Question bunk of Theory of differential equations

Thirds stage

Q1/ let matrix Abe of the form

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad a_1, a_{2,} a_3, a_4 \in R$$

Such that it two identical eigenvalues $\lambda_1 = \lambda_2 = \lambda$

a) Prove that if $a_2 \neq 0, a_3 \neq 0$ then it is always possible to put A in the form

$$A' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

By applying similarity transformations.

b) Assume that $a_1 = 1, a_2 = 1, a_3 = -1, a_4 = 3.$ Solve the linear system

c) Sketch the phase diagram for the dynamical system in (b)

Q2/ Consider the dynamical system

$$\begin{aligned} x^{\cdot} &= x + y^2 \\ y^{\cdot} &= x^3 + y \end{aligned} \qquad (1)$$

(a) determine all fixed point of the system (1)

(b) state the linearization theorem and decide for each fixed point whether it can used to draw conclusions from it with regard to the stability of the fixed point.

(c) Draw the phase portrait for the system (1).

Q3/ show that the two systems (

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2) \\ \dot{x} &= x + y(x^2 + y^2) \end{aligned} \qquad (2)$$

And

$$\begin{aligned} \dot{x} &= -y - x(x^2 + y^2) \\ \dot{x} &= x - y(x^2 + y^2) \end{aligned} \qquad (3)$$

Both have some linearized system at the origins but their phase portraits are different.

Q4/ consider the system (15 degree)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (3)$$

I. Calculate the fixed point of the (3)
II. Determine stability for (3)
III. Sketch the phase portrait for (3)

Q5/ from the system

$$\dot{x} = y$$
$$\dot{y} = -x + x^2 \qquad (4)$$

a) Find first integral
b) Sketch the phase portrait of the system (4)

Q6/ the Lotka- Volterra Predator Prey model describes a simple ecological model between two species:

$$\dot{x} = ax - xy$$
$$\dot{y} = xy - by$$

Where $a, b > 0$ are interaction parameters and $x, y \geq 0$ are variables describing the amount of predators and preys on an island

i) Find the fixed points
ii) Determine the linear stability of the fixed point at the origin
iii) Determine the linear stability of the second fixed point and find the roots $\lambda_{1,2}$. What type is the second fixed point according to linear stability analysis.

Q7/ Consider the coupled system of differential equations:

$$\dot{x} = y$$
$$\dot{y} = -x - \epsilon y^3(1 + x^2), \quad \epsilon > 0$$

1) Find the fixed point of the system . determine the eigenvalues of the fixed point
2) What can you conclude about the stability of the fixed point from these eigenvalues
3) Show that $\gamma(T) \to 0$ as $T \to \infty$. What does that tell us about the fixed point.

Q8/ Consider the coupled system of differential equations:

$$\dot{x} = ax + y - x\sqrt{x^2 + y^2}$$
$$\dot{y} = -x + ay - y\sqrt{x^2 + y^2}$$

a) Multiply $\dot{x}$ *by* y *and* $\dot{y}$ *by* x and subtract to show that (0,0) is the only fixed point
b) Determine the Jacobian in (0,0) and determine the eigenvalues of the fixed point
c) What can you conclude about the stability of the fixed point for $a > 0, a = 0, a < 0$ from these eigenvalues

Q9/ consider the two dimensional differential equation system

$$\dot{x} = ax + y + x^3$$
$$\dot{y} = x - y$$

a) Find the fixed point as a function of a
b) Find the Jacobian matrix
c) Determine the eigenvalues of the fixed points as a function of a
d) Evaluate the stability of the fixed points in $a = -2$ and the stability for the trivial fixed point in $a = 2$

Q10/ consider the two dimensional differential equation system

$$\dot{x} = -x + y + xy$$
$$\dot{y} = x - y - x^2 - y^3$$

1) The system has one fixed point (x^*, y^*), find it
2) Derive the Jacobian matrix
3) Determine the eigenvalues and corresponding eigenvectors for(x^*, y^*),
4) What can you say about the stability of (x^*, y^*),

Q11/ consider the predator prey system

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x - a) \qquad 0 < a < 1$$

1)find the fixed point

2) discuss the stability of the fixed points where one or two of the species are extinct

3) show that the fixed point $x^* > 0,\ y^* > 0$ is unstable for $0 < a < a_c$

Q12/ a protein x can repress itself by binding at its promotor at DNA and produce the associated mRNA y leading to the coupled system of differential equation:

$$\dot{x} = y - x$$
$$\dot{y} = \frac{a}{1+x} - y$$

Where a is a real parameter

1) Find the two fixed points as a function of a
2) Derive the Jacobian matrix as a function of a
3) Show that the eigenvalues of the fixed points can be written as $\lambda_+ = b \pm f_+(a)$ and $\lambda_- = b \pm f_-(a)$ and determine the constant b and the functions $f_+(a), f_-(a)$

Q13/ A two- dimensional differential equation system is defined by:

$$\dot{x} = x(1 - (ax + y^2)$$
$$\dot{y} = y(1 - (x + y)$$

Where $a > 0$ is real positive parameter

1) Now we consider $(x \geq 0, y \geq 0)$. Find the fixed point
2) Derive the Jacobian matrix
3) In the following we consider $a = \frac{3}{2}$. Find the eigenvalues of the fixed points and classify the fixed points
4) Sketch the flow around the fixed points in $(x \geq 0, y \geq 0)$.

Q14/ a two-dimensional differential equation system is defined by

$$\dot{x} = -2x - ay$$
$$\dot{y} = -ax - 2by$$

Where a, b are real parameters and $b > 0$

1) Determine the Jacobian of system and derive the eigenvalues of the fixed point as a function of a and b.
2) For $b = 1$ describe the nature of the fixed point for all values of a
3) For $a = \sqrt{7}$ *and* $b = 4$ find the eigenvalues and eigenvectors and sketch the flow
4) For $a = 4$ *and* $b = 1$ find the eigenvalues and eigenvectors and sketch the flow

Q15/ consider the nonlinear dynamic system

$$\dot{x} = ax - bxy$$
$$\dot{y} = -cx - dxy$$

Where a, b, c, d are positive constant

1) Show that (0,0) and $(c/d, -a/b)$ are the only equilibria of the system and investigate their linear stability
2) Let

$$H(x, y) = dx - clnx + by - alny + k$$

Where k is constant. verify that $\frac{dH}{dt} = 0$

Along solution of the dynamic system.

Q16/ calculate the fixed point of the cat map

$$f(x, y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} mod\ 1$$

And determine their stability

Q17/ consider the dynamic system defined by the equations

$$\frac{dx}{dt} = -x^3 + xy$$
$$\frac{dy}{dt} = -y^3 - ax^2$$

Where a is a real number

1) Find the equilibrium points of the dynamical system as functions of the parameter a . in particular, show that $a > 0$ $(x, y) = (0,0)$ is the only equilibrium of the dynamical system
2) Investigate the linear stability of all equilibria. Comment on the linear stability of the equilibrium $(x, y) = (0,0)$
3) Show that the equilibrium $(x, y) = (0,0)$ is asymptotically stable for all $a > 0$

Q18/ consider the linear dynamical system of the form

$$\dot{x_1} = ax_1 + bx_2$$
$$\dot{x_2} = cx_1 + dx_2$$

With $a, b, c, d \in R$

Assume that the Jacobian matrix can be brought into the Jordan normal form

$$J = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \ with \ \lambda_+, \lambda_- \in R$$

Prove that for $\lambda_+ > \lambda_- > 0$ the origin is an unstable node, whereas for $\lambda_- < \lambda_+ < 0$ the origin is a stable node

Q19/ consider the dynamical system of the form

$$\dot{x_1} = x_2 - {x_2}^3$$
$$\dot{x_2} = -x_1 - {x_2}^2$$

i) Determine all fixed points of the system
ii) Draw the phase portrait for a neighborhood around the fixed points for which the linearization theorem applies
iii) Use the information form (i)-(ii) to sketch the phase portrait for the above system

Q20/ consider the dynamical system

$$\dot{x} = x(3 - 2x - y)$$
$$\dot{y} = y(4 - x - 3y)$$

In the (x, y) – plane

a) Determine the fixed points

b) Give the Jacobian matrix, determine the linear stability of all fixed points

c) Draw the phase portrait of the system

Q21/ consider the system of differential equation

$$\dot{x} = -x^3 + y$$
$$\dot{y} = -ax - by$$

a) Show that (0,0) is a fixed point. Find the Jacobian and calculate the eigenvalues of the fixed point (0,0) as a function of a, b

b) Show that the fixed point (0,0) is linearly stable for all values $a > 0, b > 0$

c) Insert these fixed points into the Jacobian and find trace and the determinant. What kind of stability at $a = 0$