

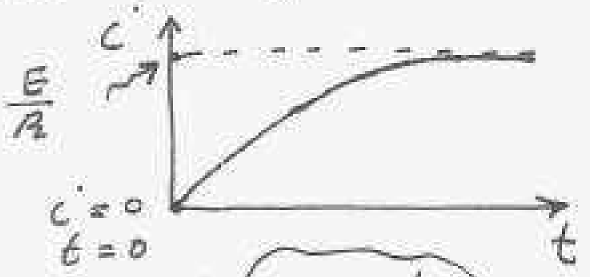
$-\frac{L}{R} \ln[E - Ri] = t - \frac{L}{R} \ln E$

$$\frac{L}{R}\left[\ln(E-Ri) - \ln E\right] = -t$$

$$\therefore \ln \frac{E-Ri}{E} = -\frac{R}{L}t \Rightarrow \frac{E-Ri}{E} = e^{-\frac{R}{L}t}$$

$$E - Ri = E\, e^{-\frac{R}{L}t} \Rightarrow Ri = E(1 - e^{-\frac{R}{L}t})$$

$$\therefore i = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$$



let $\frac{R}{L} = \frac{1}{\tau}$, τ inductive Time constant $\left(\therefore \tau = \frac{L}{R}\right)$

Example (2):- show that $y = A\cos x + B\sin x$ is a solution of (D.E) $\frac{d^2y}{dx^2} + y = 0$ A & B const.

sol:- we have

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

Differentiate again, we get

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x = -\underbrace{(A\cos x + B\sin x)}_{y}$$

$$= -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

H.W

show that $y = a\cos 2x + b\sin 2x$ is a solution of D.E $d^2y/dx^2 + 4y = 0$

Example (3):- verify that each of the following eqns has indicated solution for all values of the const. a & b

a) $Y'' - 4Y = 0$ $Y = a e^{2x} + b e^{-2x}$

$$Y = a e^{2x} + b e^{-2x} \longrightarrow Y' = \frac{dy}{dx} = 2a e^{2x} - 2b e^{-2x}$$

$$y'' = 4ae^{2x} + 4be^{-2x}$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 4(ae^{2x} + be^{-2x}) - 4(ae^{2x} + be^{-2x}) = 0$$

b) $y'' + 3y' + 2y = 12e^{x}$ $\qquad y = ae^{-x} + be^{-2x} + e^{2x}$

$$y' = dy/dx = -ae^{-x} - 2be^{-2x} + 2e^{2x}$$

$$y'' = d^2y/dx^2 = ae^{-x} + 4be^{-2x} + 4e^{2x}$$

$$\therefore y'' + 3y' + 2y =$$

$$ae^{-x} + 4be^{-2x} + 4e^{2x} + 3(-ae^{-x} - 2be^{-2x} + 2e^{2x})$$

$$+ 2(ae^{-x} + be^{-2x} + e^{2x}$$

$$= \underline{ae^{-x}} + 4be^{-2x} + 4e^{2x} \underline{- 3ae^{-x}} - 6be^{-2x} + 6e^{2x}$$

$$\underline{+ 2ae^{-x}} + 2be^{-2x} + 2e^{2x}$$

$$\therefore 12e^{2x} = 12e^{2x}$$

(1-1) * Homogeneous First ORDER Eqns :-

a differential Eqns of the Form:-

$$f(x,y)\,dx + F(x,y)\,dy = 0 \qquad \text{------} (3)$$

where: $f(x,y)$ & $F(x,y)$ are homogenous functions of x and y of the same degree called Homogeneous D.E

Such eqns Can be solved by introducing a new dependent Variable:

$$y = vx \qquad \text{---------} (4)$$

v- also Variable

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{--------} (5)$$

eqn (3) becomes:

$$v + x\frac{dv}{dx} = F(v) \qquad \text{----} (6)$$

Eqn (6) Can be solved By separting of varible:

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0 \quad \text{------- (7)}$$

Example(4):-

solve the D.E $(x+2y)\frac{dy}{dx} = 2x - y$

sol:

$$\frac{dy}{dx} = \frac{2x-y}{x+2y} \quad \text{---------- (7-a)}$$

let $y = vx \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

and put in eqn (7-a)

$$\therefore v + x\frac{dv}{dx} = \frac{2x - vx}{x + 2vx} = \frac{x(2-v)}{x(1+2v)}$$

$$\therefore x\frac{dv}{dx} = \frac{2-v}{1+2v} - v = \frac{2-v-v-2v^2}{1+2v}$$

$$x\frac{dv}{dx} = \frac{2-2v-2v^2}{1+2v}$$ "separting the varible"

$$\frac{dx}{x} = \frac{1+2v}{2-2v-2v^2}\,dv$$ Integrating

$$\int \frac{1+2v}{2-2v-2v^2} = \int \frac{1}{x}\,dx + c$$

$$-\frac{1}{2}\int \frac{-(1+2v)}{1-v-v^2}\,dv = \ln x + c$$ $\int \frac{f'}{f} \Rightarrow \ln f$

$$= -\frac{1}{2}\ln(1-v-v^2) = \ln x + c$$

$$\ln(1-v-v^2) + 2\ln x = -2c$$

let $-2c = \ln c_1$

$2\ln x = \ln x^2$

$\ln A + \ln B = \ln A * B$

$$\ln(1-v-v^2) + 2\ln x = \ln c_1$$

$$\therefore (1-v-v^2) * x^2 = c_1$$

Now $v = \frac{y}{x}$ $\therefore (1 - \frac{y}{x} - \frac{y^2}{x^2})x^2 = c$

$$\boxed{\therefore x^2 - xy - y^2 = c} \longrightarrow \text{is the solution}$$

$$\ln c^2 X^2 + \ln(1+v^2) = 2\tan^{-1} v \Rightarrow \ln c^2 X^2 (1+v^2) = 2\tan^{-1} v$$

$$\therefore \ln c^2 X^2 \left(1 + \frac{Y^2}{X^2}\right) = 2\tan^{-1}\frac{Y}{X} \quad \text{.......(11-c)} \quad v = Y/X$$

$$\ln c^2 (X^2 + Y^2) = 2\tan^{-1}\frac{Y}{X}$$

Now $X = x - h = x + \frac{1}{2}$

$Y = y - k = y + \frac{1}{2}$

substituting in

$$\ln c^2 \left\{ (x + \tfrac{1}{2})^2 + (y + \tfrac{1}{2})^2 \right\} = 2\tan^{-1}\frac{y + \frac{1}{2}}{2x+1}$$

or

$$\ln c^2 (x^2 + y^2 + x + y + \tfrac{1}{2}) = 2\tan^{-1}\frac{2y+1}{2x+1}$$

is the solution

1-3 Exact Differential Eqns :-

A First order D.E of the Form:

$$M(x, y)\,dx + N(x, y)\,dy = 0 \quad \text{.....(12)}$$

is said to be EXACT if the left-hand side is the total or Exact differential of some function

$$u = u(x, y)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{.......(13)}$$

$$du = M\,dx + N\,dy$$

comparing Eqn (12) & (13) we see that (12) is Exact, if there is some function (u) such that:

$$\frac{\partial u}{\partial x} = M \quad \text{.......(14-a)}$$

$$\frac{\partial u}{\partial y} = N \quad \text{.........(14-b)}$$

suppose that M & N are defined and have (13)
continuous first partial derivitive in a region in the x - y plane, then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \, \partial y}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \, \partial y}$$

with the assumption of continuity Thus:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{--------- (15)}$$

IF Eqn (12) be Exact, then it can be solved by Integrating as follows:-

1- Integrate M w.r. to x regarding (y = constant

$$u_1 = \int M(x, y)\, dx + k(y) \quad \text{----- (16)}$$

2- Integrate with respect to y those terms in N which do not involves x

$$u_2 = \int N(x_0, y)\, dy \quad \text{--------- (17)}$$

Note: $x_0 = 0$, exept when the terms ($\frac{1}{x}$ or $\ln x$) occur in the Integral, in that case take $x_0 = 1$

* Adding Eqns (16 & 17) for ($u_1 + u_2$)

Equating to a constant c, the solution is obtain

Example(7): solve the D.E

$$\underbrace{(3x^2+2y\sin 2x)}_{M}\,dx+\underbrace{(2\sin^2 x+3y^2)}_{N}\,dy=0$$

sol:-
Here

$$M=3x^2+2y\sin 2x$$
$$N=2\sin^2 x+3y^2$$

$\sin A.\cos B$
$=\frac{1}{2}\{\sin(A+B)-$
$\sin(A+B)$

$$\frac{\partial M}{\partial y}=2\sin 2x$$

$$\frac{\partial N}{\partial x}=2*2\sin x*\cos x=2\sin 2x$$

$$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Longrightarrow \text{the Eqn is EXACT}$$

it's solution is

$$\int(3x^2+2y\sin 2x)\,dx+\int 3y^2\,dy=C$$

$$x^3-y\cos 2x+y^3=C \Longrightarrow \underbrace{x^3+2y\sin^2 x+y^3=C}_{\text{is the sol.}}$$

Example(8):-
show that the following eqn is Exact and solve it

$$\underbrace{(x^4-2xy^2+y^4)}_{M}\,dx-\underbrace{(2x^2y-4xy^3+\sin y)}_{N}\,dy=0$$

Here
$$M=x^4-2xy^2+y^4$$
$$N=-(2x^2y-4xy^3+\sin y)$$
$$\frac{\partial M}{\partial y}=-4xy+4y^3$$
$$\frac{\partial N}{\partial x}=-4xy+4y^2$$

since $\partial M/dy=\partial N/x \Longrightarrow$ the Eqn is Exact

1- * Integrate M (w.r. to) x regarding (y = constant;

$$\int(x^4-2xy^2-y^4)\,dx=\frac{1}{5}x^5-x^2y^2+xy^4$$

2- * Now only term in N which not involve x, It's Integral is: $\int -\sin y \, dy = \cos y$

The solution of our eqn is:-

$$\frac{1}{5}x^5 - x^2y^2 + xy^4 + \cos y = C$$

1-4 Linear differential Equation:-

A (D.E) of any order in which dependent variable and it's derivatives occur only in the 1st degree and are not multiplied together is called a LINEAR D.E

* The most general form of L D.E of 1st order is

$$\frac{dy}{dx} + Ay = B \quad \text{-------- --(18)}$$

A & B → constants
→ or function of x

It is Linear in (y) and is commonly known as LeibniTz L.D.E.

* Integrating Factor (I.F):-

The expression on multiplying makes the L.H.S of Eqn (18) an Exact differential Co-efficient of some Single function of x is called the Integrating Factor (I.F)

Thus $\quad I.F = e^{\int A\,dx}$

Method of solution:- (L D.E)

1-) put the D.E in the form: $\boxed{\dfrac{dy}{dx} + Ay = B}$

2- Find H & coff. of y_n and $I.F = e^{\int A\,dx}$

3- The general solution is:

$$\boxed{y*(I.F) = \int B*(I.F)\,dx + c} \quad \ldots\ldots(19)$$

Example (9):- solve $\frac{dy}{dx} + y = e^{x}$

Sol:-

$A = 1 \quad , \quad B = e^{x}$

$I.F = e^{\int A\,dx} = e^{\int 1\,dx} = e^{x}$

$\therefore \; y*e^{x} = \int e^{x}*e^{x}\,dx + c \Rightarrow y*e^{x} = \int e^{2x}\,dx + c$

$= \frac{1}{2}e^{2x} + c$

$\div e^{x}$

$$\boxed{\therefore \; y = \frac{1}{2}e^{x} + c\,e^{-x}}$$

Example (10):- solve $x\,y' + y + 4 = 0$

Sol:-

write the eqn in the Form of (18):-

$$\frac{dy}{dx} + \frac{1}{x}\,y = -\frac{4}{x} \qquad (\div x)$$

Hence

$A = \frac{1}{x} \quad , \quad B = -\frac{4}{x}$

therefore

$I.F = e^{\int A\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

$y*(I.F) = \int B*(I.F)\,dx + c$

$y*x = \int x*(-\frac{4}{x})\,dx + c$

$y\,x = -4x + c \Longrightarrow \boxed{y = \frac{c}{x} - 4}$

is the solution

Example (11) :- solve

$$x^2 \frac{dy}{dx} = 3x^2 - 2xy + 1$$

Sol:-

Rewrite the eqn as:

$$x^2 \frac{dy}{dx} + 2xy = 3x^2 + 1$$

$(\div x^2)$ to get the Form of (18)

$$\therefore \frac{dy}{dx} + \frac{2}{x} y = 3 + \frac{1}{x^2} \Rightarrow \text{is linear in } y$$

$$A = \frac{2}{x} \quad , \quad B = 3 + \frac{1}{x^2}$$

$$I.F = e^{\int A\,dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$y * (I.F) = \int B * (I.F)\,dx$$

$$y * x^2 = \int x^2 \left(3 + \frac{1}{x^2}\right) dx + c$$

$$y * x^2 = \int (3x^2 + 1)\,dx + c = \frac{3x^3}{3} + x + c$$

$$\boxed{\therefore y x^2 = x^3 + x + c} \text{ is the solution}$$

* The Initial Value problems *

— The procedure of solving Linear (D.E) can be also used to solve the Initial Value problems. Let us illustrate that with following Example.

Example (12) :- solve the Initial value problem

$$\frac{dy}{dx} + y \tan x = \sin 2x \qquad y(0) = 1$$

$$A = \tan x \quad , \quad B = \sin 2x$$

$$\text{or} \rightsquigarrow B = 2 \sin x \cos x$$

$$I.F = e^{\int A\,dx} = e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x$$

$$y * (I.F) = \int B * (I.F)\,dx + c$$

$$y \cdot \sec x = \int \left[(2 \sin x \cos x) * \sec x + c \right.$$

$$y = \frac{1}{\sec x}\left\{ \int \left[(2 \sin x \cdot \cos x)\frac{1}{\cos x} \right] + c \right\} = \cos x \left[\int 2 \sin x\, dx + c \right.$$

$$\boxed{\therefore y = C \cos x - 2 \cos^2 x} \dashrightarrow (19-a)$$

with Initial condiction $y = 1$ when $x = 0$
put in Eqn (19-a)

$$1 = C - 2 \implies C = 1 + 2 = 3$$

and the solution of the Initial Value is $\boxed{y = 3 \cos x - 2 \cos^2 x}$

Example (13):-

solve the Initial value problem

$$y' + y = \sin x \qquad y(0) = 2$$

sol:-

$$\frac{dy}{dx} + y = \sin x$$

$$A = 1 \quad , \quad B = \sin x$$

$$I.F = e^{\int A\,dx} = e^{\int 1\,dx} = e^{x}$$

$$y * e^{x} = \int \underbrace{e^{x}}_{dv} \underbrace{\sin x}_{u}\,dx + C \dashrightarrow (19-b) \qquad \int v\,du = uv - \int v\,du$$

$$\therefore \int e^{x} \sin x\,dx = \sin x \cdot e^{x} - \int e^{x} \cos x\,dx$$

Integrating by Parts again

$$= e^{x} \cdot \sin x - \left[e^{x} \cdot \cos x - \int e^{x} (-\sin x)\,dx \right.$$

$$\therefore \int e^{x} \sin x\,dx = e^{x} \sin x - e^{x} \cos x - \int e^{x} \sin x\,dx$$

or

$$2 \int e^{x} \sin x\,dx = e^{x} (\sin x - \cos x)$$

$$\therefore \int e^{x} \sin x\,dx = \frac{1}{2} e^{x} (\sin x - \cos x) \dashrightarrow (19-c)$$

I reaches 90% Imax $= \frac{90}{100} \times 2 = \frac{9}{5}$

$\therefore \frac{9}{5} = 2(1 - e^{-25t/64}) \Rightarrow \frac{9}{10} = (1 - e^{-25t/64})$

$\therefore e^{-25t/64} = 1 - \frac{9}{10} = \frac{1}{10}$

$\therefore \frac{-25t}{64} = \ln\frac{1}{10} \Rightarrow \frac{-25t}{64} = -2.3$

$\therefore t = \frac{-2.3 \times 64}{-25} = 5.89$ sec

2. SECOND ORDER DIFFERENTIAL Eqns:-

(2-1) Eqns of 2nd ORDER reducible To 1st order:

Certain Types of 2nd-order D.E of which the General Form is:-

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0 \quad \text{---(21)}$$

Can be reduced to the 1st-order by substituting

$$P = \frac{dy}{dx} \quad , \quad \frac{d^2y}{dx^2} = \frac{dP}{dx}$$

Then Eqn (21) takes the Form:

$$F(x, P, \frac{dP}{dx}) = 0 \quad \text{---(22)}$$

Eqn (22) Can be solved For P, as the 1st order in P

Ex 16:- Solve the D.E

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = 0$$

Let $P = \frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{dP}{dx}$

Now the Eqn is:-

$$x\frac{dP}{dx} + P = -x \qquad \div x$$

$$\text{or} \Rightarrow \frac{dP}{dx} + \frac{1}{x}P = -1$$

It is a Linear Equation:-

$A = \frac{1}{x}$, $I.F = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$, $B = -1$

$\therefore P * x = \int -1 * x \, dx + c \Rightarrow P * x = \frac{-x^2}{2} + c$

$\Rightarrow P = -\frac{x}{2} + \frac{c}{x}$

or $\frac{dy}{dx} = \frac{c}{x} - \frac{x}{2} \Rightarrow dy = \left(\frac{c}{x} - \frac{x}{2}\right) dx$

Integrate Both sides :-

$\therefore y = c \ln x - \frac{x^2}{4} + c_1$ $\quad$ c & c_1 constant

Example (17) :- solve the D.E $y \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

sol:- $P = \frac{dy}{dx} \Rightarrow \frac{dP}{dx} = \frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2} = \frac{dP}{dy}\left(\frac{dy}{dx}\right) = P \frac{dP}{dy}$

$\therefore y \cdot P \frac{dP}{dy} = P^2$ $(\div P)$ and separte Var.

$\therefore \frac{dP}{P} = \frac{dy}{y}$ $\quad$ Integrate Both sides

$\Rightarrow \int \frac{1}{P} dP = \int \frac{1}{y} dy + c \Rightarrow \ln P = \ln y + c_1$

$\therefore \ln P = \ln y + \ln c$ $\quad$ let $c_1 = \ln c$

$\ln P = \ln cy \Rightarrow P = cy$

i.e $\frac{dy}{dx} = cy \Rightarrow \int \frac{1}{cy} dy = dx$

$\therefore$ the solution is $\boxed{\frac{1}{c} \ln y = x + c_2}$

(2-2) Homogenous 2nd-Order D.E with constant Coeff.

The General Form is:

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0 \quad \text{-----(23)}$$

a & b constant

In operator notation Eqn (23) becomes:-

$$(D^2 + 2aD + b)y = 0 \quad \text{......(24)}$$

The characteristic Eqn of the D.E, we get it by replacing D By r

$$r^2 + 2ar + b = 0 \quad \text{--------(25)}$$

Suppose the roots of Eqn (25) are r_1 & r_2 then:

$$r^2 + 2ar + b = (r - r_1)(r - r_2)$$

$$\therefore D^2 - 2aD + b = (D - r_1)(D - r_2)$$

Hence Eqn (24) is Equivalent to:-

$$(D - r_1)(D - r_2)y = 0 \quad \text{-------(26)}$$

Let

$$(D - r_2)y = u \quad \text{------(27)}$$

$$\therefore \quad (D - r_1)u = 0 \quad (28)$$

Therefore we Can Solve Eqn (25). From Eqn (28) we Find

$$u = c_1 e^{r_1 x}$$

substitute u in Eqn (27)

$$\therefore (D - r_2)y = c_1 e^{r_1 x}$$

$$\frac{dy}{dx} - r_2 = c_1 e^{r_1 x}$$

This Eqn is Linear, its Integrating Factor (I.F) is

$$A = e^{-r_2 x}$$

and its solution is: $\boxed{e^{-r_2 x} * y = c_1 \int e^{(r_1 - r_2)x} dx + c_2}$

Case I :- IF $r_1 \neq r_2$ (26)

The solution of Eqn (26) is :-

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad r_1 \neq r_2 \quad \text{(29)}$$

Cas II : $r_1 = r_2$

The Solution Can Be Simply as:

$$y = (C_1 x + C_2) e^{r x} \quad r_1 = r_2 \quad \text{(30)}$$

Example(18) :- solve the D.E $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Sol: The chrs Eqn is:

$$r^2 + r - 2 = 0$$

$$(r-1)(r+2) = 0$$

The roots are : $r_1 = 1$, $r_2 = -2$ $\quad r_1 \neq r_2$

$\therefore$ The solution of D.E is:

$$y = C_1 e^{x} + C_2 e^{-2x}$$

Example(19): solve $2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 0$

Sol :- The chr. Eqn is :-

$$2r^2 - 7r + 6 = 0$$

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore r = \frac{-(-7) \mp \sqrt{49 - 4 \times 2 \times 6}}{2 \times 2} = \frac{+7 \mp 1}{4}$$

$$\longrightarrow \frac{7+1}{4} = \frac{8}{4} = 2$$

$$\longrightarrow \frac{7-1}{4} = 6/4 = \frac{3}{2}$$

$\therefore r_1 = 2$, $r_2 = \frac{3}{2}$

$\therefore$ The solution of D.E is:

$$y = C_1 e^{2x} + C_2 e^{3/2 x}$$

Example(20):- Solve the D.E $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+y=0$

Sol:- The Chr Eqn is

$$r^2+4r+4=0$$

$$\therefore r=\frac{-4\mp\sqrt{16-4\times1\times4}}{2\times1}=\frac{-4\mp\sqrt{16-16}}{2}=\frac{-4\mp0}{2}=-2,-2$$

$$\therefore r_1=r_2=-2$$

$\therefore$ the solution of D.E is:/ $y=(C_1+C_2x)e^{-2x}$ /

* Imaginary Roots of D.E *

IF the coefficients «a & b» in Eqn «23» are real, The Roots of the chracacteristic Eqn(25) will Either be real, or will be a pair of Complex Conjugate number

$$\left.\begin{array}{l} r_1=\alpha+i\beta \\ r_2=\alpha-i\beta \end{array}\right\} \quad \ldots\ldots\ldots\ldots (31)$$

IF $\beta\neq0$ then eqn(29) applies with result:

$$y=C_1e^{(\alpha+i\beta)x}+C_2e^{(\alpha-i\beta)x}$$

$$=e^{\alpha x}\left[C_1e^{i\beta x}+C_2e^{-i\beta x}\right] \quad \ldots\ldots(32)$$

Now

$$e^{i\beta x}=\cos\beta x+i\sin\beta x$$

$$e^{-i\beta x}=\cos\beta x-i\sin\beta x$$

Hence Eqn «32» may Be replaced By:

$$y=e^{\alpha x}\left[(c_1+c_2)\cos\beta x+i(c_1-c_2)\sin\beta x\right] \quad \ldots\ldots(33)$$

Let $C_1=c_1+c_2$ and $C_2=i(c_1-c_2)$

Eqn (33) takes The Form:-

$$y = e^{\alpha x}\left[C_1 \cos\beta x + C_2 \sin\beta x\right] \quad \text{......(34)}$$

(28)

Example(21):- solve $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$

sol:- we have the chr. Eqn as:-

$$(2r^2 + 3r + 4) = 0 \qquad \boxed{r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}}$$

$$r = \frac{-3 \mp \sqrt{9 - 4\times2\times4}}{2\times2} = \frac{-3 \mp \sqrt{-23}}{4} = \frac{1}{4}\left[-3 \mp i\sqrt{23}\right]$$

$$\therefore r_1 = \frac{-3}{4} + i\frac{\sqrt{23}}{4} \qquad r_2 = \frac{-3}{4} - \frac{i\sqrt{23}}{4}$$

$$\therefore \alpha = \frac{-3}{4} \quad \& \quad \beta = \frac{\sqrt{23}}{4}$$

therefore the General solution By Eqn «34» is:-

$$y = e^{-\frac{3}{4}x}\left[C_1 \cos\frac{\sqrt{23}}{4}x + C_2 \sin\frac{\sqrt{23}}{4}x\right]$$

Example(22):- Solve the D.E $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = 0$

sol:- The chrs Eqn is:-

$$(r^2 - 4r + 8) = 0$$

$$r = \frac{4 \mp \sqrt{(4)^2 - 4\times1\times8}}{2\times1} = \frac{4 \mp \sqrt{16 - 32}}{2} = \frac{4 \mp 4i}{2} = 2 \mp 2i$$

$$\therefore r_1 = 2 + 2i \qquad r_2 = 2 - 2i$$

$$\therefore \alpha = 2 \quad \& \quad \beta = 2$$

$\therefore$ The solution is:

$$y = e^{2t}\left[C_1 \cos 2t + C_2 \sin 2t\right]$$

$$0 = C_2 e^0 + [C_1 + 0] e^0 + \left(\frac{-R}{2L}\right) = C_2 - \frac{C_1 R}{2L}$$

$$\therefore C_2 = \frac{C_1 R}{2L} \Longrightarrow C_2 = \frac{q_0 R}{2L}$$ since $q_0 = C_1$

Hence $q = \left[q_0 + \frac{q_0 R}{2L}\right] e^{-Rt/2L}$ is the solution

2-3 NON-HOMOGENEOUS 2nd order D.E with const. Coff.

In Eqn (23) we see the Homogeneous eqn:-
Now the Non-Homogeneous Equation is:-

$$\frac{d^2y}{dx^2} + 2a \frac{dy}{dx} + by = F(x) \quad \text{------}(35)$$

To solve Eqn (35), First we obtain the General solution of the related Homogeneous Eqn (23), By replacing $\underline{|F(x) = 0|}$

$$y_h = C_1 e^{r_1 x}(x) + C_2 e^{r_2 x}(x) \qquad [C_1 \& C_2] \text{ const.}$$

The particular function:

$$y = y_p(x) \quad \text{--------}(36)$$

Eqn (36) which satisfies Eqn (35) so we can obtain the complete solution of Eqn (35) as

$$y = y_h(x) + y_p(x) \quad \text{------}(37)$$

Example (26):- solve the D.E

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6 \quad \text{-----}(1)$$

Sol:

First we solve y_h which satisfies:-

$$\frac{d^2y_h}{dx^2} + 2\frac{dy_h}{dx} - 3y_h = 0$$

The chrs Eqn is:

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1)=0 \Rightarrow \begin{matrix} r_1=-3 \\ r_2=1 \end{matrix}$$

$$\therefore y_h = C_1 e^{-3x} + C_2 e^{x}$$

$$y = y_h + y_p \Rightarrow y = C_1 e^{-3x} + C_2 e^{x} + y_p$$

* Now y_p equal to constant, provided:

$$-3y = 6 \quad \text{Hence } y_p = -2$$

$\therefore$ The complete solution is:

$$\boxed{y = C_1 e^{-3x} + C_2 e^{x} - 2}$$

N.B :- Since y_p = constant $\Rightarrow y'_p = 0$, $y''_p = 0$

i.e By substitute for y'_p and y''_p in Eqn(1)

$$1\, y''_p + 2 y'_p - 3 y_p = 6$$

$$0 + 0 - 3 y_p = 6 \Rightarrow y_p = \frac{-6}{3} = -2$$

Example(27) :- Solve the D.E

$$d^2y/dx^2 - 4\frac{dy}{dx} + 3y = 10e^{-2x} \quad \text{---}(1)$$

Sol :-

First we solve for y_h :-

$$d^2y/dx^2 - 4\frac{dy}{dx} + 3y = 0 \quad \text{---}(2)$$

The chrs Eqn

$$(r^2 - 4r + 3) = 0 \Rightarrow (r-3)(r-1) = 0$$

$$\therefore r = 3 \quad \text{and} \quad r = 1$$

$$y_h = C_1 e^{x} + C_2 e^{3x}$$

$$y = y_h + y_p \quad = C_1 e^{x} + C_2 e^{3x} + y_p$$

* The derivative of e^{x} is [constant $\times e^{x}$]

i.e $y_p = K e^{-2x} \Rightarrow y'_p = -2Ke^{-2x}$

differentiation of $y_p \Rightarrow y''_p = (-2x-2)Ke^{-2x} = 4Ke^{-2x}$

* substitution of y_p, y'_p, y''_p into Eqn (1): (34)

$$4ke^{-2x} - 4(-2ke^{-2x}) + 3ke^{-2x} = 10e^{-2x}$$

By Equating the coefficients of e^{-2x}

Hence:-

$$4k + 8k + 3k = 10 \Longrightarrow \therefore k = \frac{10}{15} = \frac{2}{3}$$

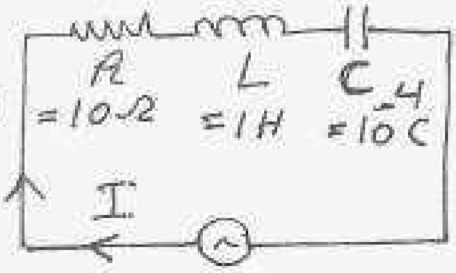
$$\therefore y_p = \frac{2}{3} e^{-2x}$$

* The complete solution is:

$$\boxed{y = c_1 e^{x} + c_2 e^{3x} + \frac{2}{3} e^{-2x}}$$

Exampl (28):-

consider the RLC connected in series to an e.m.f. E. Find the D.E Expressing the charge q at any time t.



E = 100 V

Initially at $t = 0$, no charge and current is following in the cct.

Sol:-

Let q = charge at any time t

* The condenser charge Varies as the potential acros it:

i.e $$q = CE \quad \text{-----} (1)$$

* The current is the rate of Electric charge:

$$\therefore \quad I = \frac{dq}{dt} \quad \text{-----} (2)$$

* The potential drop due to R, L, & C are:

$$RI \;,\; L\frac{dI}{dt} \;,\; \frac{q}{C}$$

By Kirchoff's voltage Low: