COMPLEX NUMBERS

1. SUMS AND PRODUCTS

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the *complex plane*, with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line. When real numbers x are displayed as points $(x, 0)$ on the *real axis*, it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called *pure imaginary numbers*. The y axis is, then, referred to as the *imaginary axis*.

It is customary to denote a complex number (x, y) by z, so that

(1) $$z = (x, y).$$

The real numbers x and y are, moreover, known as the *real and imaginary parts* of z, respectively; and we write

(2) $$\operatorname{Re} z = x, \quad \operatorname{Im} z = y.$$

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal whenever they have the same real parts and the same imaginary parts. Thus the statement $z_1 = z_2$ means that z_1 and z_2 correspond to the same point in the complex, or z, plane.

The *sum* $z_1 + z_2$ and the *product* z_1z_2 of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows:

(3) $$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

(4) $$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2).$$

Note that the operations defined by equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$
$$(x_1, 0)(x_2, 0) = (x_1x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence

$$z = (x, 0) + (0, 1)(y, 0);$$

and, if we think of a real number as either x or $(x, 0)$ and *let i denote the imaginary number* $(0, 1)$ (see Fig. 1), it is clear that*

(5) $$z = x + iy.$$

Also, with the convention $z^2 = zz$, $z^3 = zz^2$, etc., we find that

$$i^2 = (0, 1)(0, 1) = (-1, 0),$$

or

(6) $$i^2 = -1.$$

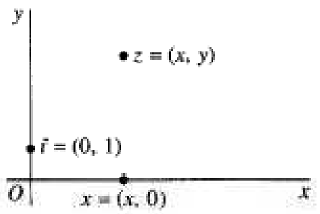


FIGURE 1

In view of expression (5), definitions (3) and (4) become

(7) $$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

(8) $$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).$$

2. BASIC ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

The commutative laws

(1) $$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1$$

and the associative laws

(2) $$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

follow easily from the definitions in Sec. 1 of addition and multiplication of complex numbers and the fact that real numbers obey these laws. For example, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1.$$

Verification of the rest of the above laws, as well as the distributive law

(3) $$z(z_1 + z_2) = zz_1 + zz_2,$$

is similar.

According to the commutative law for multiplication, $iy = yi$. Hence one can write $z = x + yi$ instead of $z = x + iy$. Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1 z_2 z_3$ is well defined without parentheses, as is the case with real numbers.

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

(4) $$z + 0 = z \quad \text{and} \quad z \cdot 1 = z$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 9).

There is associated with each complex number $z = (x, y)$ an additive inverse

(5) $$-z = (-x, -y),$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z, since the equation $(x, y) + (u, v) = (0, 0)$ implies that $u = -x$ and $v = -y$. Expression (5) can also be written $-z = -x - iy$ without ambiguity since

MODULI

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z (Sec. 1) in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

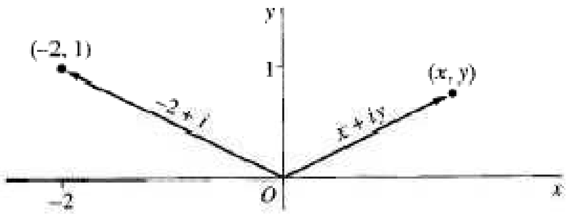


FIGURE 2

According to the definition of the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the number $z_1 + z_2$ corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 3. The difference $z_1 - z_2 = z_1 + (-z_2)$ corresponds to the sum of the vectors for z_1 and $-z_2$ (Fig. 4).

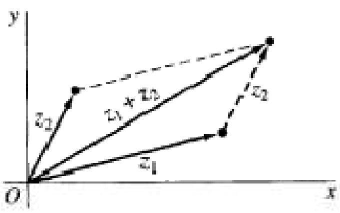


FIGURE 3

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The *modulus*, or absolute value, of a complex number $z = x + iy$ is defined as the nonnegative real

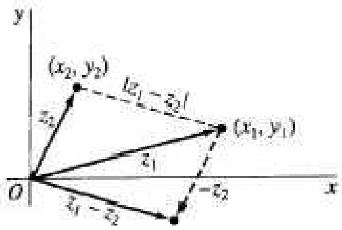


FIGURE 4

number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

(1) $$|z| = \sqrt{x^2 + y^2}.$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. Note that, while *the inequality* $z_1 < z_2$ *is meaningless unless both* z_1 *and* z_2 *are real,* the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

EXAMPLE 1. Since $|-3 + 2i| = \sqrt{13}$ and $|1 + 4i| = \sqrt{17}$, the point $-3 + 2i$ is closer to the origin than $1 + 4i$ is.

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is $|z_1 - z_2|$. This is clear from Fig. 4, since $|z_1 - z_2|$ is the length of the vector representing $z_1 - z_2$; and, by translating the radius vector $z_1 - z_2$, one can interpret $z_1 - z_2$ as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1). Alternatively, it follows from the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

and definition (1) that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of points simply as the circle $|z - z_0| = R$.

EXAMPLE 2. The equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

It also follows from definition (1) that the real numbers $|z|$, $\text{Re } z = x$, and $\text{Im } z = y$ are related by the equation

(2) $$|z|^2 = (\text{Re } z)^2 + (\text{Im } z)^2.$$

Thus

(3) $$\text{Re}\, z \leq |\text{Re}\, z| \leq |z| \quad \text{and} \quad \text{Im}\, z \leq |\text{Im}\, z| \leq |z|.$$

We turn now to the *triangle inequality*, which provides an upper bound for the modulus of the sum of two complex numbers z_1 and z_2:

(4) $$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This important inequality is geometrically evident in Fig. 3, since it is merely a statement that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. We can also see from Fig. 3 that inequality (4) is actually an equality when 0, z_1, and z_2 are collinear. Another, strictly algebraic, derivation is given in Exercise 16, Sec. 5.

An immediate consequence of the triangle inequality is the fact that

(5) $$|z_1 + z_2| \geq ||z_1| - |z_2||.$$

To derive inequality (5), we write

$$|z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2|,$$

which means that

(6) $$|z_1 + z_2| \geq |z_1| - |z_2|.$$

This is inequality (5) when $|z_1| \geq |z_2|$. If $|z_1| < |z_2|$, we need only interchange z_1 and z_2 in inequality (6) to get

$$|z_1 + z_2| \geq -(|z_1| - |z_2|),$$

which is the desired result. Inequality (5) tells us, of course, that the length of one side of a triangle is greater than or equal to the difference of the lengths of the other two sides.

Because $|-z_2| = |z_2|$, one can replace z_2 by $-z_2$ in inequalities (4) and (5) to summarize these results in a particularly useful form:

(7) $$|z_1 \pm z_2| \leq |z_1| + |z_2|,$$

(8) $$|z_1 \pm z_2| \geq ||z_1| - |z_2||.$$

EXAMPLE 3. If a point z lies on the unit circle $|z| = 1$ about the origin, then

$$|z - 2| \leq |z| + 2 = 3$$

and

$$|z - 2| \geq ||z| - 2| = 1.$$

The triangle inequality (4) can be generalized by means of mathematical induction to sums involving any finite number of terms:

(9) $$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n| \qquad (n = 2, 3, \ldots).$$

To give details of the induction proof here, we note that when $n = 2$, inequality (9) is just inequality (4). Furthermore, if inequality (9) is assumed to be valid when $n = m$, it must also hold when $n = m + 1$ since, by inequality (4),

$$\begin{aligned} |(z_1 + z_2 + \cdots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \cdots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \cdots + |z_m|) + |z_{m+1}|. \end{aligned}$$

COMPLEX CONJUGATES

The *complex conjugate*, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\overline{z}$; that is,

(1) $$\overline{z} = x - iy.$$

The number $\overline{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 5). Note that

$$\overline{\overline{z}} = z \quad \text{and} \quad |\overline{z}| = |z|$$

for all z.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

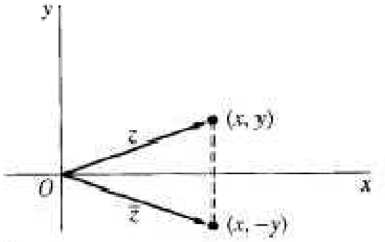


FIGURE 5

So the conjugate of the sum is the sum of the conjugates:

(2) $$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}.$$

In like manner, it is easy to show that

(3) $$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2},$$

(4) $$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2},$$

and

(5) $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0).$$

The sum $z + \overline{z}$ of a complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$ is the real number $2x$, and the difference $z - \overline{z}$ is the pure imaginary number $2iy$. Hence

(6) $$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

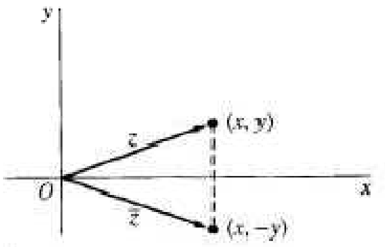
(7) $$z\overline{z} = |z|^2,$$

where each side is equal to $x^2 + y^2$. It suggests the method for determining a quotient z_1/z_2 that begins with expression (3), Sec. 3. That method is, of course, based on multiplying both the numerator and the denominator of z_1/z_2 by $\overline{z_2}$, so that the denominator becomes the real number $|z_2|^2$.

EXAMPLE 1. As an illustration,

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1+i.$$

See also the example near the end of Sec. 3.



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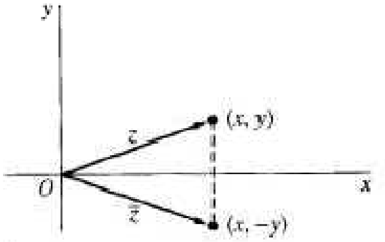

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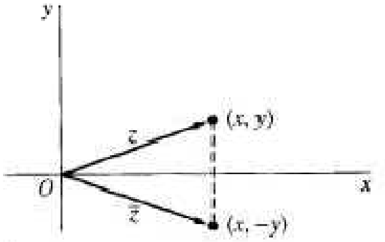

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Identity (7) is especially useful in obtaining properties of moduli from properties of conjugates noted above. We mention that

(8) $$|z_1 z_2| = |z_1||z_2|$$

and

(9) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0).$$

Property (8) can be established by writing

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\overline{z_1}\,\overline{z_2}) = (z_1 \overline{z_1})(z_2 \overline{z_2}) = |z_1|^2 |z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (9) can be verified in a similar way.

EXAMPLE 2. Property (8) tells us that $|z^2| = |z|^2$ and $|z^3| = |z|^3$. Hence if z is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, it follows from the generalized form (9) of the triangle inequality in Sec. 4 that

$$|z^3 + 3z^2 - 2z + 1| \le |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

EXPONENTIAL FORM

Let r and θ be polar coordinates of the point (x, y) that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in *polar form* as

$$(1) \qquad z = r(\cos\theta + i\sin\theta).$$

If $z = 0$, the coordinate θ is undefined; and so it is always understood that $z \neq 0$ whenever $\arg z$ is discussed.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an *argument* of z, and the set of all such values is denoted by $\arg z$. The *principal value* of $\arg z$, denoted by $\text{Arg}\, z$, is that unique value Θ such that $-\pi < \Theta \leq \pi$. Note that

$$(2) \qquad \arg z = \text{Arg}\, z + 2n\pi \qquad (n = 0, \pm1, \pm2, \ldots).$$

Also, when z is a negative real number, $\text{Arg}\, z$ has value π, not $-\pi$.

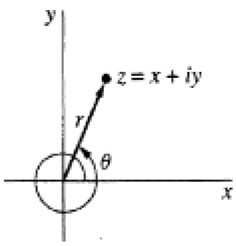

FIGURE 6

EXAMPLE 1. The complex number $-1 - i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$\text{Arg}(-1 - i) = -\frac{3\pi}{4}.$$

It must be emphasized that, because of the restriction $-\pi < \Theta \leq \pi$ of the principal argument Θ, it is *not* true that $\text{Arg}(-1 - i) = 5\pi/4$.

According to equation (2),

$$\arg(-1 - i) = -\frac{3\pi}{4} + 2n\pi \qquad (n = 0, \pm1, \pm2, \ldots).$$

Note that the term Arg z on the right-hand side of equation (2) can be replaced by any particular value of arg z and that one can write, for instance,

$$\arg(-1-i) = \frac{5\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The symbol $e^{i\theta}$, or $\exp(i\theta)$, is defined by means of *Euler's formula* as

$$(3) \qquad e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ is to be measured in radians. It enables us to write the polar form (1) more compactly in *exponential form* as

$$(4) \qquad z = re^{i\theta}.$$

The choice of the symbol $e^{i\theta}$ will be fully motivated later on in Sec. 28. Its use in Sec. 7 will, however, suggest that it is a natural choice.

EXAMPLE 2. The number $-1-i$ in Example 1 has exponential form

$$(5) \qquad -1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}\right)\right].$$

With the agreement that $e^{-i\theta} = e^{i(-\theta)}$, this can also be written $-1-i = \sqrt{2}e^{-i3\pi/4}$. Expression (5) is, of course, only one of an infinite number of possibilities for the exponential form of $-1-i$:

$$(6) \qquad -1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4} + 2n\pi\right)\right] \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Note how expression (4) with $r = 1$ tells us that the numbers $e^{i\theta}$ lie on the circle centered at the origin with radius unity, as shown in Fig. 7. Values of $e^{i\theta}$ are, then, immediate from that figure, without reference to Euler's formula. It is, for instance,

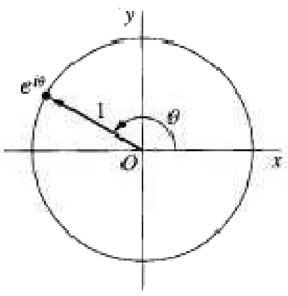

FIGURE 7

geometrically obvious that

$$e^{i\pi} = -1, \quad e^{-i\pi/2} = -i, \quad \text{and} \quad e^{-i4\pi} = 1.$$

Note, too, that the equation

$$(7) \qquad z = Re^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

is a parametric representation of the circle $|z| = R$, centered at the origin with radius R. As the parameter θ increases from $\theta = 0$ to $\theta = 2\pi$, the point z starts from the positive real axis and traverses the circle once in the counterclockwise direction. More generally, the circle $|z - z_0| = R$, whose center is z_0 and whose radius is R, has the parametric representation

$$(8) \qquad z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi).$$

This can be seen vectorially (Fig. 8) by noting that a point z traversing the circle $|z - z_0| = R$ once in the counterclockwise direction corresponds to the sum of the fixed vector z_0 and a vector of length R whose angle of inclination θ varies from $\theta = 0$ to $\theta = 2\pi$.

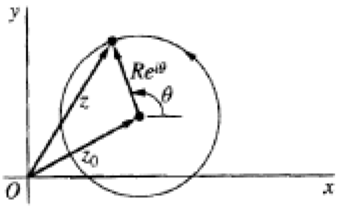

FIGURE 8

7. PRODUCTS AND QUOTIENTS IN EXPONENTIAL FORM

Simple trigonometry tells us that $e^{i\theta}$ has the familiar additive property of the exponential function in calculus:

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}. \end{aligned}$$

Thus, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, the product z_1z_2 has exponential form

$$(1) \qquad z_1z_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}.$$