Partial Differential Equation

Hero Waisi Salih

May 8, 2023

Chapter 1

Definition 1.0.1 (Partial Differential Equation) *: A partial differential equation (PDE) is a relationship between an unknown function* $u(x_1, x_2, ..., x_n)$ *and its derivatives with respect to the variables* $x_1, x_2, ..., x_n$.
Here is an example of a PDE

$$\frac{\partial u(x,y)}{\partial x} + x\frac{\partial u(x,y)}{\partial y} = \sin x \tag{1.0.1}$$

The order of a PDE *is the order of the highest derivative that occurs in the equation. The previous equation* (1.0.1) *is a first-order PDE.*

The degree of a PDE *is determined by the power of the hight derivation in the equation.*

Example 1.0.1

$$\frac{\partial u}{\partial x} + \epsilon\frac{\partial u}{\partial y} = 0 \qquad first - order PDE$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad first - order PDE$$

$$\frac{\partial^2 u}{\partial x^2} = \kappa\frac{\partial u}{\partial t} \qquad second - order PDE$$

$$\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial t^2} \qquad second - order PDE$$

We also define ***linear PDE's*** *as equations for which the dependent variable (and its derivative) appear in terms with degree at most one. Anything else is called* ***nonlinear PDE****.*

Definition 1.0.2 (Quasilinear PDE) *: A PDE is said to be quasilinear if it* *is linear with respect to all the highest order derivatives of the unknown function.*

$$A(x, y, u)p + B(x, y, u)q = C(x, y, u),$$

is quasilinear PDE of the first order.

$$A(x, y, u, p, q)r + B(x, y, u, p, q)s + C(x, y, u, p, q)t + D(x, y, u, p, q) = 0,$$

is quasilinear PDE of the second-order.

Example 1.0.2 *:*

$3p - yq^2 = u$	*nonlinear*
$xyr - 5e^y rt = 0$	*nonlinear*
$xp + yq = 3u$	*linear*
$x^2p + yq = t^3$	*nonlinear*

Formulation Of Partial Differential Equation

1.0.1 PDE By The Elimination Of Arbitrary Constant

Our aim is to see how PDEs arise mathematically. We show that such PDEs can be formed by the elimination of arbitrary constants. So in this lesson we give examples showing it is possible to associate a PDEFO with a given of functions of two variables.

Consider the relation

$$f(x, y, u, a, b) = 0, \tag{1.0.2}$$

where a *and* b *are constants. Differentiating* (1.0.2) *partially with respect to* x *and* y *separately, we get*

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u} = 0, \tag{1.0.3}$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u} = 0 \tag{1.0.4}$$

The set of Eqs. (1.0.2), (1.0.3) *and* (1.0.4) *constitute three equations involving two arbitrary constants a and b. In general, it is possible to eliminate a and b from these equations and the resulting equation is a PDEFO of the type*

$$f(x, y, u, p, q) = 0$$

Example 1.0.3 *: Form the PDE by eliminating a and b from*

$$u = ax + by \tag{1.0.5}$$

Solution *:*

$$\frac{\partial u}{\partial x} = a \Rightarrow \quad p = a,$$

$$\frac{\partial u}{\partial y} = b \Rightarrow \quad q = b,$$

substitute a and b in Eq.(1.0.5), *we get*

$$u = px + qy \quad \textit{is a PDE.}$$

Example 1.0.4 *: Find the PDE by eliminating a and b from*

$$u = ax^2 + by^2 \tag{1.0.6}$$

Solution *:*

$$\frac{\partial u}{\partial x} = 2ax \Rightarrow \quad a = \frac{1}{2x}p$$
$$\frac{\partial u}{\partial y} = 2by \Rightarrow \quad b = \frac{1}{2y}q$$

by substituting a and b in Eq. (1.0.6)*, we obtain*

$$u = \frac{1}{2x}px^2 + \frac{1}{2y}qy^2 \Rightarrow$$
$$u = \frac{1}{2}px + \frac{1}{2}yq$$

$$\therefore \quad 2u = px + qy \quad \text{is a PDE.}$$

Example 1.0.5 *: Find the PDE by eliminating constants from*

$$u = axy \tag{1.0.7}$$

Solution *:*

$$\frac{\partial u}{\partial x} = ay \quad a = \frac{1}{y}p, \tag{1.0.8}$$

$$\frac{\partial u}{\partial y} = ax \quad a = \frac{1}{x}q, \tag{1.0.9}$$

substitute Eq.(1.0.8) *in* (1.0.7)*, we get*

$$u = \frac{1}{y}pxy \Rightarrow \quad u = px$$

substitute Eq.(1.0.9) *in* (1.0.7)*, we obtain*

$$u = \frac{1}{x}qxy \Rightarrow \quad u = qy$$

so the PDE is $u = px$ *or* $u = qy$.

Remark *: If there are less arbitrary constants than the number of independent variables, the above procedure of elimination will give more than one PDE.*

Example 1.0.6 *: Eliminate a, b and c from*

$$ax + by + cu = 0$$

Solution *:*

$$a + c\frac{\partial u}{\partial x} = 0 \Rightarrow \quad c\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 0$$
$$b + c\frac{\partial u}{\partial y} = 0 \Rightarrow \quad c\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = 0$$

So the PDE is $r = 0$ *or* $t = 0$ *or* $s = 0$.

Remark *: If the number of independent variables are less than the number of arbitrary constants , then the order of PDE is greater than the first order.*

Example 1.0.7 *: Find the PDE from the following equations*

1) $u = (x + a)(y + b)$

2) $2u = (ax + y)^2 + b$

3) $u = axe^y + \frac{1}{2}a^2e^y$

Solution 1) *:*

$$\frac{\partial u}{\partial x} = 0 + (y + b)(1) = y + b$$
$$\Rightarrow p = y + b \Rightarrow \quad b = p - y$$
$$\frac{\partial u}{\partial y} = x + a \Rightarrow \quad q = x + a \Rightarrow \quad a = q - x$$

Then

$$u = (x + q - x)(y + p - y)$$
$$\Rightarrow u = pq \quad \textit{is a PDE}$$

Solution 2) *:*

$$2\frac{\partial u}{\partial x} = 2a(ax + y) \Rightarrow \quad p = a(ax + y)$$
$$2\frac{\partial u}{\partial y} = 2(ax + y) \Rightarrow \quad q = (ax + y)$$
$$xp + yq = ax(ax + y) + y(ax + y)$$
$$= (ax + y)(ax + y) = (ax + y)^2 = q^2$$

$\therefore xp + qy = q^2$ *is a PDE*

Solution 3) :

$$\frac{\partial u}{\partial x} = ae^y \Rightarrow \quad p = ae^y \Rightarrow \quad a = \frac{p}{e^y}$$

$$\frac{\partial u}{\partial y} = axe^y + a^2e^{2y} \Rightarrow \quad q = axe^y + a^2e^{2y} \tag{1.0.10}$$

substitute a in Eq.(1.0.10)

$$\therefore q = \frac{p}{e^y}xe^y + (\frac{p}{e^y})e^{2y}$$
$$\Rightarrow q = px + p^2 \quad \text{is a PDE}$$

Exercises :

1) $ax^2 + by^2 + z^2 = 1$

2) $z = (x-a)^2 + (y-b)^2$

3) $z = xy + y\sqrt{x^2 + a^2} + b$

1.0.2 PDE By The Elimination Of An arbitrary Function

If we have an equation

$$f(v, w) = 0, \tag{1.0.11}$$

where f is an arbitrary function, v and w are known functions of x, y and u. Differentiating Eq.(1.0.11) *partially with respect to x and y separately, we get*

$$\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial f}{\partial v}(v_x + pv_u) + \frac{\partial f}{\partial w}(w_x + pw_u) = 0 \tag{1.0.12}$$

$$\frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \frac{\partial f}{\partial v}(v_y + qv_u) + \frac{\partial f}{\partial w}(w_y + qw_u) \tag{1.0.13}$$

By eliminating $\frac{\partial f}{\partial v}$ *and* $\frac{\partial f}{\partial w}$ *from Eqs.* (1.0.12) *and* (1.0.13), *we obtain*

$$\begin{vmatrix} v_x + pv_u & w_x + pw_u \\ v_y + qv_u & w_y + qw_u \end{vmatrix} = 0$$

$$(v_x + pv_u)(w_y + qw_u) - (w_x + pw_u)(v_y + qv_u) = 0$$

$$v_x w_y + qv_x w_u + pv_u w_y + pqv_u w_u - w_x v_y - qw_x v_u - pw_u v_y - pqw_u v_u = 0$$

$$\Rightarrow p(v_y w_u - v_u w_y) + q(v_u w_x - v_x w_u) = v_x w_y - w_x v_y$$

$$\Rightarrow p\frac{\partial(v,w)}{\partial(y,u)} + q\frac{\partial(v,w)}{\partial(u,x)} = \frac{\partial(v,w)}{\partial(x,y)} \quad \text{is a PDE of the first order}$$

where

$$\frac{\partial(v,w)}{\partial(y,u)} = \begin{vmatrix} v_y & v_u \\ w_y & w_u \end{vmatrix},$$

$$\frac{\partial(v,w)}{\partial(u,x)} = \begin{vmatrix} v_u & v_x \\ w_u & w_x \end{vmatrix},$$

and

$$\frac{\partial(v,w)}{\partial(x,y)} = \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$$

Example 1.0.8 : *Find the PDE by eliminating the arbitrary function from*

$$f(\frac{u}{x}, \frac{x}{y^2}) = 0$$

Solution:

$$v = \frac{u}{x} \quad \text{and} \quad w = \frac{x}{y^2}$$

$$v_x = \frac{-u}{x^2} \qquad w_x = \frac{1}{y^2}$$

$$v_y = 0 \qquad w_y = \frac{-2x}{y^3}$$

$$v_u = \frac{1}{x} \qquad w_u = 0$$

$$\frac{\partial(v,w)}{\partial(y,u)} = \begin{vmatrix} v_y & v_u \\ w_y & w_u \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{x} \\ \frac{-2x}{y^3} & 0 \end{vmatrix} = \frac{2}{y^3}$$

$$\frac{\partial(v,w)}{\partial(u,x)} = \begin{vmatrix} v_u & v_x \\ w_u & w_x \end{vmatrix} = \begin{vmatrix} \frac{1}{x} & \frac{-u}{x^2} \\ 0 & \frac{1}{y^2} \end{vmatrix} = \frac{1}{xy^2}$$

$$\frac{\partial(v,w)}{\partial(x,y)} = \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = \begin{vmatrix} \frac{-u}{x^2} & 0 \\ \frac{1}{y^2} & \frac{-2x}{y^3} \end{vmatrix} = \frac{2u}{xy^3}$$

$$p(\frac{2}{y^3}) + q(\frac{1}{xy^2}) = \frac{2u}{xy^3} \Rightarrow \quad 2xp + yq = 2u \quad \textit{is a PDE.}$$

Example 1.0.9 *: Form the PDE by eliminating the arbitrary functions in the following equations*

1) $u = xy + f(x^2 + y^2)$

2) $u = f\ (\frac{xy}{u}\ )$

3) $f(x^2 + y^2 + u^2, u^2 - 2xy) = 0$

Solution :

$$\frac{\partial u}{\partial x} = y + 2xf'(x^2+y^2)$$
$$\Rightarrow p = y + 2xf'(x^2+y^2)$$
$$\frac{\partial u}{\partial y} = x + 2yf'(x^2+y^2)$$
$$\Rightarrow q = x + 2yf'(x^2+y^2)$$

$$yp - xq = y^2 + 2xyf'(x^2+y^2) - x^2 - 2xyf'(x^2+y^2)$$

$$\therefore yp - xq = y^2 - x^2 \quad \textit{is a PDE.}$$

Also we can solve it by another way

$$u - xy = f(x^2+y^2)$$
$$g(u - xy, x^2+y^2) = 0$$

Solution 2 :

$$\frac{\partial u}{\partial x} = f'(\frac{xy}{u})\frac{uy - xyp}{u^2} \Rightarrow \quad p = f'(\frac{xy}{u})\frac{uy - xyp}{u^2}$$

$$\frac{\partial u}{\partial y} = f'(\frac{xy}{u})\frac{ux - xyq}{u^2} \Rightarrow \quad p = f'(\frac{xy}{u})\frac{ux - xyq}{u^2}$$

$$\frac{p}{q} = \frac{uy - xyp}{ux - xyq} \Rightarrow \quad pux - xypq = uyq - xypq$$

$$\Rightarrow \quad pux - uqy = 0 \quad \textit{is a PDE.}$$

Solution 3 :

$$v = x^2 + y^2 + u^2, \quad w = u^2 - 2xy$$

$$v_x = 2x \qquad w_x = 2y$$
$$v_y = 2y \qquad w_y = 2x$$
$$v_u = 2u \qquad w_u = 2u$$

$$\frac{\partial(v,w)}{\partial(y,u)} = \begin{vmatrix} v_y & v_u \\ w_y & w_u \end{vmatrix} = \begin{vmatrix} 2y & 2u \\ \\ -2x & 2u \end{vmatrix} = 4yu + 4xu = 4u(x+y)$$

$$\frac{\partial(v,w)}{\partial(u,x)} = \begin{vmatrix} v_u & v_x \\ w_u & w_x \end{vmatrix} = \begin{vmatrix} 2u & 2x \\ \\ 2u & -2y \end{vmatrix} = -4uy - 4xu = -4u(x+y)$$

$$\frac{\partial(v,w)}{\partial(x,y)} = \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ \\ -2y & -2x \end{vmatrix} = -4x^2 + 4y^2 = -4(x^2 - y^2)$$

$$\therefore 4x(x+y)p - 4u(x+y)q = -4(x^2 - y^2)$$
$$\Rightarrow up - uq = (y - x) \quad \textit{is a PDE.}$$

1.0.3 Formulation Of PDESO By Eliminating Arbitrary Functions and Constants

Example 1.0.10 *: Eliminate the arbitrary functions f and g from the relation*

$$u - f(x - aiy) + g(x + aiy),$$

where a is constants. ***Solution :***

$$\frac{\partial u}{\partial x} = f'(x - aiy) + g'(x + aiy)$$
$$\frac{\partial u}{\partial y} = -aif'(x - aiy) + aig'(x + aiy)$$
$$\frac{\partial^2 u}{\partial x^2} = f''(x - aiy) + g''(x + aiy)$$
$$\frac{\partial^2 u}{\partial x \partial y} = -aif''(x - aiy) + aig''(x + aiy)$$
$$\frac{\partial^2 u}{\partial y^2} = -a^2 f''(x - aiy) - a^2 g''(x + aiy)$$

$$p - f' - g' = 0$$
$$q + aif' - aig' = 0$$
$$r - f'' - g'' = 0$$
$$s + aif'' - aig'' = 0$$
$$t + a^2 f'' + a^2 g'' = 0$$

The result is a PDE of the second order

$$t = -a^2(f'' + g'') = -a^2 r$$
$$\therefore t + a^2 r = 0 \quad \text{is a PDE of the second order.}$$

Example 1.0.11 *: Eliminate the arbitrary functions f and g, where f and g arbitrary functions.*

$$u = xf(y + 2x) + g(y + 2x)$$

$$\frac{\partial u}{\partial x} = 2xf'(y+2x) + f(y+2x) + 2g'(y+2x)$$
$$\frac{\partial u}{\partial y} = xf'(y+2x) + g'(y+2x)$$
$$\frac{\partial^2 u}{\partial x^2} = 4xf''(y+2x) + 2f'(y+2x) + 2f'(y+2x) + 4g''(y+2x)$$
$$\frac{\partial^2 u}{\partial x \partial y} = 2xf''(y+2x) + f'(y+2x) + 2g''(y+2x)$$
$$\frac{\partial^2 u}{\partial y^2} = xf''(y+2x) + g''(y+2x)$$

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0 \quad \textit{is a PDE.}$$

Exercises :

1) $u = f(x+y).\, g(x-y)$

2) $u = (\frac{1}{x})f(y-x) + g(y-x)$

3) $u = f(xy) + g(x+y)$

4) $x = f(u) + g(y)$

Total Differential Equation
(Pfaffian Equation)

In general if

$$U(x,y,z) = C,$$

is a given functional relation involving x, y, z *and an arbitrary constant* C*, then the total differential* dU *of the function is zero. But* $dU = U_x dx + U_y dy + U_z dz$*. Therefore,* $dU = 0$ *implies* $dU = U_x dx + U_y dy + U_z dz = 0$ *that is*

$$Pdx + Qdy + Rdz = 0,$$

where P, R and Q are the functions of x, y and z. An equation of this form in three variables is said to be total differential equation (pfaffian equation) in the variables x, y and z. The problem of finding all possible solutions of a pfaffian equation is called pfaff's problem.

Definition 1.0.3 *A pfaffian differential form (in three variables) is said to be exact if we can find a continuously differentiable function $U(x, y, z)$ such that*

$$Pdx + Qdy + Rdz = dU$$

To show that a pfaffian differential is exact, we need to show that

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial U}{\partial x}) = \frac{\partial P}{\partial y}$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial U}{\partial y}) = \frac{\partial Q}{\partial x}$$

$$since \quad \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}, \; so \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

in the same way

$$since \quad \frac{\partial^2 U}{\partial z \partial x} = \frac{\partial^2 U}{\partial x \partial z}, \; so \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

also,

$$since \quad \frac{\partial^2 U}{\partial y \partial z} = \frac{\partial^2 U}{\partial z \partial y}, \; so \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$$

Example 1.0.12 *: Show that*

$$yzdx + 2xzdy - 3xydz$$

is exact or not?

Solution :

$$P = yz, \qquad Q = 2xz, \qquad R = -3xy$$

$$\frac{\partial P}{\partial y} = z \qquad \frac{\partial Q}{\partial x} = 2z \qquad \frac{\partial R}{\partial x} = -3y$$

$$\frac{\partial P}{\partial z} = y \qquad \frac{\partial Q}{\partial z} = 2x \qquad \frac{\partial R}{\partial y} = -3x$$

$$\because \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$\therefore$ *this equation is not exact.*

Definition 1.0.4 *A pfaffian DE is said to be integrable if there exists a nonzero differentiable function $v = v(x, y, z)$ such that the differential form*

$$v(x, y, z)[P(x, y, z) + Q(x, y, z) + R(x, y, z)]$$

is an exact differential Eq. and the function v is called an integrating factor, i.e $\exists U$ such that $U_x = vP$, $U_y = vQ$, $U_z = vR$. If there is no such a family, we say that the pfaffian is not integrable.

Definition 1.0.5 *If we have $v = (P, Q, R) = Pi + Qj + Rk$, then the curlv is defined by*

$$curlv = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}).$$

Example 1.0.13 *: Find the curl for the DE*

$$(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$$

Solution :

$$P = y^2 + yz, \qquad Q = xz + z^2, \qquad R = y^2 - xy$$

$$\frac{\partial P}{\partial y} = 2y + z \qquad \frac{\partial Q}{\partial x} = z \qquad \frac{\partial R}{\partial x} = -y$$

$$\frac{\partial P}{\partial z} = y \qquad \frac{\partial Q}{\partial z} = x + 2z \qquad \frac{\partial R}{\partial y} = 2y - x$$

$$\begin{aligned} curlv = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} &= (2y - x - (x + 2z), y - (-y), z - (2y + z)) \\ &= (2y - 2x - 2z, y + y, z - 2y - z) \\ &= (2y - 2x - 2z, 2y, -2y). \end{aligned}$$

Remark : *If* $curlv = (0, 0, 0) = 0$*, then*

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0 \Rightarrow \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$$

$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0 \Rightarrow \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

this means that the pfaffian DE $Pdx + Qdy + Rdz = 0$*, is exact.*

Theorem 1.0.1 *: If* v *is a vector such that* $v.\,curlv = 0$*, and* u *is an arbitrary function of* x, y *and* z*, then*

$$u.v.Curl(u, v) = 0$$

proof :

$u.v = u(P,Q,R) = (uP,uQ,uR)$

$$curl(uv) = (\frac{\partial}{\partial y}(uR) - \frac{\partial}{\partial z}(uQ), \frac{\partial}{\partial z}(uP) - \frac{\partial}{\partial x}(uR), \frac{\partial}{\partial x}(uQ) - \frac{\partial}{\partial y}(uP))$$
$$= (uR_y + Ru_y - uQ_z - Qu_z, uP_z + Pu_z - uR_x - Ru_x, uQ_x + Qu_x - uP_y - Pu_y)$$
$$= u(R_y - Q_z, P_z - R_x, Q_x - P_y) + (Ru_y - Qu_z, Pu_z - Ru_x, Qu_x - Pu_y)$$
$$= ucurlv + (Ru_y - Qu_z, Pu_z - Ru_x, Qu_x - Pu_y)$$
$$uvcurluv = u(P,Q,R).(ucurlv + (Ru_y - Qu_z, Pu_z - Ru_x, Qu_x - Pu_y))$$
$$= u^2v.curlv + u(PRu_y - PQu_z + QPu_z - QRu_x + RQu_x - RPu_y)$$
$$= u^2(0) + u(0) = 0$$

$\therefore uv.curluv = 0$

Exercise : *Is the converse of of this theorem true or not?explain?*

Theorem 1.0.2 *: A necessary and sufficient condition for $f(u,v) = 0$, where f is a relation between two functions $u(x,y)$ and $v(x,y)$, not involving x or y explicitly is that $\frac{\partial(u,v)}{\partial(x,y)} = 0$.*

proof *: The first condition is necessary i.e. $f(u,v) = 0$, we differentiate f with respect to x and y, obtain :*

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial x} = 0 \tag{1.0.14}$$

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial y} = 0 \tag{1.0.15}$$

Eliminate $\frac{\partial f}{\partial v}$ from these two equations.

from (1.0.15) $\Rightarrow$

$$\frac{\partial f}{\partial v}=\frac{\frac{-\partial f}{\partial u}.\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}},\quad \textit{then } (1.0.14) \textit{ becomes :}$$

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x}+\frac{\frac{-\partial f}{\partial u}.\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}}.\frac{\partial v}{\partial x}=0$$

$$\frac{\partial f}{\partial u}(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x})=0\Rightarrow\quad \frac{\partial f}{\partial u}(\frac{\partial(u,v)}{\partial(x,y)})=0$$

since $f(u,v)$ *involves* u *and* v, *it means that* $\frac{\partial f}{\partial u}\neq 0$, *then* $\frac{\partial(u,v)}{\partial(x,y)}=0$

conversely : *The second condition is sufficient, i.e we have* $\frac{\partial(u,v)}{\partial(x,y)}=0$, *we have to prove that the relation* $f(u,v)=0$, *where* f *is not involving* x *or* y *explicitly. Now, we may eliminate* y *in the equations* $u(x,y)$ *and* $v(x,y)$ *to get a new relation* $f(u,v,x)=0$ *and we differentiate the new relation with respect to* x *and* y, *we get*

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial x}=0 \tag{1.0.16}$$

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial y}=0 \tag{1.0.17}$$

Eliminate $\frac{\partial f}{\partial v}$ *from these two equations: from*(2.0.11) $\Rightarrow$

$$\frac{\partial f}{\partial v} = \frac{-\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}}, \quad \textit{then } (1.0.16) \textit{ becomes}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{-\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}}.\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial f}{\partial x}.\frac{\partial v}{\partial y} + \frac{\partial f}{\partial u}(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}) = 0$$

$$\frac{\partial f}{\partial x}.\frac{\partial v}{\partial y} + \frac{\partial f}{\partial u}(\frac{\partial (u,v)}{\partial (x,y)}) = 0 \Rightarrow \quad \frac{\partial f}{\partial x}.\frac{\partial v}{\partial y} = 0$$

since v *depends on* x *and* y*, then* $\frac{\partial v}{\partial y} \neq 0$ *therefore* $\frac{\partial f}{\partial x} = 0$*, which means that the function* f *does not contain the variable* x *explicitly.*

Definition 1.0.6 *: If* $f(x, y, z)$ *is a function, then*

$$gradf = \bigtriangledown f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

Remark : $curl(grad\ f\ ) = 0$
proof:

$$curlv = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x})$$

$$= (0, 0, 0) = 0$$

Theorem 1.0.3 *: A pfaffian differential equation in two variables always has* *an integrating factor.*

Theorem 1.0.4 *: A necessary and sufficient condition for the integrable pfaffian differential equation $Pdx + Qdy + Rdz = 0$ is that $v.curlv = 0$, where $v = (P, Q, R)$.*

Example 1.0.14 *: Are the following equations integrable?, find the solution for the integrable equations?*

1) $(y^2 + xz)dx + (x^2 + yz)dy + 3z^2dz = 0$

2) $2xzdx + zdy - dz = 0$

Solution :

$$P = y^2 + xz \quad , Q = x^2 + yz \quad , R = 3z^2$$

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 2x \qquad \frac{\partial R}{\partial x} = 0$$

$$\frac{\partial P}{\partial z} = x \qquad \frac{\partial Q}{\partial z} = y \qquad \frac{\partial R}{\partial y} = 0$$

$$curlv = (0 - y, x - 0, 2x - 2y) = (-y, x, 2x - 2y) \neq 0$$

$\therefore$ *the DE is not exact.*

$$v.curlv = (y^2 + xz, x^2 + yz, 3z^2)(-y, x, 2x - 2y) = -y^3 - xyz + x^3 + xyz + 6xz^2 - 6yz^2 \neq 0$$

$\therefore$ *the DE is not integrable.*

2)

$$P = 2xz \quad , Q = z \quad , R = -1$$

$$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial R}{\partial x} = 0$$

$$\frac{\partial P}{\partial z} = 2x \qquad \frac{\partial Q}{\partial z} = 1 \qquad \frac{\partial R}{\partial y} = 0$$

$$curlv = (0-1, 2x-0, 0-0) = (-1, 2x, 0) \neq 0$$

$\therefore$ *the DE is not exact.*

$$v.curlv = (2xz, z, -1)(-1, 2x, 0) = -2xz + 2xz + 0 = 0$$

$\therefore$ *the DE is integrable.*

$$\int 2xdx + \int dy - \int \frac{dz}{z} = 0$$
$$x^2 + y - \ln z = c \quad \textit{is a Gs.}$$

Working Rules For Solution Of Pfaffian DE

We suppose that the equation

$$Pdx + Qdy + Rdz = 0 \tag{1.0.18}$$

is integrable (i.e. v curlv=0).

Case 1 :

If the (1.0.18) *is integrable, it may be possible in many cases that by rearranging the terms the* (1.0.18) *becomes exact and the solution is found easily.*

Example 1.0.15 *: Solve*

1) $(y+z)dx + (z+x)dy + (x+y)dz = 0$

2) $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$

Solution 1):

$$P = y+z \quad , Q = z+x \quad , R = x+y$$

we first check if the PDE is exact

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 1 \qquad \frac{\partial R}{\partial x} = 1$$

$$\frac{\partial P}{\partial z} = 1 \qquad \frac{\partial Q}{\partial z} = 1 \qquad \frac{\partial R}{\partial y} = 1$$

$$curlv = (1-1, 1-1, 1-1) = (0,0,0) = 0$$

$\therefore$ *the PDE is exact.*

$$\begin{aligned} ydx + zdx + zdy + xdy + xdz + ydz &= 0 \\ d(xy) + d(xz) + d(yz) &= 0 \\ \int d(xy) + \int d(xz) + \int d(yz) &= c \\ xy + xz + yz = c \quad &is\ a\ Gs. \end{aligned}$$

2)

$$P = x^2z - y^3 \quad , Q = 3xy^2 \quad , R = x^3$$

$$\frac{\partial P}{\partial y} = -3y^2 \qquad \frac{\partial Q}{\partial x} = 3y^2 \qquad \frac{\partial R}{\partial x} = 3x^2$$

$$\frac{\partial P}{\partial z} = x^2 \qquad \frac{\partial Q}{\partial z} = 0 \qquad \frac{\partial R}{\partial y} = 0$$

$$curlv = (0-0, x^2 - 3x^2, 3y^2 + 3y^2) = (0, -2x^2, 6y^2) \neq 0$$

$\therefore$ *the PDE is not exact.*

$$\begin{aligned} vcurlv &= (x^2z - y^3, 3xy^2, x^3)(0, -2x^2, 6y^2) \\ &= 0 - 6x^3y + 6x^3 = 0 \end{aligned}$$

$\therefore$ *the equation is integrable.*

$$\begin{aligned} &x^2zdx + -y^3dx + 3xy^2dy + x^3dz = 0 \quad (divide\ by\ x^2) \\ &zdx + \frac{-y^3dx + 3xy^2dy}{x^2} + xdz = 0 \\ &zdx + xdz + \frac{3xy^2dy - y^3dx}{x^2} \\ &d(xz) + d(\frac{y^3}{x}) = 0 \end{aligned}$$

$$\int d(xz) + \int d(\frac{y^3}{x}) = c$$
$$xz + \frac{y^3}{x} = c \quad \textit{is a Gs.}$$

Case 2: Variable Separable

If (1.0.18) *can be written of the form*

$$P(x)dx + Q(y)dy + R(z)dz = 0$$

so the Eq. is exact and we can find the solution directly.

Example 1.0.16 *: Find the solution for the following equations*

1) $a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2dz = 0$

2) $2xzdx + zdy - dz = 0$

Solution 1):

$$a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2 = 0 \tag{1.0.19}$$

Multiplying (1.0.19) *by* $\frac{1}{x^2y^2z^2}$, *the* (1.0.19) *becomes*

$$a^2\frac{dx}{x^2} + b^2\frac{dy}{y^2} + c^2\frac{dz}{z^2} = 0 \tag{1.0.20}$$

$\therefore$ *the* (1.0.20) *is variable separable and it also exact, so we can integrate the last Eq.*

$$\int a^2\frac{dx}{x^2} + \int b^2\frac{dy}{y^2} + \int c^2\frac{dz}{z^2} = c_1$$

$$-a^2\frac{dx}{x} - b^2\frac{dy}{y} - c^2\frac{dz}{z} = c_1 \quad \textit{is a Gs.}$$

2)

$$2xzdx + +zdy - dz = 0 \tag{1.0.21}$$

Multiplying (1.0.21) *by* $\frac{1}{z}$*, it becomes*

$$2xdx + dy - \frac{dz}{z} = 0$$

$$\int 2xdx + \int dy - \int \frac{dz}{z} = c$$

$$x^2 + y - \ln z = c \quad \text{is a } Gs.$$

Case 3) One Variable Separable *If* (1.0.18) *can be written of the form*

a) $P(x, y)dx + Q(x, y)dy + R(z)dz = 0$

b) $P(x, z)dx + Q(y)dy + R(x, z)dz = 0$

c) $P(x)dx + Q(y, z)dy + R(y, z)dz = 0$

If a) *is integrable, then* $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$*, and* $Pdx + Qdy$ *becomes exact.*
If b) *is integrable, then* $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$*, and* $Pdx + Rdz$ *becomes exact.*
If c) *is integrable, then* $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$*, and* $Qdy + Rdz$ *becomes exact.*

Example 1.0.17 : *Solve*

1) $y(1 + z^2)dx - x(1 + z^2)dy + (x^2 + y^2)dz = 0$

2) $x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$

Solution 1)

$$y(1 + z^2)dx - x(1 + z^2)dy + (x^2 + y^2)dz = 0 \quad \text{since this Eq. is integrable}$$

$$\frac{y}{x^2 + y^2}dx - \frac{x}{x^2 + y^2}dy + \frac{dz}{1 + z^2} = 0 \quad \text{is one variable separable in } z.$$

$$P = \frac{y}{x^{2+y^2}} \quad Q = -\frac{x}{x^2 + y^2}$$

$$\frac{\partial P}{\partial y} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, *then* $Pdx + Qdy$ *is exact.*

$$\frac{\partial U}{\partial x} = P = \frac{y}{x^2 + y^2} \quad \Rightarrow U(x,y) = \int \frac{y}{x^2 + y^2} dx \Rightarrow$$

$$U(x,y) = \int \frac{\frac{1}{y}}{(\frac{x}{y})^2 + 1} dx = tan^{-1}(\frac{x}{y}) + G(y), \quad where\ G(y)\ is\ an\ arbitrary\ function\ of\ y.$$

$$U(x,y) = tan^{-1}(\frac{x}{y}) + G(y)$$

$$\frac{\partial U}{\partial y} = \frac{\frac{-x}{y^2}}{1 + (\frac{x}{y})^2} + \frac{dG(y)}{dy} = \frac{-x}{y^2 + x^2} + \frac{dG(y)}{dy}$$

$$\therefore \quad \frac{-x}{y^2 + x^2} = \frac{-x}{y^2 + x^2} + \frac{dG(y)}{dy} \Rightarrow \quad \frac{dG(y)}{dy} = 0 \Rightarrow \quad G(y) = c$$

$$U(x,y) = tan^{-1}(\frac{x}{y})$$

$$tan^{-1}(\frac{x}{y}) + \int \frac{1}{1 + z^2} dz = c \Rightarrow \quad tan^{-1}(\frac{x}{y}) + tan^{-1}(z) = c \quad is\ a\ Gs.$$

2)

$$x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$$

Since $curlv \neq 0$, *then the Eq. is not exact, but the Eq. is integrable because* $vcurlv = 0$.
multiply Eq. by $\frac{1}{(y^2 - a^2)(x^2 - z^2)}$, *we get*

$$\frac{dx}{x^2 - z^2} + \frac{y}{y^2 - a^2} dy - \frac{z}{x^2 - z^2} dz = 0, \quad so\ the\ Eq.\ is\ one\ variable\ separable\ in\ y.$$

$$P = \frac{x}{x^2 - z^2}, \qquad R = \frac{-z}{x^2 - z^2}$$

$$\frac{\partial P}{\partial z} = \frac{-x(-2z)}{(x^2 - z^2)^2} = \frac{2xz}{(x^2 - z^2)^2}$$

$$\frac{\partial R}{\partial x} = \frac{z(2x)}{(x^2 - z^2)^2} = \frac{2xz}{(x^2 - z^2)^2}$$

$$\therefore \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

$\therefore \exists$ *a function* $U(x, y)$ *s.t*

$$\frac{\partial U}{\partial x} = \frac{x}{x^2 - z^2} \qquad and \qquad \frac{\partial U}{\partial z} = \frac{-z}{x^2 - z^2}$$

$$\frac{\partial U}{\partial z} = \frac{-z}{x^2 - z^2} \Rightarrow \quad U(x, y) = \frac{1}{2}\ln(x^2 - z^2) + G(x)$$

to find $G(x)$

$$\frac{\partial U}{\partial x} = \frac{2x}{2(x^2 - z^2)} + \frac{dG(x)}{dx}$$

$$\frac{x}{(x^2 - z^2)} = \frac{x}{(x^2 - z^2)} = \frac{dG(x)}{dx} \Rightarrow \quad \frac{dG(x)}{dx} = 0 \Rightarrow \quad G(x) = c$$

$$U(x, y) = \frac{1}{2}\ln x^2 - z^2$$

$$\frac{1}{2}\ln x^2 - z^2 + \int \frac{y}{y^2 - a^2}dy = c$$

$$\frac{1}{2}\ln x^2 - z^2 + \frac{1}{2}\ln y^2 - a^2 = c$$

$$\ln(x^2 - z^2)(y^2 - a^2) = c_1, \quad where\ c_1 = 2c$$

$$(x^2 - z^2)(y^2 - a^2) = c_2, \quad where\ c_2 = e^{c_1}\ is\ a\ Gs.$$

Case 4: Homogeneous Equations

In this case P, Q and R are homogeneous then the equation

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{1.0.22}$$

is homogeneous, then one variable, say x may be separated from the other two variables by replacing

$$y = ux \Rightarrow \quad dy = udx + xdu \tag{1.0.23}$$

$$\tag{1.0.24}$$

$$z = vx \Rightarrow \quad dz = vdx + xdv \tag{1.0.25}$$

Now, we substitute Eq.(1.0.23) *in* (1.0.22)*, we get*

$$P(x,xu,xv)dx + Q(x,xu,xv)(udx + xdu) + R(x,xu,xv)(vdx + xdv) = 0 \quad \textit{(divide by x)}$$

$$(P(1,u,v) + uQ(1,u,v) + vR(1,u,v))dx + xQ(1,u,v)du + xR(1,u,v)dv = 0$$

$$\frac{dx}{x} + \frac{Q(1,u,v)du}{(P(1,u,v) + uQ(1,u,v) + vR(1,u,v))} + \frac{R(1,u,v)dv}{(P(1,u,v) + uQ(1,u,v) + vR(1,u,v))} = 0$$

so the last equation is one variable separable in x and

$$\frac{Q(1,u,v)du}{(P(1,u,v) + uQ(1,u,v) + vR(1,u,v))} = \frac{R(1,u,v)dv}{(P(1,u,v) + uQ(1,u,v) + vR(1,u,v))} \quad \textit{is exact.}$$

Example 1.0.18 *: Solve the following Eqs.*

1) $(y^2 + z^2)dx + xydy + xzdz = 0$

2) $yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0$

Solution 1)

$$P = y^{2+z^2} \qquad Q = xy \qquad R = xz$$

$$\therefore P, Q \textit{ and } R \textit{ are homogeneous of degree 2.}$$

$\therefore$ *the Eq. is homogeneous.*

Now let

$$y = ux \Rightarrow \quad dy = udx + xdu$$
$$z = vx \Rightarrow \quad dz = vdx + xdv$$

substitute y, z, dy *and* z *in the main Eq., we obtain*

$$(x^2u^2 + x^2v^2)dx + x^2u(udx + xdu) + x^2v(vdx + xdv) = 0$$
$$(x^2u^2 + x^2v^2 + x^2u^2 + x^2v^2)dx + x^3udu + x^3vdv = 0$$
$$2(u^2 + v^2)dx + xudu + xvdv = 0$$
$$\frac{2}{x}dx + \frac{u}{u^2+v^2}du + \frac{v}{u^2+v^2}dv = 0, \quad \textit{is one variable separable in } x.$$

$$Q = \frac{u}{u^2+v^2} \qquad R = \frac{v}{u^2+v^2}$$

$$Qdu + Rdv = 0 \quad \textit{is exact}$$

$$\int \frac{2}{x}dx + \frac{1}{2}\int \frac{2udu + 2vdv}{u^2+v^2} = c_1$$
$$2\ln x + \frac{1}{2}\ln u^2 + v^2 = c_1$$
$$\ln x^4 + \ln u^2 + v^2 = c_2$$
$$\ln x^4(u^2 + v^2) = c_2 \Rightarrow \quad x^4(\frac{y^2}{x^2} + \frac{z^2}{x^2}) = c$$

$$y^2x^2 + z^2x^2 = c \quad \textit{is a Gs.}$$

2)

$$yz(y + z)dx + xz(x + z)dy + xy(x + y)dy = 0$$

$$P = yz(y + z) \qquad Q = xz(x + z) \qquad R = xy(x + y)$$
$$\therefore P, Q \textit{ and } R \textit{ are homogeneous of degree 3.}$$

$\therefore$ *the main Eq. is homogeneous.*
Now, let

$$x = yu \Rightarrow \quad x = udy + ydu \tag{1.0.26}$$
$$z = yv \Rightarrow \quad z = vdy + ydv \tag{1.0.27}$$

substitute Eq. (1.0.26) *in the main Eq., we obtain*

$$y^2v(y + vy)(udy + ydu) + y^2uv(yu + yv)dy + y^2u(yu + y)(vdy + ydv) = 0$$

$$(y^3vu(1 + v) + y^3uv(u + v) + y^3uv(1 + u))dy + y^4v(1 + v)du + y^4u(1 + u)dv = 0$$

$$\frac{dy}{y} + \frac{v(1 + v)}{2uv(1 + u + v)}du + \frac{u(u + 1)}{2uv(1 + u + v)}dv = 0 \quad \textit{is one variable separable in y.}$$

$$\frac{dy}{y} + \frac{(1 + v)}{2u(1 + u + v)}du + \frac{(1 + u)}{2v(1 + u + v)}dv = 0$$

$$Q = \frac{1 + v}{2u(1 + u + v)} \Rightarrow \quad \frac{\partial Q}{\partial v} = \frac{2u(1 + u + v) - (1 + v)(2u)}{4u^2(1 + u + v)^2}$$

$$R = \frac{1 + u}{2u(1 + u + v)} \Rightarrow \quad \frac{\partial R}{\partial u} = \frac{2v(1 + u + v) - (1 + u)(2v)}{4v^2(1 + u + v)^2}$$

$$\therefore \frac{(1 + v)}{2u(1 + u + v)}du + \frac{(1 + u)}{2v(1 + u + v)}dv \quad \textit{is exact.}$$

$\exists$ *a function* $U(x, y)$ *s.t*

$$\frac{\partial U}{\partial u} = \frac{1 + v}{2u(1 + u + v)} \qquad \frac{\partial U}{\partial v} = \frac{1 + u}{2v(1 + u + v)}$$

$$\therefore \frac{\partial U}{\partial u} = \frac{1 + v}{2u(1 + u + v)} \Rightarrow \quad U(x, y) = \int \frac{1 + v}{2u(1 + u + v)}du$$

$$\frac{1+v}{2u(1+u+v)} = \frac{A}{2u} + \frac{B}{1+u+v} = \frac{A+uA+vA+2Bu}{2u(1+u+v)}$$

$$1+v = A + Av \Rightarrow \quad A(1+v) = 1+v \Rightarrow \quad A = 1$$

$$A + 2B = 0 \Rightarrow \quad B = \frac{-1}{2}$$

$$U(u,v) = \frac{1}{2}\ln u - \frac{1}{2}\ln(1+u+v) + G(v)$$

to find $G(v)$

$$\frac{\partial U}{\partial v} = \frac{-1}{2(1+u+v)} + \frac{dG(v)}{dv}$$

$$\frac{dG(v)}{dv} = \frac{(1+u+v)}{2v(1+u+v)} = \frac{1}{2v} \Rightarrow \quad G(v) = \frac{1}{2}\ln v$$

$$U(u,v) = \frac{1}{2}\ln\frac{u}{1+u+v} + \frac{1}{2}\ln v$$

$$\therefore \int \frac{dy}{y} + \frac{1}{2}\ln\frac{u}{1+u+v} + \frac{1}{2}\ln v = c$$

$$\ln y + \frac{1}{2}\ln\frac{u}{1+u+v} + \frac{1}{2}\ln v = c \Rightarrow \quad y^2 v(\frac{u}{1+u+v}) = c_1, \quad where\ c_1 = e^c$$

$$y^2.\frac{z}{y}(\frac{\frac{x}{y}}{1+\frac{x}{y}+\frac{z}{y}}) = c_1 \Rightarrow \quad yz(\frac{x}{y+x+z}) = c_1$$

$$2\ln x + \frac{1}{2}\ln u^2 + v^2 = c_1$$

$$\ln x^4 + \ln u^2 + v^2 = c_2$$

$$\ln x^4(u^2 + v^2) = c_2 \Rightarrow \quad x^4(u^2 + v^2) = c$$

$$x^4(\frac{y^2}{x^2} + \frac{z^2}{x^2}) = c \Rightarrow \quad x^2y^2 + x^2z^2 = c \quad \textit{is a Gs.}$$

Case 5 : One Variable is Regarded as Constant (Natani's Method)
In this case , we take one of the variables, say z, as constant, so that $dz = 0$, and we solve two terms $Pdx + Qdy = 0$. Let $\phi(x, y, z) = c_1$ be the solution of $Pdx+Qdy \ldots (*$, where c_1 is constant, so the solution of (1.0.18), it will be of the form $U(\phi, z) = c_2$, where c_2 is constant, and we can write the solution of (1.0.18) of the form $U(z) = \phi(x, y, z) \ldots (3$, where U is a function only of z.
After that, to determine $U(z)$, we let $x = n$, where n is a fix number, then $\phi(n, y, z) = U(z)$ is a solution to the DE, substitute x in Eq. $*)$ $Q(n, y, z)dy + R(n, y, z)dz = 0 \ldots (4$ and find the solution $k(y, z) = c \ldots 5)$, where c is constant. Since Eq.3) and Eq.5) are the general solution to the same DE, so Eq.3) and Eq.5) are equivalent, and $U(z)$, and substitute $U(z)$ in 3) we get the solution of 1.0.18.

Example 1.0.19 *: Solve*

1) $(y + z)dx + (z + x)dy + (x + y)dz = 0$

2) $z(z + y^2)dx + z(z + x^2)dy - xy(x + y)dz = 0$

Solution 1)

$$(y + z)dx + (z + x)dy + (x + y)dz = 0$$

$$P = y + z \qquad Q = z + x \qquad R = x + y$$

$$curlv = 0 \quad , \textit{so the Eq. is exact.}$$

$$vcurlv = 0 \quad , \textit{so the Eq. is integrable.}$$

let z be a constant, so that $dz = 0$ and 1.0.18 becomes:

$$(y+z)dx + (z+x)dy = 0 \qquad (\frac{1}{(y+z)(z+x)})$$

$$\frac{dx}{z+x} + \frac{dy}{y+z} = 0$$

$$\int \frac{dx}{z+x} + \int \frac{dy}{y+z} = f_1(z)$$

$$\ln(z+x) + \ln(y+z) = f_1(z) \Rightarrow \quad \ln((z+x)(z+y))f_1(z)$$

$$\Rightarrow \quad (z+x)(z+y) = f(z)$$

To find $f(z)$: let $x = 0$ then $z(z+y) = f(z) \ldots (3$

substitute x in the main Eq., we obtain

$$zdy + ydz = 0 \Rightarrow \quad d(zy) = 0$$

$$zy = c \ldots (5)$$

Eliminate y in (3 and (5

in Eq.(5 $y = \frac{c}{z}$, substitute y in (3, we get

$$z(z + \frac{c}{z}) = f(z) \Rightarrow \quad f(z) = z^2 + c$$

$$(z+x)(z+y) = z^2 + c \Rightarrow \quad z^2 + yz + xz + xy = z^2 + c$$

$$\therefore yz + xz + xy = c \quad \textit{is a Gs.}$$

2)

$$z(z+y^2)dx + z(z+x^2)dy - xy(x+y)dz = 0$$

$$P = z(z+y^2) \qquad Q = z(z+x^2) \qquad R = -xy(x+y)$$

$$curlv \neq 0 \quad , \textit{then the Eq. is not exact.}$$

$$vcurlv = 0 \qquad , \textit{then the Eq. is integrable.}$$

let y *be a constant, then the Eq. becomes:*

$$z(z+y^2)dx - xy(x+y)dz = 0$$

$$\frac{dx}{xy(x+y)} - \frac{dz}{z(z+y^2)} = 0$$

$$\frac{1}{xy(x+y)} = \frac{A}{xy} + \frac{B}{x+y} = \frac{Ax+Ay+Bxy}{xy(x+y)} = \frac{(A+B)x+Ay}{xy(x+y)}$$

$$A + By = 0 \ldots (i$$

$$Ay = 1 \Rightarrow \quad A = \frac{1}{y} \ldots (ii$$

substitute Eq.(ii in (i, we obtain

$$\frac{1}{y} + By = 0 \Rightarrow \quad B = \frac{-1}{y^2}$$

$$\therefore \frac{1}{xy(x+y)} = \frac{\frac{1}{y}}{xy} + \frac{\frac{-1}{y^2}}{(x+y)} = \frac{1}{xy^2} - \frac{1}{y^2(x+y)}$$

$$\frac{1}{z(z+y^2)} = \frac{A}{z} + \frac{B}{z+y^2} = \frac{Az + Ay^2 + Bz}{z(z+y^2)} = \frac{(A+B)z + Ay^2}{z(z+y^2)}$$

$$\therefore A + B = 0 \Rightarrow \quad B = -A$$

$$Ay^2 = 1 \Rightarrow \quad A = \frac{1}{y^2}, \quad then \quad B = \frac{-1}{y^2}$$

$$\therefore \frac{1}{z(z+y^2)} = \frac{\frac{1}{y^2}}{z} + \frac{\frac{-1}{y^2}}{z+y^2} = \frac{1}{zy^2} - \frac{1}{y^2(z+y^2)}$$

$$\therefore [\frac{1}{xy^2} - \frac{1}{y^2(x+y)}]dx - [\frac{1}{y^2 z} - \frac{1}{y^2(z+y^2)}]dz = 0$$

$$[\frac{1}{x} - \frac{1}{x+y}]dx - [\frac{1}{z} - \frac{1}{z+y^2}]dz = 0$$

$$\ln x - \ln(x+y) - \ln z + \ln(z+y^2) = f_1(y)$$

$$\ln \frac{x(z+y^2)}{z(x+y)} = f_1(y) \Rightarrow \quad \frac{x(z+y^2)}{z(x+y)} = f(y)$$

To find f(y): Let $z = 1$

then $\frac{x(1+y^2)}{x+y} = f(y) \ldots (3$

substitute $z = 1$ in the main Eq.

$$(1 + y^2)dx + (1 + x^2)dy = 0$$

$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0 \Rightarrow \quad \tan^{-1}(x) + \tan^{-1}(y) = constant$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}\left(\frac{1}{c}\right), \quad constant = \tan^{-1}\left(\frac{1}{c}\right)$$

$$\frac{x+y}{1-xy} = \frac{1}{c} \ldots (5$$

Eliminate x in Eqs. 5) and 3)
from Eq.5):

$$cx + cy = 1 - xy \Rightarrow \quad cx + xy = 1 - cy$$

$$x = \frac{1-cy}{c+y}$$

$$\frac{1-cy}{c+y} \cdot \frac{(1+y^2)}{\frac{1-cy}{c+y} + y} = f(y)$$

$$\frac{1-cy}{c+y} \cdot \frac{(1+y^2)}{\frac{1-cy+cy+y^2}{c+y}} = f(y) \Rightarrow \quad f(y) = 1 - cy$$

$$\therefore \frac{x(z+y^2)}{z(x+y)} = 1 - cy \quad \textit{is Gs.}$$

Exercise: *Solve the following Pfaff differential equations?*

1) $yz(1 + 4xz)dx - xz(1 + 2xz)dy - xydz = 0$

2) $yzdx + xzdy + xydz = 0$

3) $(1 + yz)dx + x(z - x)dy - (1 + xy)dz = 0$

4) $(y^2 - z^2)dx + (x^2 - z^2)dy + (x + y)(x + y + 2z)dz = 0$

Chapter 2

First Order Partial Differential Equation

A partial differential equation of order one in its most general form is an equation of the form

$$A(x, y)p + B(x, y)q + C(x, y)z = D(x, y) \tag{2.0.1}$$

Here, we will not consider problems of such generality but will focus instead on a smaller class of problems. For example, the equation 2.0.1 is said to be quasilinear equation in two variables if it is of the form

$$A(x, y, z)p + B(x, y, z)q = D(x, y, z)$$

Theorem 2.0.1 *: The general solution of quasilinear PDE of the first order*

$$A(x, y, z)p + B(x, y, z)q = C(x, y, z) \tag{2.0.2}$$

is $f(u, v) = 0$, where f is an arbitrary function, $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent first integrals of ODE

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$$

proof *: Since $u(x, y, z) = c_1 \Rightarrow \quad du = 0$*

$$u_x dx + u_y dy + u_z dz = 0 \tag{2.0.3}$$

and $v(x,y,z) = c_2$

$$v_x dx + v_y dy + v_z dz = 0 \tag{2.0.4}$$

from Eqs. 2.0.3 and 2.0.10, we get:

$$dx = \frac{u_y dy + u_z dz}{-u_x} \quad , dx = \frac{v_y dy + v_z dz}{-v_x} \Rightarrow$$

$$\frac{u_y dy + u_z dz}{-u_x} = \frac{v_y dy + v_z dz}{-v_x} \Rightarrow \quad u_y v_x dy + u_z v_x dz = v_y u_x dy + v_z u_x dz \Rightarrow$$

$$(u_y v_x - v_y u_x) dy = (v_z u_x - u_z v_x) dz$$

$$\frac{dy}{v_z u_x - u_z v_x} = \frac{dz}{u_y v_x - v_y u_x} \tag{2.0.5}$$

from Eqs. 2.0.3 and 2.0.10, also we can get

$$dy = \frac{u_x dx + u_z dz}{-u_y} \quad , dy = \frac{v_x dx + v_z dz}{-v_y} \Rightarrow$$

$$\frac{u_x dx + u_z dz}{-u_y} = \frac{v_x dx + v_z dz}{-v_y} \Rightarrow \quad u_x v_y dx + u_z v_y dz = v_x u_y dx + v_z u_y dz \Rightarrow$$

$$-(u_x v_y - v_x u_y) dx = -(v_z u_y - u_z v_y) dz$$

$$\frac{dx}{-v_z u_y + u_z v_y} = \frac{dz}{-u_x v_y + v_x u_y} \tag{2.0.6}$$

from Eqs. 2.0.5 and 2.0.6, we obtain

$$\frac{dx}{u_z v_y - v_z u_y} = \frac{dy}{v_z u_x - u_z v_x} = \frac{dz}{v_x u_y - u_x v_y} \tag{2.0.7}$$

By comparing 2.0.8 with $\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$, *we get*

$$\begin{aligned} A(x,y,z) &= u_z v_y - v_z u_y \\ B(x,y,z) &= v_z u_x - u_z v_x \\ C(x,y,z) &= v_x u_y - u_x v_y \end{aligned}$$

substitute in Eq. 2.0.2

$$(u_z v_y - v_z u_y)p + (v_z u_x - u_z v_x)q = v_x u_y - u_x v_y \qquad \textit{(multiplied by -1)}$$

$$\Rightarrow \quad \frac{\partial(u,v)}{\partial(y,z)}p + \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

By eliminating of arbitrary function $f(u,v) = 0$ *is a Gs.*

Example 2.0.1 *: Find the general solution(integral) of the following equations:*

1) $3p - 2yq = -1$

2) $y^2 z\frac{\partial z}{\partial x} - x^2 z\frac{\partial z}{\partial y} - x^2 y = 0$

Solution 1)

$$A = 3 \qquad B = -2y \qquad C = -1$$

The auxiliary equations are $\frac{dx}{3} = \frac{dy}{-2y} = \frac{dz}{-1}$

$$\begin{aligned} &\textit{since} \quad \frac{dx}{3} = \frac{dy}{-2y} \Rightarrow \quad 2dx + 3\frac{dy}{y} = 0 \\ &\Rightarrow \quad 2x + 3\ln y = c_1 \Rightarrow \quad \therefore v(x,y,z) = 2x + 3\ln y \\ &\textit{since} \quad \frac{dx}{3} = \frac{dz}{-1} \Rightarrow \quad dx + 3dz = 0 \\ &\Rightarrow \quad x + 3z = c_2 \Rightarrow \quad \therefore v(x,y,z) = x + 3z \end{aligned}$$

$$\therefore f(u,v) = 0 \Rightarrow \quad f(2x + 3\ln y, x + 3z) = 0 \quad \textit{is Gs.}$$

2)

$$y^2 z\frac{\partial z}{\partial x} - x^2 z\frac{\partial z}{\partial y} = x^2 y \qquad A = y^2 z \qquad B = -x^2 z \qquad C = x^2 y$$

The auxiliary equations are $\dfrac{dx}{y^2 z} = \dfrac{dy}{-x^2 z} = \dfrac{dz}{x^2 y}$

$$\text{since} \quad \frac{dx}{y^2 z} = \frac{dy}{-x^2 z} \Rightarrow \quad x^2 dx + y^2 dy = 0$$

$$\Rightarrow \quad x^3 + y^3 = c_1 \Rightarrow \quad \therefore x^3 + y^3 = u(x, y, z)$$

$$\text{since} \quad \frac{dy}{-x^2 z} = \frac{dz}{x^2 y} \Rightarrow \quad ydy + zdz = 0$$

$$y^2 + z^2 = c_2 \Rightarrow \quad \therefore y^2 + z^2 = v(x, y, z)$$

$$\therefore f(u, v) = 0 \Rightarrow \quad f(x^3 + y^3, y^2 + z^2) = 0 \quad \text{is Gs.}$$

Methods for solving $$\frac{dx}{A(x, y, z)} = \frac{dy}{B(x, y, z)} = \frac{dz}{C(x, y, z)} \qquad (2.0.8)$$

Case 1) *If one of the variables x, y and z is not appear in one of the functions A, B and C, so we solve Eq. 2.0.8 by the following way*

$$\text{suppose} \quad \frac{dy}{B} = \frac{dz}{C} \quad \text{is not contain } x. \qquad (2.0.9)$$

Then we rearrange Eq. 2.0.9 to get $\dfrac{dy}{dz} = f(y, z)$, *and solve the last equation for z and substitute z in A and B, so we obtain*

$$\frac{dx}{A} = \frac{dy}{B} \quad \text{or we rearrange to get}$$

$$\frac{dy}{dx} = g(x, y, c_1) \quad \textit{and we find the solution.}$$

Example 2.0.2 *: Find the integral curves of the following equation*

$$\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}$$

Solution : *First we find*

$$\frac{dy}{y} = \frac{dz}{z+y^2} \Rightarrow \quad \frac{dz}{dy} = \frac{z+y^2}{y} \Rightarrow \quad \frac{dz}{dy} = \frac{z}{y} + y$$

$$\Rightarrow \quad \frac{dz}{dy} - \frac{1}{y}z = y \quad \textit{is first order LDE.}$$

$$z = \frac{\int e^{\int \frac{-1}{y}dy} ydy + c_1}{e^{\int \frac{-1}{y}dy}} = \frac{\int e^{-\ln y} ydy + c_1}{e^{-\ln y}} = \frac{\int \frac{1}{y} ydy + c_1}{\frac{1}{y}}$$

$$z = y[\int dy + c_1] = y^2 + c_1 y \tag{2.0.10}$$

substitute z in $\dfrac{dx}{x+z} = \dfrac{dy}{y}$

$$\frac{dx}{x+y^2+c_1y} = \frac{dy}{y} \Rightarrow \quad \frac{dx}{dy} = \frac{x+y^2+c_1y}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x}{y} + y + c_1 \Rightarrow \quad \frac{dx}{dy} - \frac{1}{y}x = y + c_1 \quad \textit{is first order LDE}$$

$$x = \frac{\int e^{\int -\frac{1}{y}dy}(y+c_1)dy + c_2}{e^{\int -\frac{1}{y}dy}} = \frac{\int e^{-\ln y}(y+c_1)+c_2}{e^{-\ln y}}$$

$$= y[\int (1+\frac{c_1}{y}) + c_2] = y[y + c_1 \ln y + c_2]$$

$$x = y^2 + c_1 y \ln y + c_2 y \tag{2.0.11}$$

Then the solution of the equation is determined by Eqs. 2.0.10 and 2.0.11.

Case 2) *If* $v(x,y,z) = c$ *is a solution of Eq. (2.0.8),* $v_x dx + v_y dy + v_z dz = 0 \Rightarrow$ $v_x A + v_y B + v_z C = 0$*, after that we try to find the functions* A_1, B_1 *and* C_1 *s.t*

$AA_1 + BB_1 + CC_1 = 0,$ *and s.t* $\exists$ *a function* v *with properties*

$v_x = A_1, v_y = B_1$ *and* $v_z = C_1,$

and

$A_1dx + B_1dy + C_1dz = 0$ *is an exact Eq.*

Example 2.0.3 *: Find the integral curves of the following Eq.*

1) $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

2) $\dfrac{dx}{y-z} = \dfrac{dy}{x-y} = \dfrac{dz}{z-x}$

Solution 1)

$$A = x(y-z) \qquad B = y(z-x) \qquad C = z(x-y)$$

$$1.A + 1.B + 1.C = xy - zx + yz - yx + zx - zy = 0$$

$$\therefore dx + dy + dz = 0 \Rightarrow \quad x + y + z = c_1$$

$$\therefore u(x,y,z) = x + y + z$$

$$\frac{1}{x}A + \frac{1}{y}B + \frac{1}{z}C = \frac{1}{x}x(y-z) + \frac{1}{y}y(z-x) + \frac{1}{z}z(x-y)$$

$$y - z + z - x + x - y = 0$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \Rightarrow \quad \ln x + \ln y + \ln z = c$$

$$\Rightarrow \quad \ln xyz = c \Rightarrow \quad xyz = c_2,\ \textit{where } c_2 = e^c$$

$$v(x,y,z) = xyz$$

2)

$A = y - z \qquad B = x - y \qquad C = z - x$

The auxiliary Eqs. $\dfrac{dx}{y-z} = \dfrac{dy}{x-y} = \dfrac{dz}{z-x}$

since $A + B + C = y - z + x - y + z - x = 0$

$\Rightarrow \quad \therefore dx + dy + dz = 0 \Rightarrow \quad x + y + z = c_1$

$\therefore u(x, y, z) = x + y + z$

$xA + zB + yC = x(y - z) + z(x - y) + y(z - x) = xy - xz + zx - zy + yz - yx = 0$

$\therefore xdx + zdy + ydz = 0 \Rightarrow \quad xdx + d(zy) = 0 \Rightarrow \quad \frac{1}{2}x^2 + yz = c_2$

$\therefore v(x, y, z) = \frac{1}{2}x^2 + yz$

$f(u, v) = 0 \Rightarrow \quad f(x + y + z, \frac{1}{2}x^2 + yz) = 0$ *is a Gs.*

Example 2.0.4 *: Solve the following pffaf equations?*

1) $(y(x + y) + \alpha z)p + (x(x + y) - \alpha z)q = u(x, y)$

2) $(xz - y)p + (yz - x)q = (1 - z^2)$

3) $x^2(y^3 - z^3)p + y^2(z^3 - x^3)q - z^2(x^3 - y^3) = 0$

4) $x^2p + y^2q = (x + y)z$

5) $(y + z)p + (z + x)q = x + y$

Solution 2) *The auxiliary equations are* $\dfrac{dx}{xz-y} = \dfrac{dy}{yz-x} = \dfrac{dz}{1-z^2}$

$$A = xz - y, \qquad B = yz - x, \qquad C = 1 - z^2$$

$$A + B = xz - y + yz - x = z(x+y) - (x+y)$$

$$\therefore \frac{dx+dy}{(x+y)(z-1)} = \frac{dz}{1-z^2} \Rightarrow \quad \frac{dx+dy}{x+y} = \frac{-dz}{z+1}$$

$$\Rightarrow \quad \ln(x+y) = -\ln(z+1) + c \Rightarrow \quad \ln(x+y) + \ln(z+1) = c$$

$$\Rightarrow \quad \ln(x+y)(z+1) = c \Rightarrow \quad (x+y)(z+1) = c_1$$

$$\therefore \; u(x,y,z) = (x+y)(z+1)$$

$$A - B = xz - y - yz + x = z(x-y) + (x-y) = (x-y)(z+1)$$

$$\frac{dx-dy}{(x-y)(z+1)} = \frac{dz}{1-z^2} \Rightarrow \quad \frac{dx-dy}{x-y} = \frac{dz}{1-z}$$

$$\ln(x-y) = -\ln(z-1) + c \Rightarrow \quad \ln(x-y) + \ln(z-1) = c$$

$$\Rightarrow \quad \ln(x-y)(z-1) = c \Rightarrow \quad (x-y)(z-1) = c_2$$

$$\therefore \; v(x,y,z) = (x-y)(1-z)$$

Integral Surface Passing Through A given Curve

Suppose we are given a quasilinear PDE of the form

$$A(x,y,z)p + B(x,y,z)q = C(x,y,z) \tag{2.0.12}$$

We try to determine the integral surfaces passing through a given curve of Eq. 2.0.12

Assume the given curve has a parameter equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t) \tag{2.0.13}$$

where t is a real parameter.
Now, we find the Gs. to the Eq. 2.0.12, so

$$let \quad u(x, y, z) = c_1 \quad and \quad v(x, y, z) = c_2$$

are two solution to the auxiliary DE.

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$$

then the Gs. is $f(u, v) = 0$.

To find the integral surfaces passing through the given curve which has a parameter Eq. 2.0.13.

$$u(x(t), y(t), z(t)) = c_1 \tag{2.0.14}$$
$$v(x(t), y(t), z(t)) = c_2 \tag{2.0.15}$$

and we eliminate parameter t from Eq. 2.0.14 to get a relation.

Example 2.0.5 *: Find the integral surface of the equation $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$, passing through the circle $z = 1, \quad x^2 + y^2 = 4$*
Solution :

$$A = y, \qquad B = -x, \qquad C = 0$$

$$\text{The auxiliary Eqs.} \qquad \frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$$

$$\text{since} \quad \frac{dx}{y} = \frac{dy}{-x} \Rightarrow \quad x^2 + y^2 = c_1$$
$$dz = 0 \Rightarrow \quad z = c_2$$

The parameter equations : $z = 1, \quad x = t$

$$y^2 = 4 - t^2$$
$$\therefore \quad c_1 = 4$$
$$c_2 = 1$$

$$c_1 + c_2 = 5$$
$$x^2 + y^2 = 5 \quad \textit{is Gs.}$$

Example 2.0.6 : *Find the integral surface of the Eq.*

$$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$$

which contain the straight line $x + y = 0, \quad z = 1$

Solution :

$$A = x(y^2 + z), \quad B = -y(x^2 + z), \quad C = (x^2 - y^2)z$$

The auxiliary equations are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$$

$$\textit{since} \quad xA + yB - C = x^2y^2 + x^2z - y^2x^2 - y^2z - x^2z + y^2z = 0$$

$$\therefore \quad xdx + ydy - dz = 0 \Rightarrow \quad x^2 + y^2 - 2z = c_1$$

$$\textit{since} \quad \frac{1}{x}A + \frac{1}{y}B + \frac{1}{z}C = y^2 + z - x^2 - z + x^2 - y^2 = 0$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \Rightarrow \quad xyz = c^2$$

$$\therefore \quad f(x^2 + y^2 - 2z, xyz) = 0 \quad \textit{is Gs.}$$

The parameter Eqs. are $z = 1, \quad x = t, \quad y = -t$

$$\text{since} \quad xyz = c_2 \Rightarrow \quad t(-t).1 = c_2 \Rightarrow \quad c_2 = -t^2$$

$$x^2 + y^2 - 2z = c_1 \Rightarrow \quad t^2 + (-t)^2 - 2(1) = c_1$$

$$\Rightarrow \quad t^2 + t^2 - 2 = c_1 \Rightarrow \quad c_1 = 2t^2 - 2$$

$$c_1 + 2c_2 = 2t^2 - 2 - 2t^2 = -2 \Rightarrow \quad c_1 + 2c_2 = -2$$

$$\therefore \quad x^2 + y^2 - 2z + 2xyz = -2 \quad \text{is integral surface.}$$

Exercise : *Find the integral surface passing through a given curves:*

1) $2y(z-3)p + (2x-z)q = y(2x-3) \qquad \text{for} \quad z = 0, \quad x^2 + y^2 = 2x$

2) $yzp + q = 0 \qquad \text{for} \quad x = 0, \quad u = y^2$

Simultaneous equations of first order PDE

In this section we suppose two general PDE of the form

$$f(x, y, z, p, q) = 0$$
$$g(x, y, z, p, q) = 0$$

Assume the last system can be solved with respect to p and q

$$\text{i.e.} \qquad \frac{\partial z}{\partial x} = A(x, y, z)$$
$$\frac{\partial z}{\partial y} = B(x, y, z)$$

$$since \quad \frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}\frac{\partial z}{\partial y} \Rightarrow \quad \frac{\partial^2 z}{\partial y \partial x} = A_y + BA_z$$

$$\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial z}\frac{\partial z}{\partial x} \Rightarrow \quad \frac{\partial^2 z}{\partial x \partial y} = B_x + AB_z$$

Consider A, B, A_y, A_z, B_x *and* B_z *are contain in the region*

Then $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} \Rightarrow \quad A_y + BA_z = B_x + AB_z$

$\therefore$ *The system is integrable if satisfies last equation.*

Example 2.0.7 *: Find the Gs. for the following systems*

1) $\dfrac{\partial z}{\partial x} = 3y^2, \quad \dfrac{\partial z}{\partial y} = \dfrac{2}{y^2} + \dfrac{2z}{y} - 3y^2$

2) $\dfrac{\partial z}{\partial x} = 2y^2, \quad \dfrac{\partial z}{\partial y}\dfrac{b}{2y^2} + \dfrac{2z}{y} - ay^2$

Solution 1):

$A = 3y^2, \quad B = \dfrac{2}{y^2} + \dfrac{2z}{y} - 3y^2$

$A_y = 6y \qquad B_x = 0$

$A_z = 0 \qquad B_z = \dfrac{2}{y}$

$A_y + BA_z = 6y + 0 = 6y$

$B_x + AB_z = 0 + 3y^2.\dfrac{2}{y} = 6y$

$\therefore A_y + BA_z = B_x + AB_z,$ *so the system is integrable*

$\dfrac{\partial z}{\partial x} = 3y^2 \Rightarrow \quad z = \displaystyle\int 3y^2 dx = 3y^2 x + G(y),$ *where* $G(y)$ *is a function of* y *only*

$\textit{To find } G(y): \dfrac{\partial z}{\partial y} = 6yx + \dfrac{dG(y)}{dy} \Rightarrow \quad \dfrac{2}{y^2} + \dfrac{2z}{y} - 3y^2 = 6yx + \dfrac{dG(y)}{dy}$

$\dfrac{dG(y)}{dy} - \dfrac{2z}{y} + 6yx = \dfrac{2}{y^2} - 3y^2$

$\dfrac{dG(y)}{dy} - \dfrac{2}{y}(z - 3y^2x) = \dfrac{2}{y^2} - 3y^2$

$\dfrac{dG(y)}{dy} - \dfrac{2}{y}G(y) = \dfrac{2}{y^2} - 3y^2$

$$G(y) = \frac{\int e^{\int \frac{-2}{y}} (\frac{2}{y^2} - 3y^2)dy}{e^{\int \frac{-2}{y}}}$$

$$= y^2[\int \frac{1}{y^2}(\frac{2}{y^2} - 3y^2)dy + c]$$

$$= y^2[\int (\frac{2}{y^4} - 3)dy + c]$$

$$= y^2[\frac{-2}{3y^3} - 3y + c]$$

$$\Rightarrow \quad G(y) = \frac{-2}{3y} - 3y^3 + cy^2$$

$$z = 3xy^2x + \frac{-2}{3y} - 3y^3 + cy^2$$

2)

$$A = ay^2, \qquad B = \frac{b}{2y^2} + \frac{2z}{y} + ay^2$$

$$A_y = 2ay \qquad B_x = 0$$

$$A_z = 0 \qquad B_z = \frac{2}{y}$$

$$A_y + BA_z = 2ay + 0 = 2ay$$

$$B_x + AB_z = 0 + ay^2(\frac{2}{y}) = 2ay$$

$\therefore A_y + BA_z = B_x + AB_z,$ *so the system is integrable.*

$$\frac{\partial z}{\partial x} = ay^2 \Rightarrow \quad z = \int ay^2 dx \Rightarrow \quad z = ay^2 x + G(y), \text{ where } G(y) \text{ is a function of } y \text{ only.}$$

$$\frac{\partial z}{\partial y} = 2ayx + \frac{dG(y)}{dy}$$

$$\frac{b}{2y^2} + \frac{2z}{y} + ay^2 = 2ayx + \frac{dG(y)}{dy}$$

$$\frac{dG(y)}{dy} = \frac{b}{2y^2} + \frac{2}{y}(z - axy^2) - ay^2$$

$$\frac{dG(y)}{dy} = \frac{b}{2y^2} + \frac{2}{y}G(y) - ay^2$$

$$\frac{dG(y)}{dy} - \frac{2}{y}G(y) = \frac{b}{2y^2} - ay^2 \text{ is first order LDE.}$$

Nonlinear First Order Equations(Charpit's Method)

The most general PDE of order one can be written by

$$f(x, y, z, p, q) = 0 \tag{2.0.16}$$

The fundamental idea in this method is the introduction of a second PDE of order one

$$g(x, y, z, p, q) = a \tag{2.0.17}$$

where a is a constant. Now, solving Eqs. (2.0.16) and (2.0.17) with respect to p and q.

$$i.e. \quad \frac{\partial z}{\partial x} = p = A(x, y, z)$$

$$\frac{\partial z}{\partial y} = q = B(x, y, z)$$

and the system is integrable iff

$$A_y + BA_z = B_x + AB_z \Rightarrow \quad \frac{\partial p}{\partial y} + q\frac{\partial p}{\partial z} - \frac{\partial q}{\partial x} - p\frac{\partial q}{\partial z} = 0 \tag{2.0.18}$$

we differentiate Eqs. (2.0.16) and (2.0.17) with respect to z

$$f_z + f_p\frac{\partial p}{\partial z} + f_q\frac{\partial q}{\partial z} = 0$$

$$g_z + g_p\frac{\partial p}{\partial z} + g_q\frac{\partial q}{\partial z} = 0$$

$$\begin{pmatrix} f_p & f_q \\ \\ g_p & g_q \end{pmatrix} \begin{pmatrix} \dfrac{\partial p}{\partial z} \\ \\ \dfrac{\partial q}{\partial z} \end{pmatrix} = - \begin{pmatrix} f_z \\ \\ g_z \end{pmatrix}$$

by using Gramer's rule we get

$$\frac{\partial p}{\partial z} = \frac{-\begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}} \quad and \quad \frac{\partial q}{\partial z} = \frac{-\begin{vmatrix} f_p & f_z \\ g_p & g_z \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}}$$

we also differentiate Eqs. (2.0.16) and (2.0.17) with respect to x and y

$$f_x + f_p \frac{\partial p}{\partial x} + f_q \frac{\partial q}{\partial x} = 0$$

$$g_x + g_p \frac{\partial p}{\partial x} + g_q \frac{\partial q}{\partial x} = 0$$

$$\begin{pmatrix} f_p & f_q \\ g_p & g_q \end{pmatrix} \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial q}{\partial x} \end{pmatrix} = - \begin{pmatrix} f_x \\ g_x \end{pmatrix}$$

$$\frac{\partial p}{\partial x} = \frac{-\begin{vmatrix} f_x & f_q \\ g_x & g_q \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}} \quad and \quad \frac{\partial q}{\partial x} = \frac{-\begin{vmatrix} f_p & f_x \\ g_p & g_x \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}}$$

and

$$f_y + f_p \frac{\partial p}{\partial y} + f_q \frac{\partial q}{\partial y} = 0$$

$$g_y + g_p \frac{\partial p}{\partial y} + g_q \frac{\partial q}{\partial y} = 0$$

$$\begin{pmatrix} f_p & f_q \\ \\ g_p & g_q \end{pmatrix} \begin{pmatrix} \dfrac{\partial p}{\partial y} \\ \\ \dfrac{\partial q}{\partial y} \end{pmatrix} = - \begin{pmatrix} f_y \\ \\ g_y \end{pmatrix}$$

$$\frac{\partial p}{\partial y} = \frac{-\begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}} \quad \textit{and} \quad \frac{\partial q}{\partial y} = \frac{-\begin{vmatrix} f_p & f_y \\ g_p & g_y \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}}$$

substitute $\dfrac{\partial p}{\partial y}, \dfrac{\partial p}{\partial z}, \dfrac{\partial q}{\partial x}$ *and* $\dfrac{\partial q}{\partial z}$ *in Eq. (2.0.18)*

$$\frac{-\begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}} + q\frac{-\begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}} - \frac{-\begin{vmatrix} f_p & f_x \\ g_p & g_x \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}} - p\frac{-\begin{vmatrix} f_p & f_z \\ g_p & g_z \end{vmatrix}}{\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix}} = 0$$

$$f_p \frac{\partial g}{\partial x} + f_q \frac{\partial g}{\partial y} + (pf_p + qf_q) \frac{\partial g}{\partial z} - (f_x + pf_z) \frac{\partial g}{\partial p} - (f_y + qf_z) \frac{\partial g}{\partial q} = 0 \quad (2.0.19)$$

Now, we can solve Eq. (2.0.19) auxiliary system:

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z} \quad (2.0.20)$$

by using Eq. (2.0.20) we can find p and q, say p, let $p = \phi_1(x, y, z, a)$*, substituting p in Eq. (2.0.16) we obtain* $q = \phi_2(x, y, z, a)$*, where* ϕ_1 *and* ϕ_2 *are arbitrary functions.*

since z is a function of x, y

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \Rightarrow \quad dz = pdx + qdy$$

$$dz = \phi_1 dx + \phi_2 dy \tag{2.0.21}$$

The solution of (2.0.21) *is called a complete integral of* (2.0.16).
Remark : *To find the singular solution if it exists, let $\phi(x, y, z, a, b) = 0$ be a complete integral of* (2.0.16) *and we differentiate ϕ_1 and ϕ_2 with respect to a and b respectively, i.e.*

$$\frac{\partial \phi_1}{\partial a} = 0 \quad and \quad \frac{\partial \phi_2}{\partial b} = 0$$

from the last two equations we eliminate constants a and b and we get a singular solution.

Example 2.0.8 *: Find a complete integral of the following equation.*

$$p = (z + qy)^2 \tag{2.0.22}$$

Solution

$f = p - (z + qy)^2 = 0$

$f_p = 1 \qquad , f_q = -2(z + qy)\, y \qquad , f_x = 0$

$f_y = -2q(z + qy) \qquad , f_z = -2(z + qy)$

$$\frac{dx}{1} = \frac{dy}{-2y(z + qy)} = \frac{dz}{p - 2yq(z + qy)} = \frac{-dp}{0 - 2p(z + qy)} = \frac{-dq}{-2q(z + qy) - 2q(z + qy)}$$

$$\frac{dy}{-2y(z + qy)} = \frac{-dq}{-4q(z + qy)} \Rightarrow \quad \frac{-dy}{y} = \frac{dq}{2q} \Rightarrow$$

$$-2\frac{dy}{y} = \frac{dq}{q} \Rightarrow \quad \ln q = -2\ln y + c \Rightarrow \quad q = \frac{a}{y^2} \tag{2.0.23}$$

$dz = pdx + qdy$

$dz = (z + \frac{a}{y})^2 dx + \frac{a}{y^2} dy = 0$

$(z + \frac{a}{y})^2 dx + \frac{a}{y^2} dy - dz = 0$ *is pfaffian Eq. in one variable separable.*

$\phi(x, y, z, a, b) = x + \frac{1}{z + \frac{a}{y}} + b = 0$ *is complete integral.*

$\frac{\partial \phi}{\partial b} = 1 \neq 0,$ *so the Eq. has no singular solution.*

Example 2.0.9 *: Solve*

$$q = -xp + p^2 \tag{2.0.24}$$

Solution :

$f(x, y, z, p, q) = q + xp - p^2$

$f_x = p \quad , f_y = 0 \quad , f_z = 0 \quad , f_p = x - 2p \quad , f_q = 1$

$\frac{dx}{x - 2p} = \frac{dy}{1} = \frac{dz}{p(x - 2p)} = \frac{-dp}{p} = \frac{-dq}{0}$

$\frac{dy}{1} = \frac{-dp}{p} \Rightarrow \quad y + \ln p = c \Rightarrow \quad \ln p = c - y$

$p = e^{c-y} = ae^{-y}$

Substitute p *in Eq.* (2.0.24), *we get*

$q = x(ae^{-y}) + a^2e^{-2y}$

Since $dz = pdx + qdy$

$= ae^{-y}dx + (-axe^{-y} + a^2e^{-2y})dy$

$ae^{-y}dx + (-axe^{-y} + a^2e^{-2y})dy - dz = 0$ *is one variable separable pfaffian Eq.*

$ae^{-y}dx + (-axe^{-y} + a^2e^{-2y})dy$ *is exact* $\exists$ *a function* $U(x, y)$ *s.t.*

$\frac{\partial U}{\partial x} = ae^{-y} \Rightarrow \quad U = \int ae^{-y}dx \Rightarrow \quad U = axe^{-y} + G(y)$

$\frac{\partial U}{\partial y} = -axe^{-y} + \frac{dG(y)}{dy} \Rightarrow \quad -axe^{-y} + a^2e^{-2y} = -axe^{-y} + \frac{dG(y)}{dy}$

$\frac{dG(y)}{dy} = a^2e^{-2y} \Rightarrow \quad G(y) = \frac{-a^2}{2}e^{-2y}$

$U = axe^{-y} - \frac{a^2}{2}e^{-2y}$

Substitute U *in pfaffian Eq.*

$$axe^{-y} - \frac{a^2}{2}e^{-2y} - \int dz = c$$

$$axe^{-y} - \frac{a^2}{2}e^{-2y} - z = c \quad \text{is } Gs.$$

Exercise :

1) $p^2x + q^2y = z$

2) $2(z + xp + yq) = yp^2$

3) $z + xp - x^2yq^2 - x^3pq = 0$

Special Types Of The Nonlinear PDEFO

Case 1) Equations belonging to this form involving only p ***and*** q***,*** $f(p, q) = 0$ *since* f_x, f_y *and* f_z *are all equal to zero. Then the charpit's Eqs. are*

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{0} = \frac{-dq}{0}$$

Therefore $dp = 0 \Rightarrow \quad p = a$, *where* a *is a constant, substitute* p *in* $f(p, q) = 0$ *get* $f(a, q) = 0$ *or* $q = h(a)$. *Since* $dz = pdx + qdy \Rightarrow \quad dz = adx + h(a)dy$ *is separable.* $\therefore u = ax + h(a)y + c$, *is the complete integral and has no singular solution.*

Example 2.0.10 *: Solve*

1) $pq = 1$

2) $(\frac{\partial z}{\partial x})^2 \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

Solution 1) *Since the equation involving* p *and* q *only.*
let $p = a$, *where* a *is a constant, substitute* p *in the main equation.*

$$aq = 1 \Rightarrow \quad q = \frac{1}{a}, a \neq 0$$

$$dz = pdx + qdy \Rightarrow \quad dz = adx + \frac{1}{a}dy \quad \textit{is separable.}$$

$$z = ax + \frac{1}{a}y + c \quad \textit{is the complete integral.}$$

2) $p^2q + p = q$, *since the equation just involving* p *and* q, *let* $p = a$, *substitute* p *in the Eq.*

$$a^2q + a = q \Rightarrow \quad q(a^2 - 1) = -a \Rightarrow \quad q = \frac{-a}{(a^2 - 1)}$$

$$\therefore z = ax - \frac{a}{a^2 - 1}y + c, \quad \textit{is the complete integral.}$$

Case 2) Equations Not involving the independent variables

Let PDE be of the form $f(p, q, z) = 0 \ldots 1)$*, then* $f_x = f_y = 0$*, the charpit's equations are*

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

$$since \quad \frac{-dp}{pf_z} = \frac{-dq}{qf_z} \Rightarrow \quad \frac{dp}{p} = \frac{dq}{q} \Rightarrow \quad \ln p = \ln q + c$$

$$\ln \frac{p}{q} = c \Rightarrow \quad \frac{p}{q} = a \Rightarrow \quad p = aq, \quad a \text{ is constant.}$$

substituting in Eq. 1) obtain $f(aq, q, z) = 0$*, and we can find* q*, i.e.* $q = h(a, z) \Rightarrow p = ah(a, z)$*, since* $dz = pdx + qdy = ah(a, z)dx + h(a, z)dy \Rightarrow \quad \frac{dz}{h(a, z)} = adx + dy$ *is separable Eq. and has no singular solution.*

Example 2.0.11 *: Solve the PDEs*

1) $zpq = p + q$

2) $pq = z$

Solution 1) *Since the Eq. not involving independent variables.*
Let $p = aq$*, and substitute in the Eq.*

$$azq^2 = (a + 1)q \Rightarrow \quad azq = a + 1 \Rightarrow \quad q = \frac{a + 1}{az}$$

$$\therefore p = a\,\frac{a + 1}{az} = \frac{a + 1}{z}$$

$$dz = pdx + qdy \Rightarrow \quad dz = \frac{a + 1}{z}dx + \frac{a + 1}{az}dy \Rightarrow$$

$$azdz = a(a + 1)dx + (a + 1)dy, \quad \text{is separable Eq.}$$

$$\frac{a}{2}z^2 = a(a + 1)x + (a + 1)y + c \quad \text{is the complete integral.}$$

2) *Since the Eq. is just involving p, q and z.*

Let $p = a^2q$, substitute in the Eq.

$$a^2q^2 = z \Rightarrow \quad q^2 = \frac{z}{a^2} \Rightarrow \quad q = \frac{1}{a}\sqrt{z}$$

$$\therefore p = a^2\frac{1}{a}\sqrt{z} \Rightarrow \quad p = a\sqrt{z}$$

$$dz = pdx + qdy \Rightarrow \quad dz = a\sqrt{z}dx + \frac{1}{a}\sqrt{z}dy$$

$$\frac{adz}{\sqrt{z}} = a^2dx + dy, \quad \textit{is separable Eq.}$$

$$2a\sqrt{z} = a^2x + y + c \quad \textit{is the complete integral.}$$

Case 3) Separable Equations

If the PDE can be written of the form

$$g(x, p) = h(y, q) \tag{2.0.25}$$

$$\textit{Let} \quad f = g - h$$

$$f_x = g_x - 0 = g_x$$
$$f_y = 0 - h_y = -h_y$$
$$f_z = 0$$
$$f_p = g_p - 0 = g_p$$
$$f_q = 0 - h_q = -h_q$$

$$\frac{dx}{g_p} = \frac{dy}{-h_q} = \frac{dz}{pg_p + qh_q} = \frac{-dp}{g_x} = \frac{-dq}{-h_q}$$

$$\textit{from} \quad \frac{dx}{g_p} = \frac{-dp}{g_x} \Rightarrow \quad g_x dx + g_p dp = 0 \Rightarrow$$

$$d(g(x,p)) = 0 \ldots 2) \Rightarrow \quad g(x,p) = a, \quad \textit{where } a \textit{ is a constant.}$$

$$\textit{from} \quad \frac{dy}{-h_q} = \frac{-dq}{-h_y} \Rightarrow \quad h_y\, dy + h_q\, dq = 0$$

$$d(h(y,q)) = 0 \Rightarrow \quad h(y,q) = a \ldots 3)$$

Eqs. 2) and 3) may be solved to get

$$p = g_1(x,a)$$
$$q = h_1(y,a)$$

then $dz = g_1(x,a)dx + h_1(y,a)dy$, *is separable Eq. and has no singular solution.*

Example 2.0.12 *: Find the complete integral for the following Eqs.*

1) $p^2y(1+x^2) = qx^2$

2) $p^2 - q + y^2 - x^2 = 0$

Solution :

$$p^2y(1+x^2) = qx^2$$

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} \quad \textit{is separable equation.}$$

Let $\frac{q}{y} = a^2$, *where a is a constant.*

$$q = a^2y$$

$$p^2\frac{(1+x^2)}{x^2} = a^2 \Rightarrow \quad p^2 = \frac{a^2x^2}{1+x^2} \Rightarrow \quad p = \frac{ax}{\sqrt{1+x^2}}$$

$$dz = pdx + qdy$$

$$dz = \frac{ax}{\sqrt{1+x^2}}dx + a^2ydy \quad \textit{is separable Eq.}$$

$$\int dz = \int (1+x^2)^{\frac{-1}{2}} xdx + a^2 \int ydy$$

$$z = 2a\sqrt{1+x^2} + \frac{a^2}{2}y^2 + c \quad \textit{is the complete integral.}$$

$2) p^2 - q + y^2 - x^2 = 0$

$p^2 - x^2 = q - y^2 \quad$ *is separable Eq.*

Let $q - y^2 = a^2$*, where* a *is a constant.*

$q = a^2 + y^2$

$p^2 - x^2 = a^2 \Rightarrow \quad p^2 = a^2 + x^2 \Rightarrow \quad p = \sqrt{a^2 + x^2}$

$dz = pdx + qdy$

$dz = \sqrt{a^2+x^2}dx + (a^2+y^2)dy \quad$ *is separable Eq.*

$$\int dz = \int \sqrt{a^2+x^2}dx + \int (a^2+y^2)dy + c$$

$\int \sqrt{a^2+x^2}dx \qquad let\ x = atan\theta \Rightarrow \quad dx = asec^2\theta \Rightarrow \quad \therefore a^2 + x^2 = a^2 + a^2tan^2\theta$

$$\int a\,sec\theta\ a\,sec^2\theta d\theta$$

$$a^2 \int sec^3\theta \, d\theta \qquad u = sec\theta \Rightarrow \quad du = sec\theta \; tan\theta \, d\theta$$

$$dv = sec^2\theta \, d\theta \Rightarrow \quad v = tan\theta$$

$$= a^2(sec\theta \; tan\theta - \int tan\theta \, (sec\theta \, tan\theta))$$

$$= a^2(sec\theta \; tan\theta - \int sec\theta \; tan^2\theta \, d\theta)$$

$$= a^2(sec\theta \; tan\theta - \int sec\theta \; (sec^2\theta - 1) \, d\theta)$$

$$= a^2(sec\theta \; tan\theta - \int sec^3\theta d\theta + \int sec\theta d\theta)$$

$$= \frac{a^2}{2}(sec\theta \; tan\theta + \int sec\theta \; \frac{sec\theta + tan\theta}{sec\theta + tan\theta} d\theta)$$

$$= \frac{a^2}{2}(sec\theta \; tan\theta + \ln(sec\theta + tan\theta))$$

$$z = \frac{a^2}{2}(sec\theta \; tan\theta + \ln(sec\theta + tan\theta)) + a^2 y + \frac{y^3}{3} + c$$

Case 4 *A PDE of the first order is said to be of the Clairaut's type, if it can be written of the form*

$$z = px + qy + g(p, q) \tag{2.0.26}$$

$$let \quad f = px + qy - z + g(p, q)$$

$$f_x = p \qquad f_p = x + g_p$$

$$f_y = q \qquad f_q = y + g_q$$

$$f_z = -1$$

$$\frac{dx}{x+g_p} = \frac{dy}{y+g_q} = \frac{dz}{xp+pg_p+qy+qg_q} = \frac{-dp}{0} = \frac{-dq}{0}$$

$$since \quad dp = 0 \Rightarrow p = a$$

$$dq = 0 \Rightarrow q = b$$

where a and b are constants.
substitute these two Eqs. in (2.0.26) obtain the complete integral.

example : *Find a complete integral of the following equations:*

1) $(p+q)(z-xp-yq) = 1$

2) $pqz = p^2(xq+p^2) + q^2(yp+q^2)$

3) $(p^2-q^2)(z-qy-px) = 2$

Solution 1)

$$z - xp - yq = \frac{1}{p+q} \Rightarrow \quad z = xp + yq + \frac{1}{p+q} \quad \text{is Clairaut's equation.}$$

let $p = a$ *and* $q = b$, *where* a *and* b *are constants.*

$$\therefore \; z = ax + by + \frac{1}{a+b} \text{ is the complete integral.}$$

2)

$$z = \frac{p^2}{qp}(xq+p^2) + \frac{q^2}{pq}(yp+q^2)$$

$$= xp + \frac{p^3}{q} + qy + \frac{q^3}{p}$$

$$z = xp + qy + + \frac{p^3}{q} + \frac{q^3}{p}$$

let $p = a,\ q = b,$ *where* a *and* b *are constants.*

$$\therefore z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$$

3)

$$z - qy - px = \frac{2}{p^2 - q^2} \Rightarrow \quad z = xp + yq + \frac{2}{p^2 - q^2} \quad \text{is Clairaut's equation.}$$

suppose a *and* b *are constants.*

$$z = ax + by + \frac{2}{a^2 + b^2} \quad \text{is the complete integral.}$$

Example 2.0.13 *: Find the integral surface of the equation*

$$(z + 3y)p + 3(z - x)q + (x + 3y) = 0$$

passing through the curves $x = z, y = 1.$

Solution :

$$(z + 3y)p + 3(z - x)q = -(x + 3y)$$

The auxiliary equations are

$$\frac{dx}{z+3y} = \frac{dy}{3(z-x)} = \frac{dz}{-(x+3y)}$$

$$A = z+3y \qquad B = 3(z-x) \qquad C = -(x+3y)$$

$$\textit{since} \quad A - \frac{1}{3}B + C = z + 3y - \frac{1}{3}\,3(z-x) - (x+3y)$$

$$= z + 3y - z + x - x - 3y = 0$$

$$\therefore dx - \frac{1}{3}dy + dz = 0 \Rightarrow \quad x - \frac{1}{3}y + z = c \Rightarrow$$

$3x - y + 3z = c_1 \ldots (1,$ *where* c_1 *is a constant.*

$$xA + yB + zC = x(z+3y) + 3y(z-x) - z(x+3y) = xz + 3yx + 3yz - 3yx - xz - 3yz = 0$$

$$\therefore xdx + ydy + zdz = 0 \Rightarrow \quad x^2 + y^2 + z^2 = c_2, \textit{where } c_2 \textit{ is a constant.}$$

The parameter equations are $y = 1, x = z, z = t$ *and substitute in Eq. 1) and 2).*

$$3t - 1 + 3t = c_1 \Rightarrow \quad 6t - 1 = c_1 \Rightarrow \quad t = \frac{1}{6}(c_1 + 1)$$

$$t^2 + 1 + t^2 = c^2 \Rightarrow \quad 2t^2 + 1 = c_2 \Rightarrow \quad 2\,\frac{1}{36}(c_1+1)^2 + 1 = c_2$$

$$\frac{1}{18}(c_1+1)^2 + 1 = c^2 \Rightarrow \quad \frac{1}{18}(3x - y + 3z + 1)^2 + 1 = x^2 + y^2 + z^2$$

is a solution passing through the curves $x = z, y = 1$.

Example 2.0.14 *: Find the complete integral of the equation*

$$2(z + xp + yq) = yp^2$$

$$Let \quad f = 2(z + xp + yq) - yp^2$$

$$f_x = 2p \qquad f_y = 2q - p^2 \qquad f_p = 2x - 2yp$$

$$f_q = 2y \qquad f_z = 2$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{dq}{f_y + pf_z}$$

$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{2xp - 2yp^2 + 2yq} = \frac{-dp}{2p + 2p} = \frac{-dq}{2q - p^2 + 2q}$$

$$since \quad \frac{dy}{2y} = \frac{-dp}{4p} \Rightarrow \quad \frac{dp}{p} + 2\frac{dy}{y} = 0 \Rightarrow$$

$$\ln p + \ln y^2 = c_1 \Rightarrow \quad py^2 = c \Rightarrow \quad p = \frac{c}{y^2}$$

Substitute in the main equation

$$2(z + \frac{cx}{y^2} + yq) = y\ \frac{c^2}{y^4} \Rightarrow \quad 2yq = \frac{c^2}{y^3} - \frac{2cx}{y^2} - 2z$$

$$q = \frac{c^2}{2y^4} - \frac{2cx}{y^3} - \frac{z}{y}$$

$$dz = pdx + qdy \Rightarrow \quad dz = \frac{c}{y^2}dx + (\frac{c^2}{2y^4} - \frac{2cx}{y^3} - \frac{z}{y})dy$$

$$\frac{c}{y^2}dx + \frac{c^2}{2y^4}dy - \frac{2cx}{y^3}dy - \frac{z}{y}dy - dz = 0$$

$$\frac{c}{y}dx + \frac{c^2}{2y^3}dy - \frac{cx}{y^2}dy - zdy - ydz = 0$$

$$\frac{cydx - cxdy}{y^2} + \frac{c^2}{2y^3}dy - (zdy + ydz) = 0$$

$$d(\frac{cx}{y}) + d(\frac{-c^2}{4y^2}) - d(zy) = 0$$

$\frac{cx}{y} - \frac{c^2}{4y^2} - zy = c_1$ *is the complete integral.*

Chapter 3

In this chapter, we consider linear partial differential equation of the second order. A PDE, which is linear with respect to the dependent variable and its partial differential coefficients, and in which the coefficients are constants; called LPDE with constant coefficients.
The general coefficients can be written as

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial y^2} + C\frac{\partial^2 z}{\partial x \partial y} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz = G(x, y) \tag{3.0.1}$$

where A, B, C, D, E *and* F *are constants.*
The Eq. (3.0.1) *is called homogeneous if* $G(x, y) = 0$ *and nonhomogeneous otherwise.*

Example 3.0.1 *:*

1) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + 12z = 0$ *is homo. LPDE with constant coefficients.*

2) $25x\,\dfrac{\partial^2 z}{\partial y^2} - 2e^{(x+y)}\dfrac{\partial^2 z}{\partial x \partial y} - 3z = 0$ *is homo. LPDE with variable coefficients.*

3) $\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial y^2} + y^2\dfrac{\partial z}{\partial x} = 5\cos(x + y)$ *is nonhomo. LPDE with variable coefficients.*

4) $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 13$ *is nonhomo. LPDE with constants coefficients.*

The Eq.(3.0.1) *can also be written as*

$$f(D_x, D_y)z = G(x, y)$$

where $f(D_x, D_y)$ is some polynomial function of D_x and D_y and can be written as

$$f(D_x, D_y)z = \sum_{n=0}\sum_{m=0} C_{nm} D_x^n D_y^m z = G(x, y)$$

where C_{nm} is a constant.
The general solution of the homogeneous LPDE with constant coefficients is called complementary function (z_c). And any other solution of the nonhomogeneous LPDE is called particular integral (z_p).

Theorem 3.0.1 *: If z_c the complementary function and z_p a particular integral of a linear partial differential equation, then $z = z_c + z_p$ is a general solution of the equation.*

Proof : *Since z_c is the complementary function and z_p is a particular integral of $f(D_x, D_y)z = G(x, y)$.*

$$\therefore f(D_x, D_y)z_c = 0 \quad \textit{and} \quad f(D_x, D_y)z_p = G(x, y)$$

$$f(D_x, D_y)(z_c + z_p) = f(D_x, D_y)z_c + f(D_x, D_y)z_p$$

$$= \quad 0 \qquad\qquad + \quad G(x, y)$$

$$\Rightarrow \quad f(D_x, D_y)(z_c + z_p) = G(x, y)$$

$$\therefore z_c + z_p \quad \textit{is a general solution of} \quad f(D_x, D_y)z = G(x, y).$$

Theorem 3.0.2 *: If $z_1, z_2, z_3, \ldots, z_n$ are solutions of the homogeneous linear PDE: $f(D_x, D_y)z = 0$, then $u = \sum_{i=1}^{n} C_i z_i$, where C_i's are arbitrary constants, is also a solution.*

Proof : *Since z_i is a solution of $f(D_x, D_y)z = 0$*

$$\therefore f(D_x, D_y)z_i = 0 \quad , \textit{ for } \quad i = 1, 2, 3, \ldots, n$$

$$f(D_x, D_y)u = \sum_{i=1}^{n} f(D_x, D_y)C_i z_i = \sum_{i=1}^{n} C_i f(D_x, D_y) z_i$$

$$= \sum_{i=1}^{n} C_i(0) \Rightarrow \quad f(D_x, D_y)u = 0$$

$$\therefore \sum_{i=1}^{n} C_i z_i \quad \textit{is also a solution of} \quad f(D_x, D_y)z = 0$$

We classify the operator $f(D_x, D_y)$ ***into two types :***

1) Reducible : *The operator $f(D_x, D_y)$ is said to be reducible if it can be factorized into the factors of the type $a_i D_x + b_i D_y + c_i$.*
i.e.

$$f(D_x, D_y) = (a_1 D_x + b_1 D_y + c_1)(a_2 D_x + b_2 D_y + c_2) \ldots (a_n D_x + b_n D_y + c_n)$$

$$= \Pi_{i=1}^{n}(a_i D_x + b_i D_y + c_i)$$

where a_i, b_i and c_i are constants.
Irreducible : *The operator $f(D_x, D_y)$ is said to be irreducible if it is not reducible.*

Example 3.0.2 *:*

1) $(D_x^2 - 3D_x D_y + 2D_y^2)z = 0$

2) $(D_x D_y + D_y^3)z = 0$

Solution :

$$1) \quad (D_x - 2D_y)(D_x - D_y)z = 0$$
$$a_1 = 1, b_1 = -2, c_1 = 0 \quad a_2 = 1, b_2 = -1, c_2 = 0$$

is reducible.

$$2) \quad D_y(D_x + D_y^2)z = 0$$

is irreducible.

1) Reducible Equations :

Theorem 3.0.3 *: If the operator $f(D_x, D_y)$ is reducible, then the order in which the linear factor occur is not important.*

Proof : *We must show that*

$$(a_iD_x + b_iD_y + c_i)(a_jD_x + b_jD_y + c_j) = (a_jD_x + b_jD_y + c_j)(a_iD_x + b_iD_y + c_i)$$

$$LHS = (a_iD_x + b_iD_y + c_i)(a_jD_x + b_jD_y + c_j)$$

$$= a_ia_jD_x^2 + (a_ib_j + b_i)D_xD_y + (0 + c_ia_j)D_x + b_ib_jD_y^2 + (0 + c_ib_j)D_y + c_ic_j$$

$$RHS = (a_jD_x + b_jD_y + c_j)(a_iD_x + b_iD_y + c_i)$$

$$= a_ja_iD_x^2 + (a_jb_i + b_ja_i)D_xD_y + (0 + a_jc_i)D_x + b_jb_iD_y^2 + (0 + b_jc_i)D_y + c_jc_i$$

$$= a_ia_jD_x^2 + (a_ib_j + b_ia_j)D_xD_y + (0 + c_ia_j)D_x + b_ib_jD_y^2 + (0 + c_ib_j)D_y + c_ic_j$$

therefore the order is not important.

Example 3.0.3 *: Is*

$$(3xD_x - 3y^2xD_xD_y + 3)(D_x + 2xD_y + 1) = (D_x + 2xD_y + 1)(3xD_x - 3y^2xD_xD_y + 3)?$$

Theorem 3.0.4 *If $(a_iD_x + b_iD_y + c_i)$ is a factor of $f(D_x, D_y)$ and if $a_i \neq 0$ then*

$$z_i = e^{\frac{-c_i}{a_i}x}\phi(b_ix - a_iy)$$

is a solution of $f(D_x, D_y)z = 0$, where ϕ is an arbitrary function.

Proof : *We start by solving* $(a_iD_x + b_iD_y + c_i)z_i = 0$ *by auxiliary equations*

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz_i}{-c_iz_i}$$

$$since \quad \frac{dx}{a_i} = \frac{dy}{b_i} \Rightarrow \quad b_idx - a_idy = 0 \Rightarrow$$

$b_ix - a_iy = k,$ *where* k *is a constant.*

$$since \quad \frac{dx}{a_i} = \frac{dz_i}{-c_iz_i} \Rightarrow \quad \frac{dz_i}{z_i} = \frac{-c_i}{a_i}dx$$

$\Rightarrow \quad \ln z_i = \dfrac{-c_i}{a_i}x + A,$ *where* A *is a constant.*

$$z_i = ke^{\frac{-c_i}{a_i}x}$$

$$\therefore z_i = e^{\frac{-c_i}{a_i}x}\phi(b_ix - a_iy)$$

$$since \quad f(D_x, D_y)z = \Pi_{j=1,\, j\neq i}^{n}(a_jD_x + b_jD_y + c_j)(a_iD_x + b_iD_y + c_i)z$$

$$\therefore f(D_x, D_y)z_i = \Pi_{j=1,\, j\neq i}^{n}(a_jD_x + b_jD_y + c_j)(a_iD_x + b_iD_y + c_i)z_i$$

$$and \quad D_xz_i = b_ie^{\frac{-c_i}{a_i}x}\phi'(b_ix - a_iy) - \frac{c_i}{a_i}e^{\frac{-c_i}{a_i}x}\phi(b_ix - a_iy)$$

$$= b_ie^{\frac{-c_i}{a_i}x}\phi'(b_ix - a_iy) - \frac{c_i}{a_i}z_i$$

$$D_y z_i = -a_i e^{\frac{-c_i}{a_i}x} \phi'(b_i x - a_i y)$$

$$\therefore (a_i D_x + b_i D_y + c_i) z_i = a_i(b_i e^{\frac{-c_i}{a_i}x} \phi'(b_i x - a_i y) - \frac{c_i}{a_i} z_i) + b_i(-a_i$$

$$e^{\frac{-c_i}{a_i}x} \phi'(b_i x - a_i y)) + c_i z_i = 0$$

$$f(D_x, D_y) z_i = \Pi_{j=1,\, j \neq i}^{n} (a_j D_x + b_j D_y + c_j)(0) = 0$$

$$f(D_x, D_y) z_i = 0$$

$$\therefore z_i = e^{\frac{-c_i}{a_i}x} \phi(b_i x - a_i y) \quad \textit{is the solution of } f(D_x, D_y) z = 0$$

Theorem 3.0.5 *If $(b_i D_y + c_i)$ is a factor of $f(D_x, D_y)$ and if $b_i \neq 0$ then $z_i = e^{\frac{-c_i}{b_i}y} \phi(b_i x)$ is the solution of $f(D_x, D_y) z = 0$, where ϕ is an arbitrary function.*

Proof :

$$D_y z_i = \frac{-c_i}{b_i} e^{\frac{-c_i}{b_i}y} \phi(b_i x) = \frac{-c_i}{b_i} z_i, \text{ so that}$$

$$(b_i D_y + c_i) z_i = b_i(\frac{-c_i}{b_i} z_i) + c_i z_i = 0$$

$$\textit{since} \quad f(D_x, D_y) z = \Pi_{j=1,\, j \neq i}^{n} (0 + b_j D_x + c_j)(0 + b_i D_x + c_i) z$$

$$\textit{then} \quad f(D_x, D_y) z_i = \Pi_{j=0,\, j \neq i}^{n} (0 + b_j D_x + c_j)(0 + b_i D_x + c_i) z_i$$

$$= \Pi_{j=0,\ j\neq i}^{n}(0 + b_j D_x + c_j)(0) = 0$$

$$\therefore z_i = e^{\frac{-c_i}{b_i}} \phi(b_i x) \quad \textit{is the solution } f(D_x, D_y)z = 0$$

Theorem 3.0.6 *If* $(a_i D_x + b_i D_y + c_i)^2$ *is a factor of* $f(D_x, D_y)$. *Then* $z_i = e^{\frac{-c_i}{a_i}} (\phi_1(b_i x - a_i y) + x\phi_2(b_i x - a_i y)$ *is the solution of* $f(D_x, D_y)z = 0$ *where* $a_i \neq 0$ *and* ϕ_1 *and* ϕ_2 *are arbitrary functions.*

Proof : *We start by solving*

$$(a_i D_x + b_i D_y + c_i)^2 z_i = 0 \ldots 1)$$

$$\textit{let} \quad (a_i D_x + b_i D_y + c_i)z_i = z \ldots 2) \quad \textit{then} \quad Eq.1) \textit{ becomes}$$

$$(a_i D_x + b_i D_y + c_i)z = 0$$

$$\textit{since} \quad a_i \neq 0 \textit{ and by theorem}(3.0.4) \Rightarrow$$

$$z = e^{\frac{-c_i}{a_i}} \phi(b_i x - a_i y) \quad \textit{substitute in Eq. 2)}$$

$$(a_i D_x + b_i D_y + c_i)z_i = e^{\frac{-c_i}{a_i}} \phi(b_i x - a_i y)$$

$$a_i D_x z_i + b_i D_y z_i = e^{\frac{-c_i}{a_i}} \phi(b_i x - a_i y) - c_i z_i$$

can be solved by auxiliary Eqs.

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz_i}{e^{\frac{-c_i}{a_i}} \phi(b_i x - a_i y) - c_i z_i}$$

$$\frac{dz_i}{dx} = \frac{1}{a_i}(e^{\frac{-c_i}{a_i}}\phi(k) - c_i z_i)$$

$$\frac{dz_i}{dx} + \frac{c_i}{a_i}z_i = \frac{1}{a_i}e^{\frac{-c_i}{a_i}}\phi(k) \quad \textit{is first order LDE.}$$

$$z_i = \frac{\int e^{\int \frac{c_i}{a_i}dx}(\frac{1}{a_i}e^{\frac{-c_i}{a_i}}\phi(k))dx + k_1}{e^{\int \frac{c_i}{a_i}dx}}$$

$$= \frac{\frac{1}{a_i}\int e^{\frac{c_i}{a_i}x}e^{\frac{-c_i}{a_i}x}\phi(k)dx + k_1}{e^{\frac{c_i}{a_i}x}}$$

$$z_i = \frac{1}{a_i}e^{\frac{-c_i}{a_i}x}\int \phi(k)dx + k_1 e^{\frac{-c_i}{a_i}x}$$

$$z_i = e^{\frac{-c_i}{a_i}x}(\frac{1}{a_i}\phi(k)x + k_1)$$

then

$$z_i = e^{\frac{-c_i}{a_i}x}(\phi_1(b_i x - a_i y) + x\phi_2(b_i x - a_i y) \quad \textit{is the solution of} \quad (a_i D_x + b_i D_y + c_i)^2 z = 0$$

$$\textit{since} \quad f(D_x, D_y)z = \Pi_{j=1,\, j\neq i}^{n}(a_j D_x + b_j D_y + c_j)(a_i D_x + b_i D_y + c_i)^2 z$$

$$f(D_x, D_y)z_i = \Pi_{j=1,\, j\neq i}^{n}(a_j D_x + b_j D_y + c_j)(a_i D_x + b_i D_y + c_i)^2 z_i = 0$$

$$\therefore z_i = e^{\frac{-c_i}{a_i}x}(\phi_1(b_i x - a_i y) + x\phi_2(b_i x - a_i y) \quad \textit{is the solution of} \quad f(D_x, D_y)z = 0$$

Theorem 3.0.7 *: If* $(b_iD_y+c_i)^2$ *is a factor of* $f(D_x, D_y)$*, then* $z_i = e^{\frac{-c_i}{b_i}y}(\phi_1(b_ix)+ y\,\phi_2(b_ix)$ *is the solution of* $f(D_x, D_y)z = 0$*, where* ϕ_1 *and* ϕ_2 *are arbitrary function and* $b_i \neq 0$*.*

Proof : *We start by solving*

$$(b_iD_y + c_i)^2 z_i = 0 \ldots 1)$$

let $(b_iD_y + c_i)z_i = z \ldots 2)$*, then the Eq. 1) becomes*

$$(b_iD_y + c_i)z = 0$$

since $b_i \neq 0$ *then by theorem* $(3.0.5) \Rightarrow \quad z = e^{\frac{-c_i}{b_i}y}\phi(b_ix)$

substitute in Eq. 2) obtains:

$$(b_iD_y + c_i)z_i = e^{\frac{-c_i}{b_i}y}\phi(b_ix) \Rightarrow \quad b_iD_yz_i = e^{\frac{-c_i}{b_i}y}\phi(b_ix) - c_iz_i$$

Can be solved by auxiliary Eqs.

$$\frac{dy}{b_i} = \frac{dz_i}{e^{\frac{-c_i}{b_i}y}\phi(b_ix) - c_iz_i}$$

$$\frac{dz_i}{dy} + \frac{c_i}{b_i}z_i = \frac{1}{b_i}e^{\frac{-c_i}{b_i}y}\phi(b_ix) \quad \text{is first order LDE.}$$

$$z_i = \frac{\int e^{\int \frac{c_i}{b_i} dy} (\frac{1}{b_i} e^{\frac{-c_i}{b_i} y} \phi(b_i x))dy + k}{e^{\int \frac{c_i}{b_i} dy}}$$

$$= \frac{\frac{1}{b_i} \int e^{\frac{c_i}{b_i} y} e^{\frac{-c_i}{b_i} y} \phi(b_i x)dy + k}{e^{\frac{c_i}{b_i} y}}$$

$$= e^{\frac{-c_i}{b_i} y} (y\phi(b_i x) + k)$$

$z_i = e^{\frac{-c_i}{b_i} y} (\phi_1(b_i x) + y\ \phi_2(b_i x))$ *is the solution of* $(b_i D_y + c_i)^2 z = 0$

since $f(D_x, D_y)z = \Pi_{j=1,\, j\neq i}^{n} (b_j D_y + c_j)(b_i D_y + c_i)^2 z,$ *then*

$f(D_x, D_y)z_i = \Pi_{j=1,\, j\neq i}^{n} (b_j D_y + c_j)(b_i D_y + c_i)^2 z_i = 0$

$\therefore z_i = e^{\frac{-c_i}{b_i} y} (\phi_1(b_i x) + y\ \phi_2(b_i x))$ *is the solution of* $f(D_x, D_y)z = 0.$

Theorem 3.0.8 *: If* $(a_i D_x + b_i D_y + c_i)^n$ *is a factor of* $f(D_x, D_y)$ *and if* $a_i \neq 0$ *then* $z_i = e^{\frac{-c_i}{a_i} x} (\sum_{j=1}^{n} x^{j-1} \phi_j(b_i x - a_i y))$ *is the solution of* $f(D_x, D_y)z = 0$*, where* ϕ_j *is an arbitrary function for* $n \in z^+$*.*

Proof : *By using mathematical induction, we try to solve this theorem.*

By theorems (3.0.5) *and* (3.0.6) *the result is true for* $n = 1, 2$*.*

Suppose the theorem is true for $n = k - 1$

i.e.

if $((a_iD_x + b_iD_y + c_i)^{k-1})$ *is a factor of* $f(D_x, D_y)$, *then*

$z_i = e^{\frac{-c_i}{a_i}x}(\sum_{j=1}^{k-1} x^{j-1}\phi_j(b_ix - a_iy))$ *is solution of* $f(D_x, D_y)z = 0$

Now, to solve $(a_iD_x + b_iD_y + c_i)^k z = 0 \ldots 1)$ *let* $(a_iD_x + b_iD_y + c_i)z = u \cdots *)$

Then, the Eq. 1) becomes

$(a_iD_x + b_iD_y + c_i)^{k-1}u = 0$

By hypothesis

$u = e^{\frac{-c_i}{a_i}x}(\sum_{j=1}^{k-1} x^{j-1}\phi_j(b_ix - a_iy))$

substitute in Eq.) obtain*

$$((a_iD_x + b_iD_y + c_i)z = e^{\frac{-c_i}{a_i}x}(\sum_{j=1}^{k-1} x^{j-1}\phi_j(b_ix - a_iy))$$

$$a_iD_xz + b_iD_yz = e^{\frac{-c_i}{a_i}x}(\sum_{j=1}^{k-1} x^{j-1}\phi_j(b_ix - a_iy)) - c_iz$$

then the last equation can be solved by auxiliary equations

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz}{e^{\frac{-c_i}{a_i}x}\left(\sum_{j=1}^{k-1} x^{j-1}\phi_j(b_i x - a_i y)\right) - c_i z}$$

Since $$\frac{dx}{a_i} = \frac{dz}{e^{\frac{-c_i}{a_i}x}\left(\sum_{j=1}^{k-1} x^{j-1}\phi_j(b_i x - a_i y)\right) - c_i z}$$

$$\frac{dz}{dx} = \frac{1}{a_i} e^{\frac{-c_i}{a_i}x} \sum_{j=1}^{k-1} x^{j-1}\phi_j(A) - \frac{c_i}{a_i} z$$

$$\frac{dz}{dx} + \frac{c_i}{a_i} z = \frac{1}{a_i} e^{\frac{-c_i}{a_i}x} \sum_{j=1}^{k-1} x^{j-1}\phi_j(A) \quad \textit{is first order LDE.}$$

$$z = \frac{\int e^{\int \frac{c_i}{a_i}dx} \left(\frac{1}{a_i} e^{\frac{-c_i}{a_i}x} \sum_{j=1}^{k-1} x^{j-1}\phi_j(A)\right)dx + A_1}{e^{\int \frac{c_i}{a_i}dx}}$$

$$z = \frac{\frac{1}{a_i}\int e^{\frac{c_i}{a_i}x} \sum_{j=1}^{k-1} e^{\frac{-c_i}{a_i}x} x^{j-1}\phi_j(A)dx + A_1}{e^{\frac{c_i}{a_i}x}}$$

$$z = \frac{\frac{1}{a_i}\int \sum_{j=1}^{k-1} x^{j-1}\phi_j(A)dx + A_1}{e^{\frac{c_i}{a_i}x}}$$

$$z = e^{\frac{-c_i}{a_i}x}\left(\frac{1}{a_i}\sum_{j=1}^{k-1} \phi_j(A) \int x^{j-1}dx + A_1\right)$$

$$z = e^{\frac{-c_i}{a_i}x}\left(\frac{1}{a_i}\sum_{j=1}^{k-1} \phi_j(A)\frac{x^j}{j} + A_1\right)$$

$$z = e^{\frac{-c_i}{a_i}x} (\sum_{j=1}^{k-1} \phi_j(b_i x - a_i y)x^j + \phi_1(b_i x - a_i y)x^0)$$

$$z = e^{\frac{-c_i}{a_i}x} (\sum_{j=1}^{k-1} \phi_j(b_i x - a_i y)x^{j-1})$$

$$\therefore z_i = e^{\frac{-c_i}{a_i}x} (\sum_{j=1}^{k} \phi_j(b_i x - a_i y)x^{j-1}) \quad \textit{is the solution of Eq. 1)}$$

since $\quad f(D_x, D_y)z = \Pi_{j=1,\, j\neq i}^{m}(a_j x + b_j y + c_j)(a_i x + b_i y + c_i)^n z$

$\therefore$ *the theorem is true for* $n = k$.

and

$$f(D_x, D_y)z_i = \Pi_{j=1,\, j\neq i}^{m}(a_j x + b_j y + c_j)(a_i x + b_i y + c_i)^k z_i = 0$$

$$\therefore z_i = e^{\frac{-c_i}{a_i}x} (\sum_{j=1}^{n} \phi_j(b_i x - a_i y)x^{j-1}) \quad \textit{is the solution of } f(D_x, D_y)z = 0.$$

Theorem 3.0.9 *: If* $(b_i D_y + c_i)^n$ *is a factor of* $f(D_x, D_y)$*, and if* $b_i \neq 0$*, then*

$$z_i = e^{\frac{-c_i}{b_i}y} (\sum_{j=1}^{n} \phi_j(b_i x)y^{j-1})$$

is the solution of $f(D_x, D_y)z = 0$*, where* ϕ_j *is an arbitrary functions for* $n = z^+$.

Proof: *We use mathematical induction to prove this theorem. By theorem* (3.0.5) *and* (3.0.7) *the theorem is true for* $n = 1, 2$*. Suppose the theorem is true* *for $n = k-1$ i.e. if $(b_iD_y + c_i)^{k-1}$ is a factor of $f(D_x, D_y)$, then*

$$z_i = e^{\frac{-c_i}{b_i}y}\left(\sum_{j=1}^{k-1} y^{j-1}\phi_j(b_ix)\right)$$

is a solution of $f(D_x, D_y)z = 0$.
Now, to solve $(b_iD_y + c_i)^k z = 0 \ldots (1$, let $(b_iD_y + c_i)z = u \ldots (2$, then Eq.(1 becomes:

$$(b_iD_y + c_i)^{k-1}u = 0$$

By hypothesis $u = e^{\frac{-c_i}{b_i}y}\left(\sum_{j=1}^{k-1} y^{j-1}\phi_j(b_ix)\right)$

Substitute in Eq.(2 obtain

$$(b_iD_y + c_i)^k z = e^{\frac{-c_i}{b_i}y}\left(\sum_{j=1}^{k-1} y^{j-1}\phi_j(b_ix)\right)$$

$$b_iD_yz = e^{\frac{-c_i}{b_i}y}\left(\sum_{j=1}^{k-1} y^{j-1}\phi_j(b_ix)\right) - c_iz$$

Can be solved by auxiliary equations:

$$\frac{dy}{b_i} = \frac{dz}{e^{\frac{-c_i}{b_i}y}\left(\sum_{j=1}^{k-1} y^{j-1}\phi_j(b_ix)\right) - c_iz}$$

$$\frac{dz}{dy} = \frac{1}{b_i}\, e^{\frac{-c_i}{b_i}y}\left(\sum_{j=1}^{k-1} y^{j-1}\phi_j(b_ix)\right) - \frac{c_i}{b_i}z$$

$$\frac{dz}{dy} + \frac{c_i}{b_i}z = \frac{1}{b_i}\, e^{\frac{-c_i}{b_i}y}\left(\sum_{j=1}^{k-1} y^{j-1}\phi_j(b_ix)\right) \text{ is first order LDE.}$$

$$z = \frac{\int e^{\int \frac{c_i}{b_i} dy} \frac{1}{b_i} e^{\frac{-c_i}{b_i} y} \sum_{j=1}^{k-1} y^{j-1} \phi_j(b_i x) dy + k}{e^{\int \frac{c_i}{b_i} dy}}$$

$$z = \frac{\frac{1}{b_i} \sum_{j=1}^{k} \phi_j(b_i x) \int y^{j-1} dy + k}{e^{\frac{c_i}{b_i} y}}$$

$$z = \frac{1}{b_i} \; e^{\frac{-c_i}{b_i} y} \; \sum_{j=1}^{k} \phi_j(b_i x) \; \frac{y^j}{j} + k \; e^{\frac{-c_i}{b_i} y}$$

$$z = e^{\frac{-c_i}{b_i} y} (\sum_{j=1}^{k} \phi_j(b_i x) \; y^j + \phi_j(b_i x) \; y^0)$$

$$z = e^{\frac{-c_i}{b_i} y} (\sum_{j=1}^{k} \phi_j(b_i x) \; y^{j-1})$$

$$\therefore \; z_i = e^{\frac{-c_i}{b_i} y} (\sum_{j=1}^{k} \phi_j(b_i x) \; y^{j-1}) \; \textit{is a solution of Eq.(1}$$

$\therefore$ *the theorem is true for* $n = k$.

$$\textit{since} \;\; f(D_x, D_y) z = \Pi_{j=1}^{m} (b_j x + c_j)(b_i x + c_i) z$$

$$\therefore \; f(D_x, D_y) z_i = \Pi_{j=1}^{m} (b_j x + c_j)(b_i x + c_i) z_i$$

$$\therefore \; z_i = e^{\frac{-c_i}{b_i} y} \sum_{j=1}^{n} y^{j-1} \phi_j(b_i x) \; \textit{is a solution of} \;\; f(D_x, D_y) z = 0.$$

Example 3.0.4 *: Solve*

1) $3\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$

2) $\frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^3 z}{\partial y^3} + \frac{\partial^2 z}{\partial y^2} = 0$

3) $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} - 2\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0$

4) $\frac{\partial^5 z}{\partial x^2 \partial y^3} - \frac{\partial^5 z}{\partial x^3 \partial y^2} = 0$

5) $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 0$

6) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

Solution 1): $(3D_x^2 - 2D_xD_y - D_y^2)z = 0$

$(D_x - D_y)(3D_x + D_y)z = 0$ *is reducible*

then

$$z = e^{\frac{0}{1}x}\phi_1(-x-y) + e^{\frac{0}{1}x}\phi_2(x-3y)$$

$z = \phi_1(-x-y) + \phi_2(x-3y)$ *is Gs.*

where ϕ_1 *and* ϕ_2 *are arbitrary functions.*

2): $D_xD_y^2 - D_y^3 + D_y^2)z = 0$

$D_y^2(D_x - D_y + 1)z = 0$

then

$z = \phi_1(x) + y\phi_2(x) + e^{-x}\phi_3(-x-y)$ *is Gs.*

where ϕ_1, ϕ_2 *and* ϕ_3 *are arbitrary functions.*

3): $(D_x^4 + D_y^4 - 2D_x^2\, D_y^2)z = 0$

$$(D_x^4 - 2D_x^2\, D_y^2 + D_y^4)z = 0$$

$$(D_x^2 - D_y^2)(D_x^2 - D_y^2)z = 0 \Rightarrow \quad (D_x^2 - D_y^2)^2 z = 0$$

$(D_x - D_y)(D_x + D_y)z = 0$ *is reducible*

then

$z = \phi_1(-x - y) + x\phi_2(-x - y) + \phi_3(x - y) + x\phi_4(x - y)$

or $z = \phi_1(x + y) + x\phi_2(x + y) + \phi_3(x - y) + x\phi_4(x - y)$ *is Gs.*

where ϕ_1, ϕ_2, ϕ_3 *and* ϕ_4 *are arbitrary functions.*

4): $(D_x^2 D_y^3 - D_x^2 D_y^2)z = 0$

$D_x^2 D_y^2(D_y - D_x)z = 0$

then

$z = \phi_1(-y) + x\phi_2(-y) + \phi_3(x) + y\phi_4(x) + \phi_5(x + y)$

or $z = \phi_1(y) + x\phi_2(y) + \phi_3(x) + y\phi_4(x) + \phi_5(x + y)$ *is Gs.*

where $\phi_1, \phi_2, \phi_3, \phi_4$ *and* ϕ_5 *are arbitrary functions.*

5): $(D_x^2 - 4D_y^2 + D_x + 2D_y)z = 0$

$((D_x - 2D_y)(D_x + 2D_{y)}) + D_x + 2D_y)z = 0$

$(D_x + 2D_{y)})(D_x - 2D_y + 1)z = 0$ *is reducible*

then

$z = \phi_1(2x - y) + e^{-x}\phi_2(-2x - y)$

then $z = \phi_1(2x - y) + e^{-x}\phi_2(2x - y)$ *is Gs.*

where ϕ_1 *and* ϕ_2 *are arbitrary functions.*

6): $(D_x^2 - D_y^2)z = 0$

$$(D_x - D_y)(D_x + D_y)z = 0$$

then

$$z = \phi_1(-x - y) + \phi_2(x - y)$$

or $z = \phi_1(x + y) + \phi_2(x - y)$ *is Gs.*

where ϕ_1 *and* ϕ_2 *are arbitrary functions.*

b) Irreducible Equations:

When the operator $f(D_x, D_y)$ *is irreducible (in other words* $f(D_x, D_y)$ *cannot be factorized into linear factors). To solve* $f(D_x, D_y)z = 0 \ldots (1$ *we put* $z = c_i e^{a_i x + b_i y}$ *is a solution of Eq.1).*

Since $D_x(e^{a_i x}) = a_i e^{a_i x}, \ldots, D_x^n(e^{a_i x}) = a_i^n e^{a_i x}$

$$D_y(e^{b_i y}) = b_i e^{b_i y}, \ldots, D_y^m(e^{b_i y}) = b_i^m e^{b_i y}$$

$$\Rightarrow \quad D_x^n D_y^m(e^{a_i x + b_i y}) = a_i^n b_i^m e^{a_i x + b_i y}$$

$$f(D_x, D_y)(e^{a_i x + b_i y}) = f(a_i, b_i) e^{a_i x + b_i y}$$

then $f(D_x, D_y)(c_i e^{a_i x + b_i y}) = 0 \Rightarrow \quad c_i f(D_x, D_y)(e^{a_i x + b_i y}) = 0 \Rightarrow \quad f(a_i, b_i) = 0$

then $z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$ *is a solution of Eq.1) where* $f(a_i, b_i) = 0$

Example 3.0.5 *: Solve*

1) $\dfrac{\partial z}{\partial x} - 3\dfrac{\partial^2 z}{\partial y^2} = 0$

2) $\dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial^3 z}{\partial x^3} + \dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^3 z}{\partial x^2 \partial y} = 0$

Solution 1): $(D_x - 3D_y^2)z = 0$ *is irreducible* $f(a_i, b_i) = a_i - 3b_i^2$ *then* $f(a_i, b_i) = 0$ *iff* $a_i = 3b_i^2$

then

$$z = \sum_{i=1}^{\infty} c_i e^{3b_i^2 x + b_i y} = \sum_{i=1}^{\infty} c_i e^{b_i(3b_i x + y)}$$

is Gs., where b_i *and* c_i *are constants.*

2)

$$(D_y^2 - D_x^3 + D_x D_y - D_x^2 D_y)z = 0$$
$$(D_y(D_y + D_x) - D_x^2(D_y + D_x))z = 0$$
$$(D_y + D_x)(D_y - D_x^2)z = 0$$
$$a_1 = 1, b_1 = 1, c_1 = 0 \quad b_i + a_i^2 = 0 \Rightarrow \quad b_i = a_i^2$$
$$then\ y = \phi(x + y) + \sum_{i=1}^{\infty} c_i e^{a_i x + a_i^2 y}\ is\ Gs.$$

where ϕ is an arbitrary function and a_i and c_i are constants.

Finding Particular Integral For Nonhomogeneous PDE With Constant Coefficients

1) Suppose $\quad f(D_x, D_y)z = g(x, y) \ldots (1$

then the P.I. is

$$P.I. = \frac{1}{f(D_x, D_y)} g(x, y)$$

Case a) *If* $g(x, y) = e^{\alpha x + \beta y}$

Theorem 3.0.10 : $f(D_x, D_y)(e^{\alpha x + \beta y}) = f(\alpha, \beta)e^{\alpha x + \beta y}$

Proof :

$$since \quad D_x^n(e^{\alpha x + \beta y}) = \alpha^n e^{\alpha x + \beta y}$$

$$D_y^m(e^{\alpha x + \beta y}) = \beta^m e^{\alpha x + \beta y}$$

$$so\ that \quad (c_{nm} D_x^n D_y^m)(e^{\alpha x + \beta y}) = c_{nm} \alpha^n \beta^m e^{\alpha x + \beta y}$$

$$\therefore f(D_x, D_y)(e^{\alpha x + \beta y}) = f(\alpha, \beta)e^{\alpha x + \beta y}$$

i) *If* $f(\alpha,\beta)\neq 0$ *then by the last theorem*

$$z=\frac{1}{f(D_x,D_y)}\{e^{\alpha x+\beta y}\}=\frac{e^{\alpha x+\beta y}}{f(\alpha,\beta)}$$

Example 3.0.6 *: Solve*

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=3\ e^{2x+y}$$

Solution : $(D_x^2-D_y^2)z=3\ e^{2x+y}\Rightarrow\quad (D_x-D_y)(D_x+D_y)z=3\ e^{2x+y}$ *is reducible*
then

$$z_c=\phi_1(x+y)+\phi_2(x-y)\ \textit{is a complementary function.}$$

To find P.I.

$$z_p=\frac{1}{(D_x-D_y)(D_x+D_y)}\{3\ e^{2x+y}\}=\frac{3\ e^{2x+y}}{(2-1)(2+1)}=\frac{3}{3}e^{2x+y}=e^{2x+y}$$

$$\therefore z=z_c+z_p=\phi_1(x+y)+\phi_2(x-y)+e^{x+2y}\ \textit{is Gs.}$$

Theorem 3.0.11 $f(D_x,D_y)(e^{\alpha x+\beta y}\phi(x,y))=e^{\alpha x+\beta y}f(D_x+\alpha,D_y+\beta)\{\phi(x+y)\}$
Proof: *Since*

$$D_x^n(e^{\alpha x+\beta y}\phi(x,y))=\sum_{r=0}^{n}C_r^nD_x^r(e^{\alpha x+\beta y})(D_x^{n-r}\phi)$$

$$=e^{\alpha x+\beta y}\sum_{r=0}^{n}C_r^n\alpha^rD_x^{n-r}\phi$$

[by Leibnitz's theorem] $[D_x^n(y_1.y_2)=\sum_{r=0}^{n}C_r^nD_x^r[y_1]D_x^{n-r}[y_2]]$

$$=e^{\alpha x+\beta y}(D_x+\alpha)^n\phi$$

$[by\ Binomial\ rule]$

$$and \quad D_y^m(e^{\alpha x+\beta y}\phi(x,y)) = \sum_{r=0}^{m} C_r^m D_y^r(e^{\alpha x+\beta y})(D_y^{m-r}\phi)$$
$$= e^{\alpha x+\beta y}\sum_{r=0}^{m} C_r^m \beta^r(D_y^{m-r}\phi)$$
$$= e^{\alpha x+\beta y}(D_y+\beta)^m\phi$$

$$\therefore f(D_x,D_y)(e^{\alpha x+\beta y}\phi(x,y)) = e^{\alpha x+\beta y} f(D_x+\alpha, D_y+\beta)\{\phi(x+y)\}$$

ii) *If* $f(\alpha,\beta)=0$ *and* $f(D_x,D_y)$ *is reducible then* $f(D_x,D_y) = (D_x - \frac{\alpha}{\beta}D_y)^n h(D_x,D_y)$ *where* $h(\alpha,\beta)\neq 0$. *To find particular integral of* $f(D_x,D_y)z = e^{\alpha x+\beta y}$

$$z = \frac{1}{f(D_x,D_y)}\{e^{\alpha x+\beta y}\} = \frac{1}{(D_x-\frac{\alpha}{\beta}D_y)^n h(D_x,D_y)}\{e^{\alpha x+\beta y}\}$$

$$z = \frac{e^{\alpha x,\beta y}}{(D_x+\alpha-\frac{\alpha}{\beta}(D_y+\beta))^n h(D_x+\alpha, D_y+\beta)}\{1\}$$

by theorem (3.0.11)

$$z = \frac{e^{\alpha x+\beta y}}{(D_x-\frac{\alpha}{\beta}D_y)^n h(\alpha,\beta)}\{1\}$$

$$[since \quad \frac{1}{h(D_x+\alpha, D_y+\beta)}\{1\} = \frac{1}{h(\alpha,\beta)}]$$

$$= \frac{1}{h(\alpha,\beta)}.\frac{1}{(D_x-\frac{\alpha}{\beta}D_y)^n}\{1\} = \frac{e^{\alpha x+\beta y}}{h(\alpha,\beta)}.\frac{x^n}{n!}$$

$$[since \quad (D_x-\frac{\alpha}{\beta}D_y)^n\{x^n\} = (D_x^n + n\frac{\alpha}{\beta}D_x^{n-1}D_y + \cdots + (\frac{-\alpha}{\beta})^n D_y^n)\{x^n\} = n!]$$

Example 3.0.7 *: Solve*

1) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = e^{x+y}$

2) $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{\partial^3 z}{\partial y \partial x^2} - \dfrac{\partial^2 z}{\partial x^2} = e^{y}$

3) $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{\partial^3 z}{\partial y \partial x^2} - \dfrac{\partial^2 z}{\partial x^2} = -e^{x}$

Solution 1):

$$(D_x^2 - D_y^2)z = e^{x+y} \Rightarrow \quad (D_x - D_y)(D_x + D_y)z = e^{x+y}$$

then

$$z_c = \phi_1(-x-y) + \phi_2(x-y) \text{ or}$$

$$z_c = \phi_1(x+y) + \phi_2(x-y)$$

$$\text{To find } z_p : z_p = \frac{1}{(D_x - D_y)(D_x + D_y)}\{e^{x+y}\};\ \alpha = 1 = \beta$$

$$= \frac{e^{x+y}}{(D_x + 1 - D_y - 1)(D_x + D_y + 2)}\{1\} = \frac{e^{x+y}}{1+1}.\frac{x^1}{1!} = x\frac{e^{x+y}}{2}$$

$$\therefore \quad z_p = x\frac{e^{x+y}}{2}$$

$$\therefore \quad z = z_c + z_p = \phi_1(x+y) + \phi_2(x-y) + x\frac{e^{x+y}}{2}$$

2)

$$(D_x^3 - D_y D_x^2 - D_x^2)z = e^y$$

$$D_x^2(D_x - D_y - 1)z = e^y \quad \textit{is reducible}$$

$$z_c = \phi_1(y) + x\phi_2(y) + e^x\phi_3(x+y)$$

$$\textit{To find } z_p : z_p = \frac{e^y}{D_x^2(D_x - D_y - 1)}\{1\} = \frac{e^y}{-2}.\frac{x^2}{2!} = \frac{-x^2}{4}e^y$$

$$\therefore \quad z = z_c + z_p = \phi_1(y) + x\phi_2(y) + e^x\phi_3(x+y) - \frac{x^2}{4}e^y \quad \textit{is Gs.}$$

3)

$$D_x^2(D_x - D_y - 1)z = -e^x \quad \textit{is reducible}$$

$$z_c = \phi_1(y) + x\phi_2(y) + e^x\phi_3(x+y)$$

$$\textit{To find } z_p : z_p = \frac{1}{D_x^2(D_x - D_y - 1)}\{-e^x\}$$

$$= \frac{-e^x}{1^2}.\frac{x^1}{1!} \Rightarrow \quad z_p = -xe^x$$

$$\therefore \quad z = z_c + z_p = \phi_1(y) + x\phi_2(y) + e^x\phi_3(x+y) - xe^x \quad \textit{is Gs.}$$

Case iii) *If* $f(\alpha, \beta) = 0$ *and* $fD_x, D_y)$ *is irreducible, let* $z = w(x, y)e^{\alpha x + \beta y}$ *is particular integral of* $f(D_x, D_y)z = e^{\alpha x + \beta y}$

by theorem (3.0.11)*:*

$$f(D_x, D_y)(w(x,y)e^{\alpha x+\beta y}) = e^{\alpha x+\beta y} f(D_x+\alpha, D_y+\beta)\{w(x,y)\}$$

$$\therefore \quad e^{\alpha x+\beta y} f(D_x+\alpha, D_y+\beta)\{w(x,y)\}$$

$$\Rightarrow \quad w(x,y) = \frac{1}{f(D_x+\alpha, D_y+\beta)}\{1\}$$

Example 3.0.8 *: Find Gs. of the following Eqs.?*

1) $(\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial y^2}) = e^{x+y}$

2) $(D_x - D_y^2)(D_x^2 - D_y^2)z = e^{-x+y}$

Solution 1):

$$(D_x - D_y^2)z = e^{x+y}$$

$$(D_x - D_y^2)z = 0 \quad \text{is irreducible}$$

$$f(a_i, b_i) = a_i - b_i^2 = 0 \Rightarrow \quad a_i = b_i^2$$

$$\therefore \quad z_c = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y}$$

To find $z_p : \alpha = 1, \beta = 1$

$$f(\alpha, \beta) = f(1,1) = 1 - 1^2 = 0 \quad \text{and } f(D_x, D_y) \text{ irreducible}$$

let $z_p = w(x,y)e^{x+y}$ *is particular integarl.*

$$(D_x - D_y^2)(w(x,y)e^{x+y}) = e^{x+y}$$

$$D_x(w(x,y)e^{x+y}) - D_y(D_y(w(x,y)e^{x+y})) = e^{x+y}$$

$$w(x,y).e^{x+y} + e^{x+y}D_x(w(x,y)) - D_y(w(x,y))e^{x+y} + e^{x+y}D_y(w(x,y)) = e^{x+y}$$

$$w(x,y)e^{x+y} + e^{x+y}D_x(w(x,y)) - w(x,y)e^{x+y} + e^{x+y}D_y(w(x,y)) -$$

$$e^{x+y}D_y^2(w(x,y)) - D_y(w(x,y))e^{x+y} = e^{x+y}$$

$$w(x,y) + D_x(w(x,y)) - w(x,y) - D_y(w(x,y)) - D_y^2(w(x,y) - D_y(w(x,y)) = 1$$

$$(1 + D_x - 1 - 2D_y - D_y^2)w(x,y) = 1$$

$$(1 + D_x - (D_y^2 + 2D_y + 1))w(x,y) = 1$$

$$((1 + D_x) - (D_{y+1})^2)w(x,y) = 1$$

$$w(x,y) = \frac{1}{(1 + D_x) - (D_{y+1})^2}\{1\}$$

$$= \frac{1}{D_x - D_y^2 - 2D_y}\{1\}$$

$$= \frac{1}{D_x(1 - (\frac{D_y^2}{D_x} + \frac{2D_y}{D_x}))}\{1\}$$

$$= \frac{1}{D_x}[1 + (\frac{D_y^2}{D_x} + \frac{2D_y}{D_x}) + \ldots]\{1\}$$

$$= \frac{1}{D_x}[1 + 0] = x \Rightarrow \quad w(x, y) = x$$

$$\therefore z_p = xe^{x+y}$$

$$z = z_c + z_p = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y} + xe^{x+y} \quad \textit{is Gs.}$$

2)

$$(D_x - D_y^2)(D_x^2 - D_y^2)z = e^{-x+y}$$

To find z_c :

$$(D_x - D_y^2)(D_x^2 - D_y^2)z = 0 \quad \textit{is irreducible}$$

$$(D_x - D_y^2)(D_x - D_y)(D_x + D_y)z = 0$$

$$z_c = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y} + e^{\frac{-0}{1}} \phi_1(-x-y) + e^{\frac{-0}{1}} \phi_2(x-y)$$

$$a_i - b_i^2 = 0 \Rightarrow \quad a_i = b_i^2$$

$$= \sum_{i=1}^{\infty} c_i e^{b_i^2 + b_i y} + \phi_1(-x-y) + \phi_2(x-y) \quad \textit{is complementary function}$$

To find $z_p : \alpha = -1, \beta = 1$

$$f(\alpha, \beta) = f(-1, 1) = (-1 - 1^2)((-1)^2 - 1^2)$$

$f(D_x, D_y)$ *irreducible*
$\therefore$ *let*$z_p = w(x, y)e^{-x+y}$

$$w(x, y) = \frac{1}{f(D_x - 1, D_y + 1)}\{1\}$$

$$= \frac{1}{(D_x - 1 - (D_y + 1)^2(D_x - 1 + D_y + 1)(D_x - 1 - D_y - 1))}\{1\}$$

$$= \frac{1}{(D_x - (D_y^2 + 2D_y + 1))(D_x + D_y)(D_x - D_y - 2)}\{1\}$$

$\vdots$

$$or \quad z_p = \frac{1}{(D_x - D_y^2)(D_x - D_y)(D_x + D_y)}\{e^{-x+y}\}$$

$$= \frac{e^{-x+y}}{(-1-1^2)(-1-1)}.\frac{x^1}{1!} \Rightarrow \quad z_p = \frac{-1}{4}xe^{-x+y}$$

$z = z_c + z_p$ *is Gs.*

Example 3.0.9 *: Find a Gs. for the following Eqs.?*

1) $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + 3z = e^{x-y}$

2) $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} - 3z = e^{x-y}$

3) $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} - 3z = 1$

4) $\dfrac{\partial^3 z}{\partial x^2 \partial y} - \dfrac{\partial^4 z}{\partial x^4} = e^{y-x}$

Remark : *since* $D_x(\cos(ax+by)) = -a\sin(ax+by)$

$$D_x^2(\cos(ax+by)) = -a^2\cos(ax+by)$$
$$\vdots$$
$$D_x^{2n}(\cos(ax+by)) = (-a^2)^n\cos(ax+by)$$

$$D_y(\cos(ax+by)) = -b\sin(ax+by)$$
$$D_y^2(\cos(ax+by)) = -b^2\cos(ax+by)$$
$$\vdots$$
$$D_y^{2m}(\cos(ax+by)) = (-b^2)^m\cos(ax+by)$$

$$D_xD_y(\cos(ax+by)) = -ab\cos(ax+by)$$
$$D_x^2D_y^2(\cos(ax+by)) = (-a^2)(-b^2)\cos(ax+by)$$
$$\vdots$$
$$D_x^{2n}D_y^{2m}(\cos(ax+by)) = (-a^2)^n(-b^2)^m\cos(ax+by)$$

Then by mathematical induction we can prove the following theorem:

Theorem 3.0.12 *: If a and b are constants, then*

1) $f(D_x^2, D_y^2)\{\cos(ax+by)\} = f(-a^2, -b^2)\cos(ax+by)$

2) $f(D_x^2, D_y^2)\{\sin(ax+by)\} = f(-a^2, -b^2)\sin(ax+by)$

Case b) *If $g(x,y)$ is of the form sin$(ax+by)$ or cos$(ax+by)$ where a and b are constants. Then*

$$z = \frac{1}{f(D_x, D_y)}\{sin(ax+by)\} \quad or \quad z = \frac{1}{f(D_x, D_y)}\{cos(ax+by)\}$$

is obtained by putting $D_x^2 = -a^2, D_xD_y - ab, D_y^2 = -b^2$, proved the denominator is not zero.

Example 3.0.10 *: Find the particular solution for the following equations?*

1) $(D_x^2 - 3D_xD_y + D_y^2)z = \sin(x-y)$

2) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^3 z}{\partial y^3} = 2\cos(y-2x)$

Solution 1):

$$z_p = \frac{1}{D_x^2 - 3D_xD_y + D_y^2}\{\sin(x-y)\}; a = 1, b = -1$$

$$= \frac{1}{-1^2 - 3D_xD_y + (-(-1)^2)}\{sin(x-y)\}$$

$$= \frac{1}{-2 - 3D_xD_y}\{sin(x-y)\} = \frac{-2 + 3D_xD_y}{-4 - 9D_x^2D_y^2}\{sin(x-y)\}$$

$$= \frac{1}{-5}(-2\sin(x-y) + 3\sin(x-y)) = \frac{1}{-5}(\sin(x-y))$$

$$\therefore \quad z_p = \frac{-1}{5}\sin(x-y) \quad \textit{is particular integral.}$$

2)

$$(D_x^2 - D_y^3)z = 2\cos(y - 2x)$$

$$z_p = \frac{1}{D_x^2 - D_y^3}\{2\cos(y - 2x)\}; a = -2, b = 1$$

$$= \frac{1}{D_x^2 - D_y^2 D_y}\{2\cos(y - 2x)\}$$

$$= \frac{1}{-4 + D_y}\{2\cos(y - 2x)\} = \frac{1}{D_y - 4}\{2\cos(y - 2x)\}$$

$$\frac{D_y + 4}{D_y^2 - 16}\{2\cos(y - 2x)\}$$

$$z_p = \frac{1}{-17}(-2\sin(y - 2x) + 8\cos(y - 2x)) \quad \textit{is particular integral.}$$

Remark : *If* $f(-a^2, -b^2) = 0$*, then to find particular integral of* $f(D_x, D_y)z = \cos(ax + by)\ldots 1)$
we find the particular integral of $f(D_x, D_y)z = e^{i(ax+by)}$
since $e^{i(ax+by)} = \cos(ax + by) + i\sin(ax + by)\ldots 2)$
then the real part of the particular integral of Eq. 2) is P.I. of Eq. 1) and the imaginary part of the P.I. of Eq. 2) is P.I. of the equation $f(D_x, D_y)z = \sin(ax + by)$

Example 3.0.11 *: Find Gs. of the Eq.* $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \sin(x + y)$?

Solution :

$$(D_x^2 - D_y^2)z = \sin(x + y)$$
$$(D_x - D_y)(D_x + D_y)z = \sin(x + y)$$

To find z_c : $(D_x - D_y)(D_x + D_y)z = 0$

$z_c = \phi_1(x+y) + \phi_2(x-y)$ *is complementary function.*

To find z_p : $z_p = \frac{1}{(D_x^2 - D_y^2)}\{\sin(x+y)\}; a = 1, b = 1$

$f(-a^2, -b^2) = f(-1^2, -1^2) = -1^2 - (-1^2) = -1 + 1 = 0$

so we find z_p *of*

$$z_p = \frac{1}{(D_x^2 - D_y^2)}\{e^{i(x+y)}\} = \frac{1}{(D_x - D_y)(D_x + D_y)}\{e^{i(x+y)}\}$$

$$= e^{i(x+y)}\frac{x}{2i.1!}$$

$$z_p = \frac{-i}{2}x(\cos(x+y) + i\sin(x+y)) = \frac{-i}{2}x\cos(x+y) + \frac{1}{2}x\sin(x+y)$$

$\therefore \quad z_p = \frac{-1}{2}\cos(x+y)$ *is particular integral of the main Eq.*

$z = z_c + z_p$ *is Gs.*

Case c) *If* $g(x,y)$ *is of the form* $x^n y^m$ *where* m *and* n *are positive integers. Then the P.I. is*

$$P.I. = \frac{1}{f(D_x, D_y)}\{x^n y^m\}$$

After that, we expand $\frac{1}{f(D_x, D_y)}$ *as a power series of* D_x *and* D_y.

Example 3.0.12 : *Find the particular integral for each of the following equations?*

1) $(D_x^2 - 2D_xD_y)z = x^3y$

2) $(D_x^2 - 3D_y^3)z = xy^5$

3 $(D_x^2 - 3D_y^3)z = x^3y^2$

4) $(D_x^2 - D_xD_y + D_x)z = x^2 + y^2$

Solution 1):

$$z_p = \frac{1}{D_x^2 - 2D_xD_y}\{x^3y\} = \frac{1}{D_x^2(1 - \frac{2D_y}{D_x})}\{x^3y\}$$

$$= \frac{1}{D_x^2}[1 + \frac{2D_y}{D_x} + (\frac{2D_y}{D_x})^2 + \ldots]\{x^3y\}$$

$$= \frac{1}{D_x^2}[x^3y + \frac{2x^3}{D_x} + \frac{0}{D_x} + \ldots]$$

$$= \frac{1}{D_x^2}[x^3y + +\frac{1}{2}x^4] = \frac{1}{D_x}[\frac{1}{4}x^4y + \frac{1}{10}x^5]$$

$$z_p = \frac{1}{20}x^5y + \frac{1}{60}x^6 \quad \textit{is particular integral.}$$

2):

$$z_p = \frac{1}{D_x^2 - 3D_y^3}\{xy^5\} = \frac{1}{D_x^2(1 - 3\frac{D_y^3}{D_x^2})}\{xy^5\}$$

$$= \frac{1}{D_x^2}[1 + 3\frac{D_y^3}{D_x^2} + (3\frac{D_y^3}{D_x^2})^2 + \ldots]\{xy^5\}$$

$$= \frac{1}{D_x^2}[xy^5 + \frac{3(60xy^2)}{D_x^2} + \frac{9(0)}{D_x^2} + \ldots]$$

$$= \frac{1}{D_x^2}[xy^5 + 30x^3y^2 + 0]$$

$$= \frac{1}{D_x}[\frac{1}{2}x^2y^5 + \frac{15}{2}x^4y^2]$$

$$z_p = \frac{1}{6}x^3y^5 + \frac{3}{2}x^5y^3 \quad \textit{is particular integral.}$$

Case d: *If $g(x, y)$ of the form $e^{ax+by}G(x, y)$, where $G(x, y)$ is one of the cases that we have studied. Then*

$$P.I. = \frac{1}{f(D_x, D_y)}\{e^{ax+by}G(x, y)\}$$

$$= e^{ax+by}\frac{1}{f(D_x + a, D_y + b)}\{G(x, y)\}$$

Example 3.0.13 *: Solve the following equations?*

1) $(\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial y}) = e^{-y}\sin x$

2) $\frac{\partial z}{\partial x} - 3\frac{\partial^2 z}{\partial y^2} = e^x y^2$

Solution 1):

$$(D_x^2 - 2D_y)z = e^{-y}\sin x$$

$$z_p = \frac{1}{D_x^2 - 2D_y}\{e^{-y}\sin x\} = \frac{e^{-y}}{D_x^2 - 2(D_y + (-1))}\sin x$$

$$= e^{-y}\frac{1}{D_x^2 - 2D_y + 2}\{\sin x\} = e^{-y}\frac{1}{-1 - 2D_y + 2}\{\sin x\}$$

$$= e^{-y}\frac{1}{1 - 2D_y}\{\sin x\} = e^{-y}\frac{1 + 2D_y}{1 - 4D_y^2}\{\sin x\}$$

$$= e^{-y}\frac{1 + 2D_y}{1}\{\sin x\} = e^{-y}(\sin x + 0)$$

$$z_p = e^{-y}\sin x \quad \textit{is particular integral}$$

2):

$$(D_x - 3D_y^2)z = e^x y^2$$

$$z_p = \frac{1}{D_x - 3D_y^2}\{e^x y^2\} = \frac{e^x}{(D_x+1) - 3(D_y+0)^2}\{y^2\}$$

$$= e^x \frac{1}{D_x - 3D_y^2 + 1}\{y^2\} = e^x \frac{1}{1-(3D_y^2 - D_x)}\{y^2\}$$

$$e^x[1 + (3D_y^2 - D_x) + (3D_y^2 - D_x)^2 + \ldots [\{y^2\}$$

$$= e^x[y^2 + 3(2) + 0 + \ldots]$$

$$z_p = e^x(y^2+6) \quad \textit{is particular integral.}$$

Case e) *If $g(x,y)$ is of the form $\cos(ax+by)p(x,y)$ or $\sin(ax+by)p(x,y)$, where $p(x,y)$ is polynomial*

$$z_p = \frac{1}{f(D_x, D_y)}\{\sin(ax+by)p(x,y)\}$$

$$= \frac{1}{f(D_x, D_y)}\{e^{i(ax+by)}p(x,y)\}$$

$$= \frac{e^{i(ax+by)}}{f(D_x+ai, D_y+bi)}\{p(x,y)\}$$

Example 3.0.14 *: Find the particular integral of*

$$(D_x^2 - D_x D_y + 5)z = y^2 \sin(x-y)$$

Solution :

$$z_p = \frac{1}{D_x^2 - D_xD_y + 5}\{y^2 \sin(x-y)\}$$

$$= \frac{1}{D_x^2 - D_xD_y + 5}\{y^2 e^{i(x-y)}\}$$

$$= e^{i(x-y)}\frac{1}{(D_x+i)^2 - (D_x+i)(D_y-i) + 5}\{y^2\}$$

$$= e^{i(x-y)}\frac{1}{D_x^2 + 2iD_x - 1 - D_xD_y + iD_x - iD_y - 1 + 5}\{y^2\}$$

$$= e^{i(x-y)}\frac{1}{D_x^2 + 3iD_x - D_xD_y - iD_y + 3}\{y^2\}$$

$$= e^{i(x-y)}\frac{1}{3(1 - \frac{1}{3}(-D_x^2 - 3iD_x + D_xD_y + iD_y))}\{y^2\}$$

$$= \frac{e^{i(x-y)}}{3}[1 + \frac{1}{3}(-D_x^2 - 3iD_x + D_xD_y + iD_y) + \frac{1}{9}(-D_x^2 - 3iD_x + D_xD_y + iD_y)^2 + \ldots]\{y^2\}$$

$$= \frac{1}{3}e^{i(x-y)}[y^2 + \frac{1}{3}(-0 - 0 + 0 + i2y) + \frac{1}{9}(-2) + 0]$$

$$= \frac{1}{3}e^{i(x-y)}[y^2 + \frac{2i}{3}y + \frac{-2}{9}]$$

$$= \frac{1}{3}e^{i(x-y)}[(y^2 - \frac{2}{9}) + \frac{2i}{3}y]$$

$$= \frac{1}{3}[(y^2 - \frac{2}{9})\cos(x-y) - \frac{2}{3}y\sin(x-y) + i((y^2 - \frac{2}{9})\sin(x-y)) + \frac{2}{9}y\cos(x-y)]$$

$$\therefore \quad z_p = \frac{1}{3}((y^2 - \frac{2}{9})\sin(x-y) + \frac{2}{3}y\cos(x-y)) \quad is\ P.I.$$

2) Reduction of order

If $f(D_x, D_y)z = g(x, y)$ is reducible equation, then it can be written by the form

$$(a_1D_x + b_1D_y + c_1)(a_2D_x + b_2D_y + c_2)\dots(a_nD_x + b_nD_y + c_n)z = g(x, y)\dots(1$$

where a_i, b_i and c_i are constants.
Suppose

$$(a_2D_x + b_2D_y + c_2)\dots(a_nD_x + b_nD_y + c_n)z = u_1$$

then Eq.1) becomes

$$(a_1D_x + b_1D_y + c_1)u_1 = g(x, y) \Rightarrow \quad a_1D_xu_1 + b_1D_yu_1 = g(x, y) - c_1u_1$$

and by using the auxiliary equation we can solve the last equation:

$$\frac{dx}{a_1} = \frac{dy}{b_1} = \frac{du_1}{g(x, y) - c_1u_1}$$

$$\textit{since} \quad \frac{dx}{a_1} = \frac{dy}{b_1} \Rightarrow \quad b_1dx - a_1dy = 0 \Rightarrow$$

$$b_1x - a_1y = k_1, \textit{where } k_1 \textit{ is constant} \Rightarrow \quad y = \frac{b_1x + k_1}{a_1}$$

$$\frac{dx}{a_1} = \frac{du_1}{g(x, y) - c_1u_1} \Rightarrow \quad \frac{du_1}{dx} = \frac{1}{a_1}(g(x, y) - c_1u_1)$$

$$\frac{du_1}{dx} + \frac{c_1}{a_1}u_1 = \frac{1}{a_1}g(x, \frac{b_1x + k_1}{a_1}) \quad \textit{is first order LDE.}$$

After that we substitute u_1 in Eq.2), we get

$$(a_2D_x + b_2D_y + c_2)\dots(a_nD_x + b_nD_y + c_n)z = u_1(x, y)\dots(3$$

$$\textit{let} \quad (a_3D_x + b_3D_y + c_3)\dots(a_nD_x + b_nD_y + c_n)z = u_2$$

then equation (3) becomes

$$(a_2D_x + b_2D_y + c_2)u_2 = u_1(x, y)$$

continuing in this way until we find particular integral.

Example 3.0.15 *: Find the particular integral of the following equations?*

1) $(D_x - 3D_y)(D_x + 2D_y - 1)z = \sqrt{x+y}$

2) $D_y(D_x - D_y)^2 z = \dfrac{1}{x^2}$

3) $(D_x^2 + 2D_xD_y - 3D_y^2)z = \sqrt{2x+3y}$

Solution 1):

$$(D_x - 3D_y)(D_x + 2D_y - 1)z = \sqrt{x+y} \ldots (1$$

$$let \quad (D_x + 2D_y - 1)z = u_1 \ldots (2$$

substitute Eq.2) in Eq.1), we get

$$(D_x - 3D_y)u_1 = \sqrt{x+y}$$

$$D_xu_1 - 3D_yu_1 = \sqrt{x+y}$$

$$\frac{dx}{1} = \frac{dy}{-3} = \frac{du_1}{\sqrt{x+y}}$$

$$\frac{dx}{1} = \frac{dy}{-3} \Rightarrow \quad 3dx + dy = 0 \Rightarrow \quad 3x + y = k_1, where\ k_1\ is\ constant$$

$$\frac{dx}{1} = \frac{du_1}{\sqrt{x+y}} \Rightarrow \quad dx = \frac{du_1}{\sqrt{x+k_1-3x}} \Rightarrow \quad dx = \frac{du_1}{\sqrt{k-2x}}$$

$$\int \sqrt{k-2x}dx = \int du_1 \Rightarrow \quad \frac{-1}{2}\int (k-2x)^{\frac{1}{2}}(-2dx) = \int du_1$$

$$\frac{-1}{2}\frac{(k-2x)^{\frac{3}{2}}}{\frac{3}{2}} + k_2 = u_1 \Rightarrow \quad u_1 = \frac{-1}{3}(k-2x)^{\frac{3}{2}} + k_2$$

since we are just finding P.I., we take $k_2 = 0$

substitute $u_1 = \frac{-1}{3}(k-2x)^{\frac{3}{2}}$ *in Eq. 2)*

$$(D_x + 2D_y - 1)z = \frac{-1}{3}(k-2x)^{\frac{3}{2}}$$

$$D_x z + 2D_y z = \frac{-1}{3}(k-2x)^{\frac{3}{2}} + z$$

$$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{z - \frac{1}{3}(x+y)^{\frac{2}{3}} + z}$$

$$\text{since} \quad \frac{dx}{1} = \frac{dy}{2} \Rightarrow \quad 2dx - dy = 0 \Rightarrow$$

$$2x - y = k, \text{where } k \text{ is constant}$$

$$\frac{dx}{1} = \frac{dz}{z - \frac{1}{3}(x+y)^{\frac{2}{3}} + z} \Rightarrow \quad \frac{dz}{dx} = z - \frac{1}{3}(3x-k)^{\frac{2}{3}}$$

$$\frac{dz}{dx} = z - \frac{1}{3}(3x-k)^{\frac{2}{3}} \quad \text{is first order LDE.}$$

$$z = \frac{\int e^{\int -dx}(-\frac{1}{3}(3x-k)^{\frac{2}{3}})dx}{e^{\int -dx}}$$

$$= \frac{\int e^{-x}(-\frac{1}{3}(3x-k)^{\frac{2}{3}})dx}{e^{-x}} \Rightarrow \quad z_p = e^{x}[\frac{-1}{3}\int e^{-x}(3x-k)^{\frac{2}{3}}dx]$$

Example 3.0.16 *: Find P.I. for each of the following equations?*

1) $(D_x^2 - D_y)z = 1 - e^{x-y} + y^2$

2) $(D_x^2 - D_y)z = 1 - \sin(x-y)$

Solution 1):

$$z_p = \frac{1}{(D_x^2 - D_y)}\{1 - e^{x-y} + y^2\}$$

$$= \frac{1}{(D_x^2 - D_y)}\{1 - y^2\} - \frac{1}{(D_x^2 - D_y)}\{e^{x-y}\}$$

$$= \frac{1}{-D_y(1 - \frac{D_x^2}{D_y})}\{1 - y^2\} - \frac{e^{x-y}}{1^2 - (-1)}$$

$$= \frac{-1}{D_y}[1 + \frac{D_x^2}{D_y} + \dots]\{1 + y^2\} - \frac{1}{2}e^{x-y}$$

$$= \frac{-1}{D_y}[1 + y^2 + 0] - \frac{1}{2}e^{x-y} = -y - \frac{1}{3}y^3 - \frac{1}{2}e^{x-y}$$

$$\therefore \quad z_p = -y - \frac{1}{3}y^3 - \frac{1}{2}e^{x-y}$$

1) $r + s - 2t = e^{x+y}$

2) $r - s = 2q - z = x^2y^2$

3) $r + s - 2t - p - 2q = 0$

4) $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$

5) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$

Linear PDE with variable coefficients

Consider the second order LPDE wit variable coefficient of the form:

$$A(x,y)\frac{\partial^2 z}{\partial x^2} + B(x,y)\frac{\partial^2 z}{\partial x \partial y} + C(x,y)\frac{\partial^2 z}{\partial y^2} + D(x,y)\frac{\partial z}{\partial x} + E(x,y)\frac{\partial z}{\partial y} + F(x,y)z = g(x,y) \ldots (1$$

There exists some cases, we can solve Eq.1) directly:

Case 1:Cauchy Euler equation:

The PDE of the form

$$ax^2\frac{\partial^2 z}{\partial x^2} + bxy\frac{\partial^2 z}{\partial x \partial y} + cy^2\frac{\partial^2 z}{\partial y^2} + d\,x\frac{\partial z}{\partial x} + e\,y\frac{\partial z}{\partial y} + fz = g(x,y) \ldots (2$$

is called Cauchy Euler equation, where a, b, c, d, e and f are constants.
We use the transformation $x = e^u$ and $y = e^v$, i.e. $u = \ln x$ and $v = \ln y$
Now,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial u} \Rightarrow$$

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \Rightarrow \quad xD_x = D_u$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial v} \Rightarrow$$

$$yD_y = D_v$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) = \frac{\partial}{\partial x}(\frac{1}{x}\frac{\partial z}{\partial u}) = \frac{1}{x}\frac{\partial}{\partial x}(\frac{\partial z}{\partial u}) + \frac{\partial z}{\partial u}.\frac{-1}{x^2}$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} - \frac{1}{x^2}\frac{\partial z}{\partial u} \Rightarrow \quad x^2 D_x^2 = D_u^2 - D_u = D_u(D_u - 1)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) = \frac{\partial}{\partial y}(\frac{1}{y}\frac{\partial z}{\partial v}) = \frac{1}{y}\frac{\partial}{\partial y}(\frac{\partial z}{\partial v}) + \frac{\partial z}{\partial v}.\frac{-1}{y^2}$$

$$= \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} - \frac{1}{y^2}\frac{\partial z}{\partial v} \Rightarrow \quad y^2 D_y^2 = D_v^2 - D_v = D_v(D_v - 1)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) = \frac{\partial}{\partial x}(\frac{1}{y}\frac{\partial z}{\partial v}) = \frac{1}{y}\frac{\partial}{\partial x}(\frac{\partial z}{\partial v})$$

$$= \frac{1}{y}\frac{\partial}{\partial u}\frac{\partial z}{\partial v}.\frac{\partial u}{\partial x} = \frac{1}{xy}\frac{\partial^2 z}{\partial u \partial v} \Rightarrow \quad xyD_xD_y = D_uD_v$$

substitute in Eq. 2), we obtain:

$$(aD_u(D_u - 1) + bD_uD_v + cD_v(D_v - 1) + dD_u + eD_v + f)z = g(e^u, e^v)$$

is LPDE with constant coefficients.

Example 3.0.17 *: Solve*

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = cos(\ln y)$$

Solution :

$$(x^2D_x^2 - y^2D_y^2 + xD_x - yD_y)z = \cos\ln y$$

is Cauchy Euler equation.

$$let \quad x = e^u,\ y = e^v$$

$$x^2D_x^2 = D_u(D_u - 1),\ y^2D_y^2 = D_v(D_v - 1)$$

$$xD_x = D_u,\ yD_y = D_v$$

substitute in the main equation, we get

$$(D_u(D^u - 1) - D_v(D_v - 1) + D_u - D_v)z = \cos(v)$$

$$(D_u^2 - D_u - D_v^2 + D_v = D_u - D_v)z = \cos(v)$$

$$(D_u^2 - D_v^2)z = \cos(v) \quad \textit{is nonhomo. LPDE with constant coefficients.}$$

$$\textit{To find } z_c : (D_u^2 - D_v^2)z = 0$$

$$(D_u - D_v)(D_u + D_v)z = 0$$

$$z_c = \phi_1(-u - v) + \phi_2(u - v)$$

$$z_c = \phi_1(\ln x + \ln y) + \phi_2(\ln x - \ln y) \quad \textit{is complementary function.}$$

$$\textit{To find } z_p : z_p = \frac{1}{D_u^2 - D_v^2}\{\cos v\} = \frac{\cos v}{0 - (-1^2)} = \cos v \Rightarrow$$

$$z_p = \cos v = \cos \ln y \quad \textit{is particular integral.}$$

$$\therefore \quad z = z_c + z_p \quad \textit{is Gs.}$$

Remark : *To reduce Cauchy Euler equations of order* n *to LPDE with constant coefficients, we can use*

$x^n D_x^n = D_u(D_u - 1)(D_u - 2)\dots(D_u - n + 1)$
$y^m D_y^m = D_v(D_v - 1)(D_v - 2)\dots(D_v - m + 1)$
$x^n y^m D_x^n D_y^m = D_u(D_u - 1)(D_u - 2)\dots(D_u - n + 1)D_v(D_v - 1)(D_v - 2)\dots(D_v - m + 1)$

Example 3.0.18 *: Solve the following equations?*

1) $y\frac{\partial^3 z}{\partial x \partial y^2} - x\frac{\partial^3 z}{\partial x^2 \partial y} = 1$

2) $x^2\frac{\partial^2 z}{\partial x^2} - 4y^2\frac{\partial^2 z}{\partial y^2} - 4y\frac{\partial z}{\partial y} - z = \frac{y^2}{x}\ln x$

Solution 1):

$$(yD_xD_y^2 - xD_x^2D_y)z = 1$$

$(xy^2D_xD_y^2 - x^2yD_x^2D_y)z = xy$ *is Cauchy Euler equation.*

let $x = e^u$ *and* $y = e^v$

then

$$xy^2D_xD_y^2 = D_uD_v(D_{v-1})$$

$$x^2yD_x^2D_y = D_u(D_{u-1})D_v$$

substitute the last to equations in the main Eq.

$$(D_uD_v(D_{v-1}) - D_u(D_{u-1})D_v)z = e^{u+v}$$

is nonhomo. LPDE with constant coefficient.

To find z_c *:* $D_uD_v(D_v - D_u)z = 0$

$$z_c = \phi_1(v) + \phi_2(u) + \phi_3(u+v)$$

$z_c = \phi_1(\ln y) + \phi_2(\ln x) + \phi_3(\ln x + \ln y)$ *is complementary function.*

To find z_p *:* $z_p = \frac{1}{D_uD_v(D_v - D_u)}e^{u+v}; \quad a = 1, b = 1$

$$= \frac{e^{u+v}}{1.1}\frac{u}{1!} \Rightarrow \quad z_p = \ln(x)e^{\ln x + \ln y}$$

$= xy\ln(x) \Rightarrow \quad z_p = xy\ln(x)$ *is P.I.*

$z = z_c + z_p$ *is Gs.*

Case 2: *Equations belonging to this type are of the form*

$$\frac{\partial^2 z}{\partial x^2} = \frac{g}{A}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{g}{B} \qquad (1$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{g}{C}$$

Equation (1) is reducible to equations with constant coefficient.

Example 3.0.19 *: Solve*

1) $xy^2 \frac{\partial^2 z}{\partial y \partial x} = x^3 - y^3$

2) $\frac{\partial^2 z}{\partial x^2} = x^2 e^{-y}$

3) $xy^2 \frac{\partial^2 z}{\partial x \partial y} = 1 - 4x^2 y$

Solution 1):

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2}{y^2} - \frac{y}{x}$$

$$\frac{\partial z}{\partial y} = \int (\frac{x^2}{y^2} - \frac{y}{x})dx = \frac{1}{3}\frac{x^3}{y^2} - y \ln x + \phi_1(y)$$

$$z = \frac{-1}{3}\frac{x^3}{y} - \frac{1}{2}y^2 \ln x + \int \phi_1(y)dy + \phi_2(x)$$

$$z = \frac{-1}{3}\frac{x^3}{y} - \frac{1}{2}y^2 \ln x + \phi_3(y) + \phi_2(x); \quad \textit{where} \quad \phi_3(y) = \int \phi_1 dy$$

Case 3: *Equations belonging to this type are of the form*

$$A(x,y)\,\frac{\partial^2 z}{\partial x^2}+D(x,y)\,\frac{\partial z}{\partial x}=g(x,y)\Rightarrow\quad A\,\frac{\partial p}{\partial x}+D\;p=g$$

$$B(x,y)\,\frac{\partial^2 z}{\partial x\partial y}+D(x,y)\,\frac{\partial z}{\partial x}=g(x,y)\Rightarrow\quad B\,\frac{\partial p}{\partial y}+D\;p=g$$

$$B(x,y)\,\frac{\partial^2 z}{\partial x\partial y}+E(x,y)\,\frac{\partial z}{\partial y}=g(x,y)\Rightarrow\quad B\,\frac{\partial q}{\partial x}+E\;q=g$$

$$C(x,y)\,\frac{\partial^2 z}{\partial y^2}+E(x,y)\,\frac{\partial z}{\partial y}=g(x,y)\Rightarrow\quad C\,\frac{\partial q}{\partial y}+E\;q=g$$

which are linear ODEs of order one in which p(or q) is the dependent variable.

Example 3.0.20 *: Solve*

1) $y\,\frac{\partial^2 z}{\partial y^2}+2\,\frac{\partial z}{\partial y}=3ye^{2x}$

2) $x\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial z}{\partial x}=xe^{x+y}$

3) $x\frac{\partial^2 z}{\partial y^2}-x\frac{\partial z}{\partial y}=-\sin y-x\cos y$

Solution 1):

$$let\quad \frac{\partial z}{\partial y}=q$$

$$y\frac{\partial q}{\partial y}+2q=3e^{2x}y\Rightarrow\quad \frac{\partial q}{\partial y}+\frac{2}{y}q=3e^{2x}$$

is first order LDE.

$$q=\frac{\int e^{\int\frac{2}{y}dy}\;3e^{2x}dy+\phi_1(x)}{e^{\int\frac{2}{y}dy}}=\frac{\int e^{2\ln y}\;3e^{2x}dy+\phi_1(x)}{e^{2\ln y}}$$

$$=\frac{\int 3y^2e^{2x}dy+\phi_1(x)}{y^2}=y^{-2}[e^{2x}y^3+\phi_1(x)]$$

$$q = ye^{2x} + y^{-2}\phi_1(x) \Rightarrow \quad \frac{\partial z}{\partial y} = ye^{2x} + y^{-2}\phi_1(x)$$

$$z = \frac{1}{2}y^2e^{2x} - \frac{\phi_1(x)}{y} + \phi_2(x) \quad \text{is Gs.}$$

2)

$$\text{let} \quad p = \frac{\partial z}{\partial x}$$

$$x\frac{\partial p}{\partial x} - 2xy\, p = xe^{x+y} \quad \text{is first order LDE.}$$

$$p = \frac{\int e^{\int -2ydx}\, e^{x+y}\, dx + \phi_1(y)}{e^{\int -2ydx}}$$

Case 4: *Equations belonging to this type are of the form*

$$A(x,y)\,\frac{\partial^2 z}{\partial x^2} + B(x,y)\,\frac{\partial^2 z}{\partial x \partial y} + D(x,y)\,\frac{\partial z}{\partial x} = g(x,y) \Rightarrow$$

$$A\,\frac{\partial p}{\partial x} + B\,\frac{\partial p}{\partial y} = g - D\,p$$

$$B(x,y)\,\frac{\partial^2 z}{\partial x \partial y} + C(x,y)\,\frac{\partial^2 z}{\partial y^2} + E(x,y)\,\frac{\partial z}{\partial y} = g(x,y) \Rightarrow$$

$$B\,\frac{\partial q}{\partial x} + C\,\frac{\partial q}{\partial y} = g - Eq$$

These equations are linear with p(or q) as dependent variable and x and y as independent variables.

Example 3.0.21 *: Solve each of the following PDEs?*

1) $x\dfrac{\partial^2 z}{\partial x \partial y} - 2x^2\dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial z}{\partial y} = 1$

2) $y\dfrac{\partial^2 z}{\partial x \partial y} - 2x\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} = 6xy$

Solution 1):

$$let \quad \frac{\partial z}{\partial y} = q$$

$$x\,\frac{\partial q}{\partial x} - 2x^2\,\frac{\partial q}{\partial y} - q = 1 \Rightarrow \quad x\,\frac{\partial q}{\partial x} - 2x^2\,\frac{\partial q}{\partial y} = 1 + q \quad \textit{is quasi linear PDE.}$$

$$\frac{dx}{x} = \frac{dy}{-2x^2} = \frac{dq}{1+q}$$

$$\frac{dx}{x} = \frac{dy}{-2x^2} \Rightarrow \quad 2xdx + dy = 0 \Rightarrow \quad x^2 + y = c_1$$

$$\frac{dx}{x} = \frac{dq}{1+q} \Rightarrow \quad \ln(1+q) = \ln x + c_2 \Rightarrow \quad 1 + q = cx; \quad c = e^{c_1}$$

$$q = cx - 1 \Rightarrow \quad z = \int (cx-1)dy \Rightarrow \quad z = cxy - y + \phi_2(x)$$

$$z = \phi_1(x^2+y)xy - y + \phi_2(x) \quad \textit{is Gs.}$$

2)

$$let \quad \frac{\partial z}{\partial x} = p \Rightarrow \quad y\frac{\partial p}{\partial y} - 2x\frac{\partial p}{\partial x} = 6xy + 2p$$

$$\frac{dx}{-2x} = \frac{dy}{y} = \frac{dp}{6xy + 2p}$$

Case 5: *Equations belonging to this type are of the form*

$$A(x,y)\,\frac{\partial^2 z}{\partial x^2} + D(x,y)\,\frac{\partial z}{\partial x} + fz = g(x,y) \quad (\textit{consider } y \textit{ is constant.})$$

$$C(x,y)\,\frac{\partial^2 z}{\partial y^2} + E(x,y)\,\frac{\partial z}{\partial y} + fz = g(x,y) \quad (\textit{consider } x \textit{ is constant.})$$

The equations are linear ODE with x (or y) as independent variable and z as dependent variable.

Example 3.0.22 *: Solve*

1) $\dfrac{\partial^2 z}{\partial x^2} - y\dfrac{\partial z}{\partial x} - 2y^2 z = ye^{x-2y}$

2) $t - 2xq + x^2 z = (x-2)e^{3x+2y}$

Solution 1):

let y *be a constant.*

$$(D_x^2 - yD_x - 2y^2)z = ye^{x-2y}$$

$z_c = e^{2xy}\phi_1(y) + e^{-yx}\phi_2(y)$ *is complementary function.*

$$z_p = \frac{1}{(D_x - 2y)(D_x + y)}\{ye^{x-2y}\} = \frac{ye^{x-2y}}{(1-2y)(1+2y)}$$

$z = z_c + z_p$ *is Gs.*

2)

$$\frac{\partial^2 z}{\partial y^2} - 2x\frac{\partial z}{\partial y} + x^2 z = (x-2)e^{3x+2y}$$

let x *be a constant.*

$$(D_y^2 - 2xD_y + x^2)z = (x-2)e^{3x+2y}$$

Example 3.0.23 *: Solve the following equations?*

1) $x\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial z}{\partial x} = xe^{x-y}$

2) $y\dfrac{\partial^2 z}{\partial y^2} + 2\dfrac{\partial z}{\partial y} = xy^2$

3) $\dfrac{\partial^2 z}{\partial x^2} = 6xy \quad z(0,y) = y, \quad \dfrac{\partial z}{\partial x}(1,y) = 0$

4) $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}; z(0,y) = 0, \quad \dfrac{\partial z}{\partial x}(0,y) = \dfrac{1}{1+y^2}; \quad a \in R$

5) $xy\dfrac{\partial^2 z}{\partial x \partial y} - x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} + z = 0$

Solution 1):

$$let \quad \frac{\partial z}{\partial x} = p, \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}$$

$$x\frac{\partial p}{\partial x} - 2xyp = xe^{x-y}$$

$$\frac{\partial p}{\partial x} - 2yp = e^{x-y} \quad \textit{is first order LDE}$$

$$p = \frac{\int e^{\int -2ydx} e^{x-y}dx + \phi_1(y)}{e^{\int -2ydx}} = \frac{\int e^{-2yx}e^{x-y}dx + \phi_1(y)}{e^{-2yx}}$$

$$= e^{2xy}[e^{(1-2y)x}e^{-y}dx + \phi_1(y)] = e^{2xy}[\frac{e^{-y}}{1-2y}e^{(1-2y)x} + \phi_1(y)]$$

$$p = \frac{1}{1-2y}e^{x-y} + e^{2xy}\phi_1(y)$$

$$\frac{\partial z}{\partial x} = \frac{1}{1-2y}e^{x-y} + e^{2xy}\phi_1(y)$$

$$z = \frac{1}{1-2y}e^{x-y} + \frac{1}{2y}e^{2xy}\phi_1(y) + \phi_2(y)$$

3)

$$\frac{\partial^2 z}{\partial x^2} = 6xy \Rightarrow \quad \frac{\partial z}{\partial x} = \int 6xydx + \phi_1(y)$$

$$= 3x^2y + \phi_1(y)$$

$$z = \int (3x^2y + \phi_1(y))dx + \phi_2(y) = x^3y + x\phi_1(y) + \phi_2(y)$$

$$since \quad z(0, y) = y \Rightarrow \quad \phi_2 = y \ldots (1$$

$$\frac{\partial z}{\partial x} = 3x^2y + \phi_1(y)$$

$$since \quad \frac{\partial z}{\partial x}(0, y) = 0 \Rightarrow \quad 3y + \phi_1(y) = 0$$

$$\phi_1(y) = -3y \ldots (2$$

$$z = x^3y - 3yx + y \quad is\ Gs.\ of\ BVP$$

4)

$(D_x^2 - a^2 D_y^2)z = 0$ *is homo. LPDE with constant coifficients.*

$(D_x - aD_y)(D_x + aD_y)z = 0$

$z_c = \phi_1(ax + y) + \phi_2(ax - y)$

since $z(0, y) = 0$

$0 = \phi_1(y) + \phi_2(-y) \Rightarrow \quad \phi_1(y) + \phi_2(-y) = 0 \ldots (1$

$$\frac{\partial z}{\partial x} = a\phi_1'(ax + y) + a\phi_2'(ax - y)$$

since $\dfrac{\partial z}{\partial x}(0, y) = \dfrac{1}{1 + y^2}$

$$a(\phi_1'(y) + \phi_2'(-y)) = \frac{1}{1 + y^2}$$

$$\phi_1'(y) + \phi_2'(-y) = \frac{1}{a} \frac{1}{1 + y^2}$$

$$\phi_1(y) - \phi_2(-y) = \frac{1}{a} \tan^{-1}(y) \ldots (2$$

$$\underline{\phi_1(y) + \phi_2(-y) = 0} \ldots (1$$

$$2\phi_1(y) = \frac{1}{a} \tan^{-1}(y) \Rightarrow \quad \phi_1(y) = \frac{1}{2a} \tan^{-1}(y)$$

by substituting $\phi_1(y)$ in Eq.1), we get

$$\phi_2(-y) = \frac{-1}{2a}\ \tan^{-1}(y)$$

$$\phi_1(y) = \frac{1}{2a}\ \tan^{-1}(y)$$

$$\phi_1(ax+y) = \frac{1}{2a}\ \tan^{-1}(ax+y)$$

$$\phi_2(ax-y) = \phi_2(-(y-ax)) = \frac{-1}{2a}\tan -^{-1}(y-ax)$$

$$z = \frac{1}{2a}\tan^{-1}(ax+y) + \frac{-1}{2a}\tan^{-1}(y-ax)$$

Classification Of A linear PDE Of The Second Order With Variable coefficients

A linear PDE of the second order in two independent variables is given by

$$A(x,y)\frac{\partial^2 z}{\partial x^2} + B(x,y)\frac{\partial^2 z}{\partial x \partial y} + C(x,y)\frac{\partial^2 z}{\partial y^2} + D(x,y)\frac{\partial z}{\partial x} + E(x,y)\frac{\partial z}{\partial y} + F(x,y)z = g(x,y)\ldots(1$$

where A, B, C, D, E, F and g are functions of x and y then Eq. 1) can be classified into three cases:

Case 1: *If $B^2 - 4AC > 0$, then Eq.(1 is called hyperbolic equation.*

Case 2: *If $B^2 - 4AC = 0$, then Eq.(1 is called parabolic equation.*

Case 3: *If $B^2 - 4AC < 0$, then Eq.(1 is called elliptic equation.*

Example 3.0.24 *: For what values of x and y are the following PDEs hyperbolic, parabolic or elliptic?*

1) $x\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = x^2$

2) $e^x\dfrac{\partial^2 z}{\partial x^2} + e^y\dfrac{\partial^2 z}{\partial y^2} = 0$

3) $x^2\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = e^x$

Solution 1):

$$A = x, B = 0, C = 1$$

$$B^2 - 4AC = 0 - 4x = -4x$$

If $x > 0$, *then* $B^2 - 4AC < 0$, *therefore the equation is elliptic.*
If $x = 0$, *then* $B^2 - 4AC = 0$, *therefore the equation is parabolic.*
If $x < 0$, *then* $B^2 - 4AC > 0$, *therefore the equation is hyperbolic.*
2)

$$A = e^x, B = 0, C = e^y$$

$$B^2 - 4AC = 0 - 4e^x e^y = -4e^{x+y} < 0 \quad \forall (x, y) \in R$$

$$\therefore \quad \textit{the main Eq. is elliptic.}$$

3)

$$A = x^2, B = -2xy, C = y^2$$

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0 \quad \forall (x, y) \in R$$

$$\therefore \quad \textit{the main Eq. is parabolic.}$$

Canonical Forms Of Second Order Linear Equations
By using a suitable change in the independent variables all second order LPDEs of the form Eq.1) can be reduced to one of the three canonical forms:

We use the transformation $u = u(x, y)$ and $v = v(x, y)$ to reduce Eq.1):

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x})$$

$$= \frac{\partial}{\partial u}(\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x})\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}(\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x})\frac{\partial v}{\partial x}$$

$$= (\frac{\partial z}{\partial u} \cdot \frac{\partial}{\partial u}(\frac{\partial u}{\partial x}) + \frac{\partial u}{\partial x} \cdot \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial v}\frac{\partial}{\partial u}(\frac{\partial v}{\partial x}) + \frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial u \partial v})\frac{\partial u}{\partial x} + (\frac{\partial z}{\partial u} \cdot \frac{\partial}{\partial v}(\frac{\partial u}{\partial x})$$

$$+ \frac{\partial u}{\partial x} \cdot \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v}\frac{\partial}{\partial v}(\frac{\partial v}{\partial x}) + \frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v^2})\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + (\frac{\partial u}{\partial x})^2\frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial v}\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial u \partial x} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u \partial v}$$

$$+ \frac{\partial z}{\partial u}\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial v \partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2} + (\frac{\partial v}{\partial x})^2\frac{\partial^2 z}{\partial v^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = u_x^2\frac{\partial^2 z}{\partial u^2} + 2u_x v_x\frac{\partial^2 z}{\partial u \partial v} + v_x^2\frac{\partial^2 z}{\partial v^2} + u_{xx}\frac{\partial z}{\partial u} + v_{xx}\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x \partial y} = u_x u_y\frac{\partial^2 z}{\partial u^2} + (u_x u_y + u_y v_x)\frac{\partial^2 z}{\partial u \partial v} + v_x v_y\frac{\partial^2 z}{\partial v^2} + u_{xy}\frac{\partial z}{\partial u} + v_{xy}\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial y^2} = u_y^2\frac{\partial^2 z}{\partial u^2} + 2u_y v_y\frac{\partial^2 z}{\partial u \partial v} + v_y^2\frac{\partial^2 z}{\partial v^2} + u_{yy}\frac{\partial z}{\partial u} + v_{yy}\frac{\partial z}{\partial v}$$

substitute in Eq.1), we get:

$$A^*\frac{\partial^2 z}{\partial u^2}+B^*\frac{\partial^2 z}{\partial u\partial v}+C^*\frac{\partial^2 z}{\partial v^2}+D^*\frac{\partial z}{\partial u}+E^*\frac{\partial z}{\partial v}+F^*z=g^*\ldots(2$$

where

$$A^* = Au_x^2 + Bu_xu_y + Cu_y^2$$

$$B^* = 2Au_xv_x + B(u_xv_y + v_xu_y) + 2Cu_yv_y$$

$$C^* = Av_x^2 + Bv_xv_y + Cv_y^2$$

$$D^* = Au_{xx} + Bu_{yx} + Cu_{yy} + Du_x + Eu_y$$

$$E^* = Av_{xx} + Bv_{yx} + Cv_{yy} + Dv_x + Ev_y$$

but $F^* = F$ *and* $g^* = g$*. Now, the problem is to determine u and v.*

Case I) Hyperbolic Equation

The equation for which $B^2 - 4AC > 0$ *are called hyperbolic equation. The roots of* $A\lambda^2 + B\lambda + C = 0$ *are real and given by*

$$\lambda = \frac{1}{2A}(B \mp \sqrt{B^2 - 4AC}), \quad textwhere \quad A \neq 0$$

let $f_1(x,y) = c_1$ *and* $f_2(x,y) = c_2$ *be a solution of ODE* $\frac{dy}{dx} + \lambda_1 = 0$ *and* $\frac{dy}{dx} + \lambda_2 = 0$ *respectively.*
Moreover, let $u = f_1(x,y)$ *and* $v = f_1(x,y)$*. Since* $A^* = C^* = 0$ *and* $B^* \neq 0$ *and Eq.1) reduces to canonical form*

$$\frac{\partial^2 z}{\partial u\partial v} = H(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v})$$

Example 3.0.25 *: Rewrite the equation* $y\frac{\partial^2 z}{\partial x^2}+(x+y)\frac{\partial^2 z}{\partial x\partial y}+x\frac{\partial^2 z}{\partial y^2}$ *in canonical form and then solve it?*

Solution :

$$A = y, B = x + y, C = x, D = E = F = g = 0$$

$$B^2 - 4AC = (x + y)^2 - 4xy = x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 = (x - y)^2$$

$$B^2 - 4AC > 0 \ldots (1 if \;\; x \neq y, then \; Eq.(1 \; is \; hyperbolic$$

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad y\lambda^2 + (x + y)\lambda + x = 0 \Rightarrow$$

$$\lambda = \frac{-(x + y) \mp \sqrt{(x - y)^2}}{2y} = \frac{-(x + y) \mp (x - y}{2y}$$

$$\lambda = \frac{-x - y + x - y}{2y} = -1$$

$$\lambda = \frac{-x - y - x + y}{2y} = \frac{-x}{y}$$

Then we solve

$$\frac{dy}{dx} - 1 = 0 \Rightarrow \quad dy - dx = 0 \Rightarrow \quad y - x = c_1$$

and

$$\frac{dy}{dx} - \frac{x}{y} = 0 \Rightarrow \quad ydy - xdx = 0 \Rightarrow \quad y^2 - x^2 = c_2$$

$$let \quad u = y - x \quad and \quad v = y^2 - x^2$$

$$u_x = -1 \qquad\qquad v_x = -2x$$

$$u_y = 1 \qquad\qquad v_y = 2y$$

$$u_{xx} = u_{yy} = u_{xy} = 0 \qquad\qquad v_{xx} = -2$$

$$v_{yy} = 2, v_{xy} = 0$$

$$A^* = y(-1)^2 + (x+y)(-1) + x(1) = y - x - y + x = 0$$

$$B^* = 2y(-1)(-2x) + (x+y)(-2y-2x) + 2x(2y)$$

$$= 4xy - 2(x+y)^2 + 4xy = 4xy - 2x^2 - 4xy - 2y^2 + 4xy$$

$$= -2(x^2 - 2xy + y^2) = -2(x-y)^2 = -2u^2$$

$$C^* = y(4x^2) + (x+y)(-4xy) + x(4y^2) = 4x^2y - 4x^2y - 4xy^2 + 4xy^2 = 0$$

$$D^* = y(0) + (x+y)(0) + x(0) + 0 + 0 = 0$$

$$E^* = y(-2) + (x+y)(0) + x(2) + 0 + 0 = 2(x-y) = -2u$$

$$F^* = F = 0$$

$$g^* = g = 0$$

substitute in Eq.(2:

$$-2u^2\frac{\partial^2 z}{\partial u \partial v} - 2u\frac{\partial z}{\partial v} = 0 \quad \textit{is canonical form.}$$

$$u\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} = 0$$

$$\frac{\partial}{\partial u}(u\frac{\partial z}{\partial v}) = 0 \Rightarrow \quad u\frac{\partial z}{\partial v} = \phi_1(v)$$

$$\frac{\partial z}{\partial v} = \frac{1}{u}\phi_1(v) \Rightarrow \quad z = \frac{1}{u}\int \phi_1(v)dvv + \phi_2(u)$$

$$\therefore \quad z = \frac{1}{u}\phi_3(v) + \phi_2(u); \, where \quad \phi_3(v) = \int \phi_1(v)dv$$

$$z = \frac{1}{y-x}\phi_3(y^2 - x^2) + \phi_2(y-x) \quad is \ Gs. \ if \, x \neq y$$

If $y = x$ *then* $B^2 - 4AC = 0$*, then Eq.(1 is parabolic*

$$x\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x \partial y} + x\frac{\partial^2 z}{\partial y^2} = 0$$

$$\Rightarrow \quad (D_x^2 + 2D_x D_y + D_y^2)z = 0, \ if \ \ x \neq 0$$

$$(D_x + D_y)(D_x + D_y)z = 0 \Rightarrow \quad (D_x + D_y)^2 z = 0$$

$$z = \phi_1(x-y) + x\phi_2(x-y) \quad is \ Gs.$$

Example 3.0.26 *: Rewrite the equation*

$$4\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2$$

in canonical form, and then solve it? ***Solution :***

$$A = 4, B = 5, C = 1, D = 1, E = 1, g = 2$$

$$since \quad B^2 - 4AC = 25 - 16 = 9 > 0$$

$\therefore$ *the equation is hyperbolic.*

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad 4\lambda^2 + 5\lambda + 1 = 0$$

$$\lambda = \frac{-5 \mp \sqrt{9}}{8} = \frac{-5 \mp 3}{8} \quad \lambda_1 = \frac{-1}{4} \quad \lambda_2 = -1$$

Then, we solve

$$\frac{dy}{dx} - \frac{1}{4} = 0 \Rightarrow \quad 4dy - dx = 0 \Rightarrow \quad 4y - x = c_1 = u$$

$$\frac{dy}{dx} - 1 = 0 \Rightarrow \quad dy - dx = 0 \Rightarrow \quad y - x = c_2 = v$$

$$u_x = -1 \qquad\qquad v_x = -1$$

$$u_y = 4 \qquad\qquad v_y = 1$$

$$u_{xx} = u_{xy} = u_{yy} = 0 \qquad v_{xx} = v_{xy} = v_{yy} = 0$$

$$A^* = C^* = 0$$

$$B^* = 2(4)(-1)(-1) + 5(-1-4) + 2(4) = 8 - 25 + 8 = -9$$

$$D^* = 4(0) + 5(0) + 1(0) + 1(-1) + 1(4) = 3$$

$$E^* = 4(0) + 5(0) + 1(0) + 1(-1) + 1(1) = 0$$

$$F^* = 0, g^* = g = 2$$

Then Eq.(1 reduce to canonical form

$$-13\frac{\partial^2 z}{\partial u \partial v} + 3\frac{\partial z}{\partial u} = 2 \quad \textit{is nonhomo. LPDE}$$

$$(-13D_uD_v + 3D_u)z = 2$$

To find z_c:$D_u(-13D_v + 3)z = 0$

$$z_c = \phi_1(v) + e^{\frac{3}{13}v}\phi_2(13u)$$

$$z_c = \phi_1(y-x) + e^{\frac{3}{13}(y-x)}\phi_2(13(4y-x)) \quad \textit{is complementary function}$$

$$z_p = \frac{1}{3D_u(1-\frac{13}{3}D_v}\{2\} = \frac{1}{3D_u}[1+\frac{13}{3}D_v + (\frac{13}{3}D_v)^2 + \ldots]\{2\}$$

$$z_p = \frac{1}{3D_u}\{2\} = \frac{2}{3}u$$

$$z_p = \frac{2}{3}(4y-x) \quad \textit{is particular integral}$$

$$z = z_c + z_p \quad \textit{is Gs.}$$

Case II) Parabolic Equation

If $B^2 - 4AC = 0$, then the equation is parabolic and $A\lambda^2 + B\lambda + C = 0$ has one real root $\lambda = \frac{-B}{2A}$. Let $f(x,y) = c$ be a solution of ODE $\frac{dy}{dx} + \lambda = 0$. Furthermore, we suppose that $f(x,y) = u$ and we will also define another function, say $v = v(x,y)$ to be any function of x and y and independent of u. Moreover, in this case $A^* = B^* = 0$, but $C^* \neq 0$ and Eq. (1 reduce to canonical form:

$$\frac{\partial^2 z}{\partial v^2} + p(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v})$$

Example 3.0.27 *: Rewrite the equation*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = 0$$

in the canonical form and solve it?

Solution :

$$A = x^2, B = 2xy, C = y^2, D = xy, E = y^2, F = g = 0$$

$$B^2 - 4AC = (2xy)^2 - 4x^2y^2 = 4x^2y^2 - 4x^2y^2 = 0$$

$$\therefore \quad \textit{the equation is parabolic}$$

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

$$\lambda = \frac{-B}{2A} = \frac{-2xy}{2x^2} = \frac{-y}{x}$$

Now, we will solve $\dfrac{dy}{dx} + \lambda = 0$

$$\frac{dy}{dx} - \frac{y}{x} = 0 \Rightarrow \quad \frac{dy}{y} - \frac{dx}{x} = 0 \Rightarrow \quad \ln y - \ln x = c$$

$$\ln\frac{y}{x} = c \Rightarrow \quad \frac{y}{x} = c_1 = u$$

For arbitrary values of $v(x, y)$ which are functionally independent of u; For instance if $v = x$, the Jacobian does not vanish in the domain of parabolicity.

$$u = \frac{y}{x} \qquad\qquad v = x$$

$$u_x = \frac{-y}{x^2} \qquad\qquad v_x = 1$$

$$u_y = \frac{1}{x} \qquad\qquad v_y = v_{xx} = v_{yy} = v_{xy} = 0$$

$$u_{xx} = \frac{2y}{x^3} \quad u_{yy} = 0 \quad u_{xy} = \frac{-1}{x^2}$$

$$A^* = x^2(\frac{-y}{x^2})^2 + 2xy(\frac{-y}{x^2})(\frac{1}{x}) + y^2(\frac{1}{x})^2$$

$$= \frac{y^2}{x^2} - 2\frac{y^2}{x^2} + \frac{y^2}{x^2} = 0$$

$$B^* = 2(x^2)(\frac{-y}{x^2})(1) + 2xy(0 + \frac{1}{x}) + 2y^2(0) = -2y + 2y = 0$$

$$C^* = x^2(1)^2 + 2xy(0) + y^2(0) = x^2 = v^2$$

$$D^* = x^2\frac{2y}{x^2} + 2xy(\frac{-1}{x^2}) + y^2(0) + xy(\frac{-y}{x^2}) + y^2(\frac{1}{x})$$

$$= \frac{2y}{x} + \frac{-2y}{x} - \frac{y^2}{x} + \frac{y^2}{x} = 0$$

$$E^* = x^2 + 2xy(0) + y^2(0) + xy(1) + y^2(0) = xy = x^2\frac{y}{x} = uv^2$$

$F^* = F = 0 \qquad\qquad g^* = g$

Then

$$v^2\frac{\partial^2 z}{\partial v^2} + uv^2\frac{\partial z}{\partial v} = 0$$

$$\frac{\partial^2 z}{\partial v^2} + u\frac{\partial z}{\partial v} = 0$$

Let $L = \frac{\partial z}{\partial v} \Rightarrow \quad \frac{\partial L}{\partial v} + uL = 0$ *is first order LDE.*

$$L = \frac{\int e^{\int u dv}(0)dv + \phi_1(u)}{e^{\int u dv}} = e^{-uv}\phi_1(u)$$

$$\therefore \quad \frac{\partial z}{\partial v} = e^{-uv}\phi_1(u) \Rightarrow \quad z = \int e^{-uv}\phi_1(u)dv$$

$$z = \frac{-\phi_1(u)}{u}e^{-uv} + \phi_2(u)$$

$$z = \frac{-x}{y}e^{-y}\phi_1(\frac{y}{x}) + \phi_2(\frac{y}{x}) \quad \textit{is Gs.}$$

Example 3.0.28 *: Rewrite the equation*

$$x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} - e^x = 0$$

in canonical form and solve it?

$$A = x^2, B = -2xy, C = y^2, F = D = E = 0 \;\; \textit{and} \;\; g = e^x$$

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$$

$$\therefore \quad \textit{the equation is parabolic}$$

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad x^2\lambda^2 + B\lambda + y^2 = 0$$

$$\lambda = \frac{2xy}{2x^2} = \frac{y}{x}$$

$$\textit{and solve} \quad \frac{dy}{dx} + \frac{y}{x} = 0 \Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\ln x + \ln y = c \Rightarrow \quad \ln xy = c \Rightarrow \quad xy = c_1 = u$$

$xy = u \qquad\qquad v = x$

$u_x = y \qquad\qquad v_x = 1$

$u_y = x \qquad\qquad v_y = 0$

$u_{xx} = u_{yy}, u_{xy} = 1 \qquad\qquad v_{xx} = v_{yy} = v_{xy} = 0$

then $\quad A^* = B^* = 0$

$C^* = x^2(1)^2 + (-2xy)(0) + y^2(0) = x^2 = v^2$

$D^* = -2xy = -2u$

$E^* = F^* = 0$

$g^* = g = e^x = e^v$

$\therefore$ *the canonical form is*

$v^2 \dfrac{\partial^2 z}{\partial v^2} - 2u \dfrac{\partial z}{\partial u} = e^v$ *is Cauchy Euler equation*

Case III) Elliptic Equation

If $B^2-4AC < 0$, *then the equation is elliptic and the roots of* $A\lambda^2+B\lambda+C = 0$ *are complex conjugate functions of* x *and* y, *i.e.* $\lambda_1 = a + ib$ *and* $\lambda_2 = a - ib$, *where* a *and* b *are functions of* x *and* y.*Let* $f_1(x, y) = c_1$ *and* $f_2(x, y) = c_2$ *be solutions of ODE* $\dfrac{dy}{dx} + \lambda_1 = 0$ *and* $\dfrac{dy}{dx} + \lambda_2 = 0$ *respectively. Now, we assume that* $f_1(x, y) = u$ *and* $f_2(x, y) = v$.

since $u = \bar{v}$ *if* $u = \alpha + i\beta \Rightarrow \quad v = \alpha - i\beta$.

since α *=real part of* u, $\alpha = \dfrac{u+v}{2}$
since β *=imaginary part of* v $\beta = \dfrac{u-v}{2i}$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2})$$

and Eq.(1 reduces to canonical form

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = F(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta})$$

Example 3.0.29 *: Rewrite the equation*

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

in canonical form? ***Solution :***

$$A = 1, C = x^2, B = D = E = F = g = 0$$

$$B^2 - 4AC = 0 - 4x^2 = -4x^2 < 0 \ \ if \ \ x \neq 0$$

$\therefore$ *the equation elliptic.*

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad \lambda^2 + 0 + x^2 = 0 \Rightarrow \quad \lambda^2 + x^2 = 0 \Rightarrow \quad \lambda^2 = -x^2$$

$$\lambda = \mp ix$$

$$\frac{dy}{dx} + ix = 0 \Rightarrow \quad y + \frac{1}{2} ix^2 = c \Rightarrow \quad 2y + ix^2 = c_1 = u$$

$$\frac{dy}{dx} - ix = 0 \Rightarrow \quad dy - ixdx = 0 \Rightarrow \quad y - \frac{i}{2} x^2 = c \Rightarrow \quad 2y - ix^2 = c_2 = v$$

$$u_x = 2ix \qquad\qquad v_x = -2i$$

$$u_y = 2 \qquad\qquad v_y = 2$$

$u_{xx} = 2i \qquad\qquad v_{xx} = -2i$

$u_{yy} = u_{xy} = 0 \qquad v_{yy} = v_{xy} = 0$

$A^* = 1(2ix)^2 + 0() + x^2(2)^2 = -4x^2 + 4x^2 = 0$

$B^* = 2(2ix)(-2ix) + 2(0) + 2x^2(2)(2) = 8x^2 + 8x^2 = 16x^2 = 16\beta$

$C^* = 1(-2ix)^2 + 0() + x^2(2)^2 = -4x^2 + 4x^2 = 0$

$D^* = 1(2i) + 0() + x^2(0) + 0 + 0 = 2i$

$E^* = 1(-2i) + 0() + x^2(0) + 0 + 0 = -2i$

Then the Eq. becomes

$$16\beta\frac{\partial^2 z}{\partial u\partial v} + 2i\frac{\partial z}{\partial u} - 2i\frac{\partial z}{\partial v} = 0$$

$$since \quad \alpha = \frac{u+v}{2} \qquad \beta = \frac{u-v}{2i}$$

$$\frac{\partial\alpha}{\partial u} = \frac{1}{2}, \quad \frac{\partial\alpha}{\partial v} = \frac{1}{2}, \quad \frac{\partial\beta}{\partial u} = \frac{1}{2i} = \frac{-i}{2}, \quad \frac{\partial\beta}{\partial v} = \frac{-1}{2i} = \frac{i}{2}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial\alpha}\,\frac{\partial\alpha}{\partial u} + \frac{\partial z}{\partial\beta}\,\frac{\partial\beta}{\partial u}$$

$$= \frac{1}{2}\frac{\partial z}{\partial\alpha} - \frac{i}{2}\frac{\partial z}{\partial\beta} \Rightarrow \qquad \frac{\partial z}{\partial u} = \frac{1}{2}(\frac{\partial z}{\partial\alpha} - i\frac{\partial z}{\partial\beta)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial\alpha}\,\frac{\partial\alpha}{\partial v} + \frac{\partial z}{\partial\beta}\,\frac{\partial\beta}{\partial v}$$

$$= \frac{1}{2}\frac{\partial z}{\partial \alpha} + \frac{i}{2}\frac{\partial z}{\partial \beta} = \frac{1}{2}(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta})$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}(\frac{\partial z}{\partial v}) = \frac{\partial}{\partial u}(\frac{1}{2}(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta})$$

$$= \frac{\partial}{\partial u}(\frac{1}{2}\frac{\partial z}{\partial \alpha}) + \frac{\partial}{\partial u}(\frac{i}{2}\frac{\partial z}{\partial \beta})$$

$$= \frac{\partial}{\partial \alpha}(\frac{1}{2}\frac{\partial z}{\partial \alpha})\frac{\partial \alpha}{\partial u} + \frac{\partial}{\partial \beta}(\frac{i}{2}\frac{\partial z}{\partial \beta})\frac{\partial \beta}{\partial u}$$

$$= \frac{1}{4}\frac{\partial^2 z}{\partial \alpha^2} + \frac{1}{4}\frac{\partial^2 z}{\partial \beta^2} = \frac{1}{4}(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2})$$

substitute in Eq.(1

$$16\ \beta\ (\frac{1}{4}(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2})) + 2i(\frac{1}{2}(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}) - 2i(\frac{1}{2}(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}) = 0$$

$$= 4\ \beta\ (\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}) + i\frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta} - i\frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta} = 0$$

$$\therefore \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{-1}{2\beta}\frac{\partial z}{\partial \beta}$$

Example 3.0.30 *: Rewrite the following equations in canonical forms?*

1) One dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$$

2) Two dimensional diffusion equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$$

3) Two dimensional harmonic equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Solution 1):

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

$$A = 1, B = 0, C = -1, D = E = F = g = 0$$

$$B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0$$

$$\therefore \quad hyperbolic$$

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad (1)\lambda^2 + (0)\lambda + (-1) = 0$$

$$\lambda^2 - 1 = 0 \Rightarrow \quad \lambda = \mp 1 \Rightarrow \quad \lambda_1 = 1, \lambda_2 = -1$$

$$\frac{dy}{dx} + \lambda_1 = 0 \Rightarrow \quad \frac{dy}{dx} + 1 = 0 \Rightarrow \quad dy + dx = 0$$

$$y + x = c_1 = u$$

$$\frac{dy}{dx} + \lambda_2 = 0 \Rightarrow \quad \frac{dy}{dx} - 1 = 0 \Rightarrow \quad dy - dx = 0$$

$$y - x = c_2 = v$$

$$u_x = 1 \qquad v_x = -1$$

$$u_y = 1 \qquad v_y = 1$$

$$u_{xx} = u_{yy} = u_{xy} = 0 \quad v_{xx} = v_{yy} = v_{xy} = 0$$

$$A^* = (1)(1)^2 + 0 + (-1)(1)^2 = 0$$

$$B^* = 2(1)(1)(-1) + 0 + 2(-1)(1)(1) = -4$$

$$C^* = (1)(-1)^2 + 0 + (-1)(1)^2 = 0$$

$$D^* = (1)(0) + 0 + (-1)(0) + 0 + 0 = 0$$

$$E^* = (1)(0) + 0 + (-1)(0) + 0 + 0 = 0$$

$$g^* = g = 0, F^* = F = 0$$

$\therefore$ *The canonical equation is*

$$-4\frac{\partial^2 z}{\partial u \partial v} = 0 \Rightarrow \quad \frac{\partial^2 z}{\partial u \partial v} = 0 \Rightarrow \quad \frac{\partial z}{\partial u} = \phi(u)$$

$$z = \int \phi(u) du + \phi_2(v) \Rightarrow \quad z = \phi_1(u) + \phi_2(v)$$

$$z = \phi_1(y + x) + \phi_2(y - x) \quad is\ Gs.$$

2)

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0 \qquad\qquad A = 1, B = C = D = F = g = 0, E = -1$$

$$B^2 - 4AC = 0 - 4(1)(0) = 0$$

$\therefore$ *The equation is parabolic*

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad \lambda^2 = 0 \Rightarrow \quad \lambda_1 = 0, \lambda_2 = 0$$

$$\frac{dy}{dx} + 0 = 0 \Rightarrow \quad y = c_1 = u$$

$$let \quad v = x$$

$$u_x = u_{xx} = u_{yy} = u_{xy} = 0 \quad v_{xx} = v_{yy} = v_y = v_{xy} = 0$$

$$u_y = 0 \qquad\qquad v_x = 1$$

$$A^* = (1)(0) + 0 + 0(1)^2 = 0$$

$$B^* = 0 + 0 + 0 = 0$$

$$C^* = 1 + 0 + 0 = 1$$

$$D^* = 0 + (-1) = -1$$

$$E^* = F^* = g^* = 0$$

$\therefore$ *The canonical equation is*

$$\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} = 0$$

$$(D_v^2 - D_u)z = 0 \quad \textit{irreducible equation}$$

$$z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}; f(a_i, b_i) = b_i^2 - a_i = 0$$

$$z = \sum_{i=1}^{\infty} c_i e^{b_i^2 u + b_i v}$$

3)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$A = 1, B = 0, C = 1, D = E = F = g = 0$$

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

$\therefore$ *The equation is elliptic*

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \quad \lambda^2 + 0 + 1 = 0 \Rightarrow \quad \lambda^2 + 1 = 0 \Rightarrow$$

$$\lambda^2 = -1 \Rightarrow \quad \lambda = \mp i \Rightarrow \quad \lambda_1 = i, \lambda_2 = -i$$

$$\frac{dy}{dx} + i = 0 \Rightarrow \quad dy + idx = 0 \Rightarrow \quad y + ix = c_1 = u$$

$$\frac{dy}{dx} - i = 0 \Rightarrow \quad dy - idx = 0 \Rightarrow \quad y - ix = c_2 = v$$

$$u_x = i \qquad v_x = -i$$

$$u_y = 1 \qquad v_y = 1$$

$$u_{xx} = u_{yy} = u_{xy} = 0 \quad v_{xx} = v_{yy} = v_{xy} = 0$$

$$A^* = (i)^2 + 0 + 1 = 0$$

$$B^* = 2(i)(-i) + 0 + 2(1)(1) = 4$$

$$C^* = (-i)^2 + 0 + (1) = 0$$

$$D^* = 0 + 0 + 0 + 0 = 0$$

$$E^* = F^* = g^* = 0$$

$$4\frac{\partial^2 z}{\partial u \partial v} = 0 \Rightarrow \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2})$$

$$4.\ \frac{1}{4}(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}) = 0 \Rightarrow \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = 0$$

$$(D_\alpha^2 + D_\beta^2)z = 0 \Rightarrow \quad (D_\alpha + iD_\beta)(D_\alpha - iD_\beta)z = 0$$

$$z = \phi_1(i\alpha - \beta) + \phi_2(i\alpha + \beta)$$

Separation Of Variables Method

In this lesson we deal with elementary and generally useful method known as ***separation of variables****, which sometimes also called* ***method of Fourier****. The first basic idea of this method is to seek an elementary solution of PDE in several variables as a product of functions of one variable. If we are seeking a solution $z(x, y)$ to some PDE, we write*

$$z(x, y) = X(x)\, Y(y)$$

In this way the original PDE in several independent variables is broken up or separated into a set of ODEs, each involving just one independent variable.
At present, we use the separation method for finding solution of a second order PDE in two variables by reducing the original into a set of a second order ODEs. After that, we assume that the separated equation is equal to a constant, say k.

Example 3.0.31 *: Find the separated solutions of the equation*

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{a}\frac{\partial z}{\partial y}$$

where $a > 0$ is a constant. such that $z(x, y) \to 0$ as $y \to \infty$.

Solution : *Let* $z(x, y) = X(x)\,Y(y)$

$$\frac{\partial z}{\partial x} = X'(x)\,Y(y)$$

$$\frac{\partial^2 z}{\partial x^2} = X''(x)\,Y(y)$$

$$\frac{\partial z}{\partial y} = X(x)\,Y'(y)$$

and substitute in the main equation, we get

$$X''(x)\,Y(y) = \frac{1}{a}\,X(x)\,Y'(y) \Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{1}{a}\,\frac{Y'(y)}{Y(y)}$$

is separated. Now, let

$$\frac{X''(x)}{X(x)} = k \Rightarrow \quad X'' - k\,X(x) = 0$$

$$\frac{1}{a}\,\frac{Y'(y)}{Y(y)} = k \Rightarrow \quad Y'(y) - ak\,Y(y) = 0$$

where k is constant.

$$\frac{Y'(y)}{Y(y)} = a\,k \Rightarrow \quad \ln Y(y) = a\,ky + c_1 \Rightarrow$$

$$Y(y) = c\,e^{ak\,y}, \ \textit{where} \quad c = e^{c_1}$$

$$\textit{since} \quad Y(y) \to 0 \text{ as } y \to \infty \text{ when } k < 0,$$

$$\textit{and} \quad X'' - k\,X(x) = 0 \Rightarrow \quad (D_x^2 - k)X = 0, \ \textit{let} \quad k = -\alpha^2, \ 0 \neq \alpha \in R.$$

$$\lambda^2 - k = 0 \Rightarrow \quad \lambda^2 = k \Rightarrow \quad \lambda^2 = -\alpha^2 \Rightarrow \quad \lambda = \mp\, \alpha i$$

$$X(x) = A\ \cos(\alpha\ x) + iB\ \sin(\alpha\ x),\ where \quad A and\ B are\ constants$$

$$\therefore \quad z(x,y) = X(x)\ Y(y) = (A\ \cos(\alpha\ x) + iB\ \sin(\alpha\ x))c_1 e^{aky},\ \forall\ y$$

$$\therefore \quad z(x,y) = \sum_{n=0}^{\infty} c_n(A\ \cos(\alpha\ x) + iB\ \sin(\alpha\ x))e^{-a\alpha^2 y} \quad is\ the\ solution.$$

Example 3.0.32 *: Find the separated solutions of the PDE*

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 z}{\partial y^2} = 0$$

where c is a constant. Which satisfy the BCs $z(0,y) = 0$ *and* $z(a,y) = 0,\ \forall y$

Solution : *Let* $z(x,y) = X(x)\ Y(y)$

$$\frac{\partial z}{\partial x} = X'(x)\ Y(y) \qquad \frac{\partial^2 z}{\partial x^2} = X''(x)\ Y(y)$$

$$\frac{\partial z}{\partial y} = X(x)\ Y'(y) \qquad \frac{\partial^2 z}{\partial y^2} = X(x)\ Y''(y)$$

and substitute in the main equation, we get

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\ \frac{Y''(y)}{Y(y)}$$

$$\frac{X''(x)}{X(x)} = k \Rightarrow \quad X''(x) - k\ X(x) = 0$$

$$\frac{1}{c^2}\ \frac{Y''(y)}{Y(y)} = k \Rightarrow \quad Y''(y) - c^2 k\ Y(y) = 0$$

where k is a constant.

$$since \quad z(0,y) = 0 \Rightarrow \quad X(0)\ Y(y) = 0 \Rightarrow \quad X(0) = 0\ or\ Y(y) = 0$$

$$\textit{If} \quad Y(y) = 0 \Rightarrow \quad z(x,y) = 0, \ \textit{the solution is trivial.}$$

$$\textit{If} \quad Y(y) \neq 0 \Rightarrow \quad X(0) = 0$$

$$\textit{since} \quad z(a,y) = 0 \Rightarrow \quad X(a)\,Y(y) = 0 \Rightarrow \quad X(a) = 0 \ \textit{or}\ Y(y) = 0$$

$$\textit{If} \quad Y(y) = 0 \Rightarrow \quad z(x,y) = 0 \Rightarrow \quad \textit{the solution is trivial.}$$

$$\textit{If} \quad Y(y) \neq 0 \Rightarrow \quad X(a) = 0$$

$$\textit{To solve} \quad X'' - kX = 0 \quad \textit{with BCs}\ X(a) = 0 \ \textit{and}\ X(0) = 0$$

Case I) $\textit{If} \ k = 0 \Rightarrow \quad X''(x) = 0 \Rightarrow \quad X'(x) = c_1 \Rightarrow \quad X(x) = c_1 x + c_2$

$$\textit{since} \quad X(0) = 0 \Rightarrow \quad c_1(0) + c_2 = 0 \Rightarrow \quad c_2 = 0$$

$$X(a) = 0 \Rightarrow \quad c_1(a) + c_2 = 0 \Rightarrow \quad ac_1 = 0 \Rightarrow \quad c_1 = 0, \ \textit{since}\ a > 0.$$

$$\therefore \quad X(x) = 0 \Rightarrow \quad z(x,y) = 0 \quad \textit{is trivial.}$$

Case II) $\textit{If}\ k > 0, \ \textit{let} \ \ k = \alpha^2, \ 0 \neq \alpha \in R$

$$X'' - \alpha^2 X = 0 \Rightarrow \quad (D_x^2 - \alpha^2)X = 0 \Rightarrow$$

$$\lambda^2 - \alpha^2 = 0 \Rightarrow \quad \lambda^2 = \alpha^2 \Rightarrow \quad \lambda = \mp\,\alpha$$

$$\therefore \quad X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$\textit{since} \quad X(0) = 0 \Rightarrow \quad c_1 + c_2 = 0 \Rightarrow \quad c_1 = -c_2$$

$$X(a) = 0 \Rightarrow \quad c_1 e^{-a\alpha} + c_2 e^{-a\alpha} \Rightarrow \quad c_1 e^{a\alpha} - c_1 e^{-a\alpha} = 0$$

$$\Rightarrow \quad c_1(e^{a\alpha} - e^{-a\alpha}) = 0 \Rightarrow \quad c_1 = 0, \ since \ \ e^{a\alpha} - e^{-a\alpha} \neq 0$$

$$\therefore \quad c_2 = 0$$

$$\therefore \quad X(x) = 0 \Rightarrow \quad z(x, y) = 0 \quad is\ trivial.$$

Case III) *If* $k < 0$, *let* $k = -\alpha^2, \ 0 \neq \alpha \in R$

$$X'' - (-\alpha^2)X = 0 \Rightarrow \quad (D_x^2 + \alpha^2)X = 0 \Rightarrow$$

$$\lambda^2 + \alpha^2 = 0 \Rightarrow \quad \lambda = \mp\, i\alpha$$

$$X(x) = c_1 \cos(\alpha x) + i\ c_2\ \sin(\alpha x)$$

$$since \quad X(0) = 0 \Rightarrow \quad c_1 \cos 0 + ic_2 \sin 0 = 0 \Rightarrow c_1 = 0$$

$$since \quad X(a) = 0 \Rightarrow \quad c_1 \cos(a\alpha) + ic_2 \sin(a\alpha) = 0 \Rightarrow \quad c_2\ \sin(a\alpha) = 0$$

then the solution is intrivial if $c_2 \neq 0$

$$i.e. \quad \sin a\alpha = 0 \Rightarrow \quad a\alpha = n\pi \Rightarrow \quad \alpha = \frac{n}{a}\pi;\ n = \mp 1, \mp 2, \ldots$$

$$\therefore \quad X(x) = c_2\ \sin(\frac{n\pi}{a})x,\ n = \mp 1, \mp 2, \ldots$$

To solve $\quad Y''(y) - c^2kY = 0,$ *where* $k < 0$

$$k = -\alpha^2,\ 0 \neq \alpha \in R \quad \alpha = \frac{n\pi}{a},\ n = \mp 1, \mp 2, \ldots$$

$$(D_y^2 - kc^2)y = 0 \Rightarrow \quad (D_y^2 + c^2\alpha^2)y = 0 \Rightarrow \quad \lambda^2 + c^2\alpha^2 = 0 \Rightarrow \quad \lambda = \mp i\ c\ \alpha$$

$$Y(y) = A_1 \cos(c\alpha y) + iA_2 \sin(c\alpha y) = A_1 \cos(c\,\frac{n\pi}{a}y) + iA_2 \sin(c\,\frac{n\pi}{a}y)$$

$$\Rightarrow \quad z(x,y) = X(x)\,Y(y) = ic_2 \sin(\frac{n\pi}{a}x)(A_1 \cos(c\,\frac{n\pi}{a}y) + iA_2 \sin(c\,\frac{n\pi}{a}y))$$

$$z(x,y) = \sum_{n=0}^{\infty}(a_n \cos(c\,\frac{n\pi}{a}y) + ib_n \sin(c\,\frac{n\pi}{a}y))i\ \sin(\frac{n\pi}{a}x))$$

Example 3.0.33 *: Find the separated solution to the BVP*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

for $z(x,0) = 0$ *and* $z(x,a) = 0,\ x \geq 0,\ 0 \leq y \leq a$ *and* $z(x,y) \to 0$ *as* $x \to \infty$.

Solution : *Let* $z(x,y) = X(x)\,Y(y)$

$$\frac{\partial z}{\partial x} = X'(x)\,Y(y) \qquad \frac{\partial^2 z}{\partial x^2} = X''(x)\,Y(y)$$

$$\frac{\partial z}{\partial y} = X(x)\,Y'(y) \qquad \frac{\partial^2 z}{\partial y^2} = X(x)\,Y''(y)$$

and substitute in the main equation, we get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\frac{X''(x)}{X(x)} = k \Rightarrow \quad X''(x) - k\,X(x) = 0$$

$$\frac{-Y''(y)}{Y(y)} = k \Rightarrow \quad Y''(y) + k\,Y(y) = 0$$

where k *is a constant.*

$$\textit{since} \quad z(x,0) = 0 \Rightarrow \quad X(x)\,Y(0) = 0 \Rightarrow \quad X(x) = 0 \ \textit{or}\ Y(0) = 0$$

$If \quad X(x) = 0 \Rightarrow \quad z(x,y) = 0, \text{ the solution is trivial.}$

$If \quad X(x) \neq 0 \Rightarrow \quad Y(0) = 0$

$since \quad z(x,a) = 0 \Rightarrow \quad X(x)\,Y(a) = 0 \Rightarrow \quad X(x) = 0 \text{ or } Y(a) = 0$

$If \quad X(x) = 0 \Rightarrow \quad z(x,y) = 0 \Rightarrow \quad \text{the solution is trivial.}$

$If \quad X(x) \neq 0 \Rightarrow \quad Y(a) = 0$

$To\ solve \quad Y'' + kY = 0 \quad with\ BCs \quad Y(a) = 0 \text{ and } Y(0) = 0$

Case I) $If \ k = 0 \Rightarrow \quad Y''(y) = 0 \Rightarrow \quad Y'(y) = c_1 \Rightarrow \quad Y(y) = c_1 y + c_2$

$since \quad Y(0) = 0 \Rightarrow \quad c_1(0) + c_2 = 0 \Rightarrow \quad c_2 = 0$

$Y(a) = 0 \Rightarrow \quad c_1(a) + c_2 = 0 \Rightarrow \quad ac_1 = 0 \Rightarrow \quad c_1 = 0, \text{ since } a > 0.$

$\therefore \quad Y(y) = 0 \Rightarrow \quad z(x,y) = 0 \quad \text{is trivial.}$

Case II) $If\ k < 0, \ let \ k = -\alpha^2, \ 0 \neq \alpha \in R$

$$Y'' - \alpha^2\, Y = 0 \Rightarrow \quad (D_y^2 - \alpha^2)Y = 0 \Rightarrow$$

$$\lambda^2 - \alpha^2 = 0 \Rightarrow \quad \lambda^2 = \alpha^2 \Rightarrow \quad \lambda = \mp\, \alpha$$

$$\therefore \quad Y(y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y}$$

$since \quad Y(0) = 0 \Rightarrow \quad c_1 + c_2 = 0 \Rightarrow \quad c_1 = -c_2$

$$Y(a) = 0 \Rightarrow \quad c_1 e^{-a\alpha} + c_2 e^{-a\alpha} \Rightarrow \quad c_1 e^{a\alpha} - c_1 e^{-a\alpha} = 0$$

$$\Rightarrow \quad c_1(e^{a\alpha} - e^{-a\alpha}) = 0 \Rightarrow \quad c_1 = 0, \ since \ \ e^{a\alpha} - e^{-a\alpha} \neq 0$$

$$\therefore \quad c_2 = 0$$

$$\therefore \quad Y(y) = 0 \Rightarrow \quad z(x,y) = 0 \quad is\ trivial.$$

Case III) *If* $k > 0$, *let* $k = \alpha^2, \ 0 \neq \alpha \in R$

$$Y'' + (\alpha^2)Y = 0 \Rightarrow \quad (D_y^2 + \alpha^2)Y = 0 \Rightarrow$$

$$\lambda^2 + \alpha^2 = 0 \Rightarrow \quad \lambda = \mp\, i\alpha$$

$$Y(y) = c_1 \cos(\alpha y) + i\ c_2\ \sin(\alpha y)$$

$$since \quad Y(0) = 0 \Rightarrow \quad c_1 \cos 0 + ic_2 \sin 0 = 0 \Rightarrow c_1 = 0$$

$$since \quad Y(a) = 0 \Rightarrow \quad c_1 \cos(a\alpha) + ic_2 \sin(a\alpha) = 0 \Rightarrow \quad c_2\ \sin(a\alpha) = 0$$

then the solution is intrivial if $c_2 \neq 0$

$$i.e. \quad \sin a\alpha = 0 \Rightarrow \quad a\alpha = n\pi \Rightarrow \quad \alpha = \frac{n}{a}\pi; \ n = \mp 1, \mp 2, \ldots$$

$$\therefore \quad Y(y) = ic_2\ \sin(\frac{n\pi}{a})y, \ n = \mp 1, \mp 2, \ldots$$

To solve $\quad X''(x) - kX = 0, where\ k > 0$

$$k = \alpha^2, \ 0 \neq \alpha \in R \quad \alpha = \frac{n\pi}{a}, \ n = \mp 1, \mp 2, \ldots$$

$$(D_x^2 - k)X = 0 \Rightarrow \quad (D_x^2 - \alpha^2)y = 0 \Rightarrow \quad \lambda^2 - \alpha^2 = 0 \Rightarrow \quad \lambda = \mp\, \alpha$$

$$X(x) = d_1 e^{\alpha x} + d_2 e^{-\alpha x}, \ \textit{where } d_1 \textit{ and } d_2 \textit{ are constants}$$

$$\Rightarrow \quad z(x,y) = X(x)\,Y(y) = ic_2 \sin(\frac{n\pi}{a}y)(d_1 e^{\alpha x} + d_2 e^{-\alpha x})$$

$$\textit{since} \quad z(x,y) \to 0 \textit{ as } x \to \infty \textit{ then } d_1 = 0$$

$$z(x,y) = ic_2 d_2 e^{\frac{-n\pi}{a}x} \sin\frac{n\pi}{a}y$$

$$z(x,y) = \sum_{n=1}^{\infty} ib_n e^{\frac{-n\pi}{a}x} \sin\frac{n\pi}{a}y, \quad \textit{where } b_n \textit{ is constant.}$$

Chapter 4

Fourier Series

Definition 4.0.1 *: A function f is said to be periodic with period $p > 0$, if*

1) the domain of f contains $x + p$ whenever it contains x.

2) $f(x + p) = f(x)$ for every value of x.

The smallest value of p is called the fundamental period of f.

Example 4.0.1 *: The functions $\sin(x), \cos(x), \sec(x), \csc(x), \tan(x)$ and $\cot(x)$ are periodic functions of period 2π*

$$\text{since} \quad \sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x \quad \forall x \in R.$$

Theorem 4.0.1 *: If $f(x)$ is a periodic function of period p, then the function f is also periodic function with $np, \forall n = 1, 2, \dots$*

$$i.e. \qquad f(x + np) = f(x + p)f(x) \dots (1$$

Proof: *We can prove this theorem by using mathematical induction.*
- we must prove that it is true when $n = 1$.

$$f(x + np) = f(x + p) = f(x)$$

since f is a periodic function of period p.
$\therefore$ *it is true when $n = 1$.*

- *suppose that Eq.1) is true for* $n = k$.

$$i.e. \qquad f(x+np) = f(x+kp) = f(x) \ldots (2$$

- *we must prove that it is true when* $n = k+1$

$$f(x+np) = f(x+(k+1)p) = f(x+kp+p) = f(x+kp)$$

since f *is a periodic function of period* p.

$$= f(x) \quad by\ Eq.(2$$

$Eq.1)$ *is true when* $n = k+1$

$Eq.1)$ *is true when* $n = 1, 2, 3, \ldots$

Theorem 4.0.2 : *If* f *and* g *are two periodic functions of the same period* p, *and* a *and* b *any two constants. Then* $af + bg$ *is a periodic of the same period* p.
Proof:

$$(af+bg)(x+p) = (af)(x+p) + (bg)(x+p)$$

$$= a\ f(x+p) + b\ g(x+p)$$

$$= a\ f(x) + b\ g(x)$$

$$(since \quad f(x+p) = f(x) \ \ and \ \ g(x+p+g(x))$$

$$\therefore \quad (af+bg)(x+p) = (af+bg)(x)$$

$\therefore \quad af + bg$ *is periodic function with period* p.

Theorem 4.0.3 : $\sin nx$ *and* $\cos nx$ *are periodic function of period* $\frac{2\pi}{n}$, $n = 1, 2, \ldots$
Proof:

$$\sin\left(n(x + \frac{2\pi}{n})\right) = \sin(nx + 2\pi) = \sin nx$$

$$\cos\left(n(x+\frac{2\pi}{n})\right) = \cos(nx+2\pi) = \cos nx$$

Definition 4.0.2 *: A function $f(x)$ is said to be odd function if $f(-x) = -f(x) \quad \forall x$.*
or the function f is odd if the graph of f is symmetric with respect to the origin.

Definition 4.0.3 *: A function $f(x)$ is said to be even function if $f(-x) = f(x) \quad \forall x$.*
or the function $f(x)$ is even if the graph of f is symmetric with respect to the y-axis.
For example, the functions $\sin(x), \csc(x), \tan(x), \cot(x)$ *are odd functions, but the functions* $\cos(x), \sec(x)$ *are even functions.*

Remark: 1) $\int_{-a}^{a} f(x)dx = 0$, *if $f(x)$ is odd function.*

Proof:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$let \quad I_1 = \int_{-a}^{0} f(x)dx \quad and \quad I_2 = \int_{0}^{a} f(x)dx$$

$$I_1 = \int_{-a}^{0} f(x)dx \; ; \; let \quad x = -t \Rightarrow \quad dx = -dt \quad at \quad x = -a \Rightarrow \quad t = a$$

$$= \int_{a}^{0} f(-t)\,(-dt) = \int_{a}^{0} f(t)\,dt = -\int_{0}^{a} f(t)\,dt = -I_2 \quad [\int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx]$$

$$\therefore \quad \int_{-a}^{a} f(x)dx = -I_2 + I_2 = 0$$

2) $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\,(dx)$, *if $f(x)$ is even function.*

Proof:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$let \quad I_1 = \int_{-a}^{0} f(x)dx \quad and \quad I_2 = \int_{0}^{a} f(x)dx$$

$$I_1 = \int_{-a}^{0} f(x)dx \;;\; let \quad x = -t \Rightarrow \quad dx = -dt \quad at \quad x = -a \Rightarrow \quad t = a$$

$$= \int_{a}^{0} f(-t)\;(-dt) = -\int_{a}^{0} f(t)\;dt = -(-)\int_{0}^{a} f(t)\;dt = \int_{0}^{a} f(t)\;dt = I_2$$

$$\therefore \quad \int_{-a}^{a} f(x)dx = I_2 + I_2 = 2\int_{0}^{a} f(x)\;dx$$

3) $\int_{-\pi}^{\pi} \sin(nx)\;dx = 0,$ *since* $\sin(nx)$ *is odd function.*

4)

$$\int_{-\pi}^{\pi} \cos(nx)\;dx = \begin{cases} 0 & if\;\; n \neq 0 \\ 2\pi & if\;\; n = 0 \end{cases}$$

Proof:

$$-\,If \quad n \neq 0 \Rightarrow \quad \int_{-\pi}^{\pi} \cos(nx)\;dx = \frac{1}{n}\sin(nx)\Big]_{-\pi}^{\pi} = \frac{1}{n}[\sin(n\pi) + \sin(n\pi)]$$

$$= \frac{1}{n}[0+0] = 0$$

$$-\,If \quad n = 0 \Rightarrow \quad \int_{-\pi}^{\pi} \cos(0x)\;dx = \int_{-\pi}^{\pi} dx = x\;\Big]_{-\pi}^{\pi} = \pi - (-\pi) = 2\pi$$

Remark:

$$\cos(nx+mx) = \cos(nx)\ \cos(mx) - \sin(nx)\ \sin(mx)$$

$$\cos(nx-mx) = \cos(nx)\ \cos(mx) + \sin(nx)\ \sin(mx)$$

$$\sin(nx+mx) = \sin(nx)\ \cos(mx) + \cos(nx)\ \sin(mx)$$

$$\sin(nx-mx) = \sin(nx)\ \cos(mx) - \cos(nx)\ \sin(mx)$$

5)

$$\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\ dx = \begin{cases} 0 & \textit{if}\ \ n \neq m \\ \pi & \textit{if}\ \ n = m \neq 0 \\ 2\pi & \textit{if}\ \ n = m = 0 \end{cases}$$

Proof:

$$-\ If \quad n \neq m \Rightarrow \quad \int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\ dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}[\cos(n-m)x + \cos(n+m)x]dx$$

$$= \frac{1}{2}[\frac{1}{n-m}\sin(n-m)x + \frac{1}{n+m}\sin(n+m)x]\ ]_{-\pi}^{\pi}$$

$$= \frac{1}{2}[\frac{1}{n-m}(\sin(n-m)\pi + \sin(n-m)\pi)$$

$$+ \frac{1}{n+m}(\sin(n+m)\pi + \sin(n+m)\pi)] = 0$$

$$-\ If \quad n = m \Rightarrow \quad \int_{-\pi}^{\pi} \cos(nx)\ \cos(nx)\ dx = \int_{-\pi}^{\pi} \cos^2(nx)dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos(2nx))dx = \frac{1}{2}(x + \frac{1}{2n}\sin(2nx))\ \Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2}[(\pi + \frac{1}{2n}\sin 2n\pi) - (-\pi + \frac{-1}{2n}\sin 2n\pi)$$

$$= \frac{1}{2}[(\pi + 0) - (-\pi + 0)] = \frac{1}{2}(2\pi) = \pi$$

$$-\ If \quad n = m = 0 \Rightarrow \int_{-\pi}^{\pi} \cos(0x)\ \cos(0x)\ dx = \int_{-\pi}^{\pi} dx = x\ \Big]_{-\pi}^{\pi} = 2\pi$$

6)

$$\int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\ dx = \begin{cases} 0 & if\ n \neq m,\ n = m = 0 \\ \pi & if\ n = m \neq 0 \end{cases}$$

Proof:

$$-\ If \quad n \neq m \Rightarrow \quad \int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\ dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}(\cos(n-m)x - \cos(n+m)x)dx$$

$$= \frac{1}{2}[\frac{1}{n-m}\sin(n-m)x - \frac{1}{n+m}\sin(n+m)x]\ \Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2}[\frac{1}{n-m}(\sin(n-m)\pi + \sin(n-m)\pi)$$

$$-\frac{1}{n+m}(\sin(n+m)\pi + \sin(n+m)\pi)] = 0$$

$$-\ If \quad n = m = 0 \Rightarrow \quad \int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\ dx = 0$$

$$-\ If \quad n = m \neq 0 \Rightarrow \quad \int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\ dx = \int_{-\pi}^{\pi} \sin^2(nx) dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}(1 - \cos(2nx))dx$$

$$= \frac{1}{2}[x - \frac{1}{2n}(x - \frac{1}{2n}\sin(2nx))\ ]_{-\pi}^{\pi}$$

$$= \frac{1}{2}[(\pi - \frac{1}{2n}\sin(2n\pi)) - (-\pi - \frac{-1}{2n}\sin(2n\pi))]$$

$$= \frac{1}{2}(\pi + 0 + \pi) = \pi$$

7) $\int_{-\pi}^{\pi} \sin(nx)\ \cos(mx)\ dx = 0$

Proof:

$$\int_{-\pi}^{\pi} \sin(nx)\ \cos(mx)\ dx = \int_{-\pi}^{\pi} \frac{1}{2}(\sin(n-m)x + \sin(n+m)x\ dx)$$

$$= \frac{1}{2}(\frac{-1}{n-m}\cos(n-m)x - \frac{1}{n+m}\cos(n+m)x)]_{-\pi}^{\pi}$$

$$= \frac{1}{2}[\frac{-1}{n-m}(\cos(n-m)\pi - \cos(n-m)\pi)$$

$$- \frac{1}{n+m}(\cos(n+m)\pi - \cos(n+m)\pi)] = 0$$

$$- \, if \quad n = m \Rightarrow \quad \int_{-\pi}^{\pi} \sin(nx)\ \cos(mx)\ dx = \int_{-\pi}^{\pi} \sin(nx)\ \cos(nx)\ dx$$

$$= \frac{\sin^2(nx)}{2n}\Big]_{-\pi}^{\pi} = \frac{1}{2n}(\sin^2(n\pi) - \sin^2(n\pi)) = 0$$

Definition 4.0.4 *: If $f(x)$ is continuous and is periodic function with period 2π, then $f(x)$ can be represented by an infinite series:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ldots (1$$

the right side of Eq.1) is Fourier series of $f(x)$ and a_0, a_n and b_n are Fourier coefficients, where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

Fourier coefficients:

- Integrate Eq.1) from $-\pi$ to π, then

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\ dx + \int_{-\pi}^{\pi}(a_n \cos nx + b_n \sin nx)dx$$

$$\frac{1}{2}\ a_0\ x\Big]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} \cos nx\ dx + b_n \int_{-\pi}^{\pi} \sin nx\ dx)$$

$$= \frac{1}{2}\ a_0\ (\pi + \pi) + \sum_{n=1}^{\infty}(a_n(0) + b_n(0)) = \frac{1}{2}a_0(2\pi) = a_0\pi \Rightarrow$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

- Multiply both sides of Eq.1) by $\cos mx$ *and integrate the new equation from* $-\pi$ *to* π

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos mx dx + \sum_{n=1}^{\infty} (a_n \int_{-\pi}^{\pi} \cos nx \ \cos mx dx$$

$$+ b_n \int_{-\pi}^{\pi} \sin nx \ \cos mx dx)$$

If we take $n = m$*, then*

$$\int_{-\pi}^{\pi} \cos nx \ \cos mx dx = \pi \quad and \quad \int_{-\pi}^{\pi} \sin nx \ \cos mx dx = 0$$

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = a_n \ \pi \Rightarrow$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx, \ for \quad n = 1, 2, \ldots$$

- Multiply both sides of Eq.1) by $\sin mx$ *and integrate the new equation from* $-\pi$ *to* π

$$\int_{-\pi}^{\pi} f(x) \sin mx \ dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \sin mx \ dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \ \sin mx \ dx$$

$$+ b_n \int_{-\pi}^{\pi} \sin nx \ \sin mx \ dx$$

If we take $n = m$*, then*

$$\int_{-\pi}^{\pi} \cos nx \ \sin mx \ dx = 0 \quad \textit{and} \quad \int_{-\pi}^{\pi} \sin nx \ \sin mx \ dx = \pi$$

$$\int_{-\pi}^{\pi} f(x) \ \sin nx \ dx = b_n \ \pi \Rightarrow$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin nx \ dx, \quad for \quad n = 1, 2, \ldots$$

Example 4.0.2 *: Find the Fourier series for* $f(x) = x^2$ *on* $-\pi < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \ \cos nx + b_n \ \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \ dx = \frac{1}{3\pi} x^3 \ \Big]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi} (\pi^3 + \pi^3) = \frac{2}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos nx \ dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \ \cos nx \ dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \ \cos nx \ dx$$

$$u = x^2 \Rightarrow \quad du = 2x \ dx$$

$$dv = \cos nx \ dx \Rightarrow \quad v = \frac{1}{n} \ \sin nx$$

$$a_n = \frac{2}{\pi} (\frac{1}{n} \ x^2 \ \sin nx \ \Big]_{0}^{\pi} - \frac{2}{n} \int_{0}^{\pi} x \sin nx \ dx)$$

$$u = x \Rightarrow \quad du = dx$$

$$dv = \sin nx \; dx \Rightarrow \qquad v = \frac{-1}{n} \cos nx$$

$$a_n = \frac{-4}{n\pi} (\frac{-1}{n} \; x \cos nx \; \Big]_0^{\pi} - \frac{-1}{n} \int_0^{\pi} \cos nx \; dx)$$

$$= \frac{-4}{n\pi} (\frac{-1}{n} \; \pi(-1)^n + \frac{1}{n^2} \; \sin nx \; \Big]_0^{\pi})$$

$$= \frac{-4}{n\pi} (\frac{-\pi}{n} \; (-1)^n + \frac{1}{n^2} \; (0 - 0)) = \frac{4}{n^2} \; (-1)^n$$

$$a_n = \frac{4}{n^2} \; (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \; \sin nx \; dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \; \sin nx \; dx = 0 \Rightarrow$$

$$b_n = 0$$

$\therefore$ *The Fourier series $f(x)$ is*

$$\frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \; \cos nx$$

Example 4.0.3 : *Find the Fourier series of the periodic function $f(x) = x + |x|$, with period 2π on $-\pi < x < \pi$.*

Solution :

$$|x| = \begin{cases} x & : \; x \geq 0 \\ -x & : \; x < 0 \end{cases}$$

$$f(x) = x + |x| = \begin{cases} x + x & : \quad 0 \le x < \pi \\ \\ x + (-x) & : \quad -\pi < x < 0 \end{cases}$$

$$= \begin{cases} 2x & : \quad 0 \le x < \pi \\ \\ 0 & : \quad -\pi < x < 0 \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \ dx + \int_{0}^{\pi} 2x \ dx \right]$$

$$= \frac{1}{\pi} \ x^2 \Big]_0^{\pi} = \pi \Rightarrow \qquad a_0 = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos nx \ dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \ dx + \int_{0}^{\pi} 2x \ \cos nx \ dx \right]$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \ \cos nx \ dx \qquad u = x \Rightarrow \quad du = dx, \quad dv = \cos nx \ dx \Rightarrow \quad v = \frac{1}{n} \sin nx$$

$$\int_{0}^{\pi} x \ \cos nx \ dx = \frac{x}{n} \ \sin nx \Big]_0^{\pi} - \int_{0}^{\pi} \frac{1}{n} \ \sin nx \ dx$$

$$= 0 + \frac{1}{n^2} \cos nx \Big]_0^{\pi} = \frac{1}{n^2} (\cos n\pi - \cos n(0) \ ) = \frac{1}{n^2} ((-1)^n - 1)$$

$$\therefore \quad a_n = \frac{2}{n^2 \pi} ((-1)^n - 1)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin nx \ dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \ dx + \int_{0}^{\pi} 2x \ \sin nx \ dx \right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\ \sin nx\ dx \qquad u = x \Rightarrow \quad du = dx, \quad dv = \sin nx\ dx \Rightarrow \quad v = \frac{-1}{n}\cos nx$$

$$= \frac{2}{\pi}\left[\frac{-1}{n}x\ \cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\ dx\Bigg]$$

$$= \frac{2}{\pi}\left[\frac{-1}{n}\ \pi\ (-1)^n + \frac{1}{n^2}\sin nx\Big]_0^{\pi}\quad\right] = \frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n + \frac{1}{n^2}(0-0)\right]$$

$$b_n = \frac{2}{\pi}(-1)^{n+1}, \quad n = 1, 2, \ldots$$

$\therefore$ *the Fourier series of $f(x)$ is*

$$\frac{2}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos nx + \frac{2}{n}(-1)^{n+1}\ \sin nx\right]$$

Example 4.0.4 *: Find the Fourier series of the following functions*

1) $f(x) = \begin{cases} 1 & : \frac{-3\pi}{2} < x < \frac{\pi}{2} \\ -1 & : \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$

2) $f(x) = x^3 \quad -\pi < x < \pi$

3) $f(x) = \begin{cases} x & : \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & : \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$

4) $f(x) = \begin{cases} 0 & : -\pi < x < 0 \\ \pi - x & : 0 < x < \pi \end{cases}$

The Euler-Fourier Formula

If $f(x)$ *is a continuous and periodic function of period* $2L$*. Use the transformulation* $y = \frac{\pi}{L}\, x$ *for* $-L < x < L$ *and* $-\pi < y < \pi$*. Then*

$$y = \frac{\pi}{L}\, x \Rightarrow \quad x = \frac{L}{\pi}\, y \Rightarrow \quad dx = \frac{L}{\pi}\, dy$$

Now,

$$f(x) = f(\frac{L}{\pi}\, y) = F(y)$$

$\therefore$ $F(y)$ *is continuous and periodic function of period* 2π*, then the Fourier series of* $F(y)$ *is*

$$F(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \,\cos ny + b_n \,\sin ny)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\; dy = \frac{1}{\pi}\int_{-L}^{L} f(x)\; \frac{\pi}{L}\; dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\; dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\; \cos ny\; dy = \frac{1}{\pi}\int_{-L}^{L} f(x)\; \cos\frac{n\pi}{L}\; x\; \frac{\pi}{L}\; dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\; \cos\frac{n\pi}{L} x\; dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\; \sin ny\; dy = \frac{1}{\pi}\int_{-L}^{L} f(x)\; \sin\frac{n\pi}{L} x\; dx$$

Then the Fourier series of $f(x)$ is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\ \cos\frac{n\pi}{L}x+b_n\ \sin\frac{n\pi}{L}x)$$

Example 4.0.5 *: Find the Fourier series for the given function*

$$f(x)=\begin{cases}-1 & : \ -1<x<0\\ \\ 1 & : \ 0<x<1\end{cases} \qquad f(x+2L)=f(x)$$

Solution:

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)\ dx=\frac{1}{1}\Big[\int_{-1}^{0}(-1)\ dx+\int_{0}^{1}(1)\ dx\Big]$$

$$=-x\Big]_{-1}^{0}+x\Big]_{0}^{1}=(0-1)+(1-0)=0$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\ \cos\frac{n\pi}{L}x\ dx$$

$$=\frac{1}{1}\Big[\int_{-1}^{0}\ -\cos n\pi x\ dx+\int_{0}^{1}\ \cos n\pi x\ dx$$

$$=\frac{-1}{n\pi}\sin n\pi x\Big]_{-1}^{0}+\frac{1}{n\pi}\ \sin n\pi x\Big]_{0}^{1}=0\Rightarrow \quad a_n=0$$

$$b_n=\frac{1}{\pi}\int_{-L}^{L}f(x)\ \sin\frac{n\pi}{L}x\ dx$$

$$=\frac{1}{1}\Big[\int_{-1}^{0}(-1)\ \sin n\pi x\ dx+\int_{0}^{1}(1)\ \sin n\pi x\ dx\Big]$$

$$= \frac{1}{n\pi}\ \cos n\pi x\Big]_{-1}^{0} - \frac{1}{n\pi}\ \cos n\pi x\Big]_{0}^{1}$$

$$= \frac{1}{n\pi}(1-(-1)^n) - \frac{1}{n\pi}((-1)^n - 1)$$

$$b_n = \frac{2}{n\pi}(1-(-1)^n), \quad n = 1, 2, \ldots$$

$\therefore$ *the Fourier series of* $f(x)$ *is*

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\ \sin n\pi x$$

Complex Form Of Fourier Series

If $f(x)$ *is a periodic function of period* 2π*, then Fourier series of* $f(x)$ *is*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\ \cos nx + b_n\ \sin nx$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos nx\ dx, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ dx, \quad n = 1, 2, \ldots$$

By using Euler formulas

$$e^{inx} = \cos nx + i \sin nx$$
$$e^{-inx} = \cos nx - i \sin nx$$

$$\Rightarrow \quad \cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$
$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \, (\frac{e^{inx} + e^{-inx}}{2}) + b_n \, (\frac{e^{inx} - e^{-inx}}{2i})$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (\frac{a_n - ib_n}{2}) \, e^{inx} + (\frac{a_n + ib_n}{2}) \, e^{-inx}$$

$$\Rightarrow \quad f(x) = C_0 + \sum_{n=1}^{\infty} (C_{-n} \, e^{inx} + C_n \, e^{-inx})$$

where

$$C_0 = \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{i0x} dx$$

$$C_{-n} = \frac{a_n - ib_n}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, (\cos nx - i \sin nx) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

$$C_n = \frac{a_n + ib_n}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, (\cos nx + i \sin nx) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx$$

Then $f(x) = \sum_{-\infty}^{\infty} C_{-n} e^{inx}$ *is called complex Fourier series of* $f(x)$ *and* C_{-n} *is called complex Fourier coefficient of* $f(x)$

Example 4.0.6 *: Find a complex form of Fourier series of*

$$f(x) = \cosh x \qquad -\pi < x < \pi$$

Solution:

$$C_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh x \, e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\frac{e^x + e^{-x}}{2}) \, e^{-inx} \, dx = \frac{1}{4\pi} \int_{-\pi}^{\pi} (e^{(1-in)x} + e^{-(1+in)x}) \, dx$$

$$= \frac{1}{4\pi} \left[\left[\frac{1}{1 - in} \, e^{(1-in)x} \right]_{-\pi}^{\pi} + \left[\frac{1}{-(1 + in)} \, e^{-(1+in)x} \right]_{-\pi}^{\pi} \right]$$

$$= \frac{1}{4\pi}\left[\frac{1}{1-in}(e^{\pi}\ e^{-in\pi} - e^{-\pi}\ e^{in\pi}) + \frac{1}{-(1+in)}(e^{-\pi}\ e^{in\pi} - e^{\pi}\ e^{-in\pi})\right.$$

since $\quad e^{-inx} = e^{inx} = \cos n\pi = (-1)^n$

$$= \frac{(-1)^n}{2}\left[\frac{1}{1-in}\ (\frac{e^{\pi}-e^{-\pi}}{2}) + \frac{1}{1+in}\ (\frac{e^{\pi}-e^{-\pi}}{2})\right]$$

$$= \frac{(-1)^n}{2\pi}\ \sinh\pi\left[\frac{1}{1-in} + \frac{1}{1+in}\right] = \frac{(-1)^n}{2\pi}\ \sinh\pi\ (\frac{1+in+1-in}{1+n^2})$$

$$= \frac{(-1)^n\ \sinh\pi}{\pi\ (1+n^2)}$$

$$\therefore \quad f(x) = \frac{\sinh\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{(1+n^2)}\ e^{inx} \quad \textit{is complex Fourier series}$$

since $\quad e^{inx} = \cos nx + i\sin nx$

$\qquad\qquad e^{-inx} = \cos nx - i\sin nx$

it can be seen that, the imaginary part will cancel when we add these two equations and their result is $2\cos nx$. *If* $n = 0 \Rightarrow \quad e^{i0x} = 1$

$$f(x) = \frac{\sinh\pi}{\pi}\ (1 - \frac{2}{1+1^2}\cos x + \frac{2}{1+2^2}\cos 2x - \frac{2}{1+3^2}\cos 3x + \dots)$$

$$= \frac{2\sinh\pi}{\pi}\ (\frac{1}{2} - \frac{1}{1+1^2}\cos x + \frac{1}{1+2^2}\cos 2x - \frac{1}{1+3^2}\cos 3x + \dots)$$

Example 4.0.7 *: Find the Fourier series of*

$$f(x) = e^{x} \qquad , -\pi < x < \pi$$

and obtain the usual Fourier series?

Solution :

$$C_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ e^{-inx}\ dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x}\ e^{-inx}\ dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)x}\ dx$$

$$= \frac{1}{2\pi}\ \frac{1}{1-in}\ e^{(1-in)x}\ \Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\ \frac{1}{1-in}\ (e^{(1-in)\pi} - e^{-(1-in)\pi})$$

$$= \frac{1}{2\pi}\ \frac{1+in}{1+n^2}\ (e^{\pi}\ e^{-in\pi} - e^{-\pi}\ e^{in\pi})$$

Remark: $e^{-in\pi} = e^{in\pi} = \cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\ \frac{1+in}{1+n^2}\ (-1)^n\ (\frac{e^{\pi} - e^{-\pi}}{2})$$

$$C_{-n} = \frac{(-1)^n}{\pi}\ \sinh\pi\ (\frac{1+in}{1+n^2}) \qquad , n = 0, \mp 1, \mp 2, \ldots$$

$$then \quad f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi}\ \sinh\pi\ (\frac{1+in}{1+n^2})\ e^{inx}$$

$$= \frac{\sinh\pi}{\pi}\ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^2}\ (1+in)\ e^{inx}$$

$$\begin{aligned} \textit{since} \quad e^{inx} &= \cos nx + i \sin nx \\ e^{-inx} &= \cos nx - i \sin nx \end{aligned}$$

$$(1 + in)\ e^{inx} = (1 + in)\ (\cos nx + i \sin nx)$$

$$= (\cos nx - n \sin nx) + i(\sin nx + n \cos nx)$$

where n is positive.

$$(1 - in)\ e^{-inx} = (1 - in)\ (\cos nx - i \sin nx)$$

$$= (\cos nx - n \sin nx) - i(\sin nx + n \cos nx)$$

where n is negative.

We can see that when we add all the positive and negative n terms the imaginary part will vanish, so the real part will just remain.
if $\quad n = 0 \Rightarrow \quad (1 + in)\ e^{inx} = 1$

$$\textit{then} \quad f(x) = \frac{\sinh \pi}{\pi}(1 - \frac{2}{1 + 1^2}(\cos x - \sin x) + \frac{2}{1 + 2^2}(\cos 2x - 2 \sin 2x)$$

$$+ \frac{2}{1 + 3^2}(\cos 3x - 3 \sin 3x) + \dots)$$

$$f(x) = \frac{\sinh \pi}{\pi}(1 - \frac{1}{1 + 1^2}(\cos x - \sin x) + \frac{1}{1 + 2^2}(\cos 2x - 2 \sin 2x)$$

$$+ \frac{1}{1 + 3^2}(\cos 3x - 3 \sin 3x) + \dots) \quad \textit{where} \quad -\pi < x < \pi$$

Remark: *If $f(x)$ is a periodic function of period $2L$, then the complex Fourier* *series of $f(x)$ is defined by*

$$f(x) = \sum_{n=-\infty}^{\infty} C_{-n}\, e^{in\frac{\pi}{L}x}$$

where

$$C_{-n} = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\frac{\pi}{L}x}\, dx$$

Example 4.0.8 *: Find the complex Fourier series for the following functions?*

1) $f(x) = x \quad , \; 0 < x < 2\pi,$

2) $f(x) = \begin{cases} 0 & : \; -\pi < x < 0 \\ 1 & : \; 0 < x < \pi \end{cases}$

Remark: 1) *If $f(x)$ is even function on $(-L, L)$ then*

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\frac{n\pi}{L}x\, dx = \frac{2}{L}\int_{0}^{L} f(x)\,\cos\frac{n\pi}{L}x\, dx \qquad , n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\frac{n\pi}{L}x\, dx = 0$$

since $f(x)\,\sin\frac{n\pi}{L}x$ *is odd function, so the Fourier series of $f(x)$ is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\,\cos\frac{n\pi}{L}x \qquad for \;\; n = 1, 2, \ldots$$

this kind of Fourier series is called cosine Fourier series.

2) *If $f(x)$ is odd function on $(-L, L)$ then*

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \cos\frac{n\pi}{L}x\ dx = 0, because$$

$$f(x)\ \cos\frac{n\pi}{L}x \quad is\ odd\ for\ \ n = 0, 1, 2, \ldots$$

but

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \sin\frac{n\pi}{L}x\ dx = \frac{2}{L}\int_{0}^{L} f(x)\ \sin\frac{n\pi}{L}x\ dx$$

then the Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n\ \sin\frac{n\pi}{L}x$$

this kind of Fourier series is called sine Fourier series.

Example 4.0.9 *: Find the Fourier series of the periodic function of period 4,*

$$f(x) = -x \qquad\qquad -2 < x < 2$$

Solution: *Since $f(x)$ is odd function, so*

$$a_n = 0 \qquad , n = 0, 1, 2, \ldots$$

$$2L = 4 \Rightarrow \quad L = 2$$

$$since \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\ \sin\frac{n\pi}{L}x\ dx$$

$$= \frac{2}{2}\int_{0}^{2} (-x)\ \sin\frac{n\pi}{2}x\ dx$$

$$b_n = -\int_0^2 x\ \sin\frac{n\pi}{2}x\ dx \qquad \textit{let} \quad u = x \Rightarrow \quad du = dx$$

$$dv = \sin\frac{n\pi}{2}x \Rightarrow \qquad v = \frac{-2}{n\pi}\ \cos\frac{n\pi}{2}x$$

$$= -\left[\frac{-2x}{n\pi}\cos\frac{n\pi}{2}x\right]_0^2 - \int_0^2 \frac{-2}{n\pi}\cos\frac{n\pi}{2}x\ dx\ \Big]$$

$$= -\left[\frac{-4}{n\pi}\ \cos n\pi - 0 + \frac{4}{n^2\pi^2}\ \sin\frac{n\pi}{2}x\Big]_0^2\ \right]$$

$$= -\left[\frac{-4}{n\pi}\ \cos n\pi + \frac{4}{n^2\pi^2}(0-0)\right]$$

$$b_n = \frac{4}{n\pi}(-1)^n \quad , n = 1, 2, \ldots$$

Then the Fourier sine series of $f(x)$ is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\ \sin\frac{n\pi}{2}x$$

Example 4.0.10 *:1) Find the Fourier cosine series of*

$$f(x) = |x| \qquad , 0 < x < \pi, f(x + 2\pi) = f(x)$$

2) Find the Fourier series for each of the following function with period 2π (assumed).

$$a)\ f(x) = \begin{cases} k & : \ \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0 & : \ \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

$$b)\ f(x) = \begin{cases} x & : \ \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & : \ \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

$$c)\ f(x) = \begin{cases} -x & : \ -\pi < x < 0 \\ \\ x & : \ 0 < x < \pi \end{cases}$$

$$d)\ f(x) = x + \pi \qquad -\pi < x < \pi$$

Definition 4.0.5 *: A function $f(x)$ is said to be removable discontinuity at a point $x = a$, if the limit of f at a exists and is finite, but $f(a)$ is not defined.*

Example 4.0.11 *:* $f(x) = \dfrac{\sin x}{x}$

$$since \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$but \quad f(0) \ \ is\ not\ defined.$$

therefore f is removable discontinuity.

Definition 4.0.6 *: A function $f(x)$ is said to be jump discontinuity at a point a, if the left hand and right hand limits at a exist, are finite but they are different (not equal).*
i.e.

$$\lim_{x \to a^+} f(x) \quad and \quad \lim_{x \to a^-} f(x) \quad exist$$

$$, but$$

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

Example 4.0.12 *:* $f(x) = \begin{cases} 1 & : \ 0 < x \\ \\ -1 & : \ x < 0 \end{cases}$

$$\lim_{x \to 0^+} f(x) = 1 \quad and \quad \lim_{0^-} f(x) = -1$$

$$, but \quad \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$$

$\therefore \quad f(x)$ *at* $x = 0$ *is jump discontinuity.*

Definition 4.0.7 *: A function $f(x)$ is said to be bad discontinuity at a point $x = a$, if $f(x)$ is not removable and jump discontinuity at a and also f is not continuous at a.*

Example 4.0.13 *:* $f(x) = e^{\frac{1}{x}}$

$x = 0$ is bad discontinuity, since $e^{\frac{1}{x}}$ is unbounded function.

Definition 4.0.8 *: A function $f(x)$ is said to be sectionally continuous (or piecewise continuous) on an interval $a \leq x \leq b$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.*

Example 4.0.14 *:*

1) $f(x) = \dfrac{\sin x}{x}$ *on $[-1, 1]$ is sectionally continuous, since f is continuous on $[-1, 1]$ except at $x = 0$ ($x = 0$ is removable discontinuity)*

2) $$f(x) = f(x) = \begin{cases} 2 & : \quad -1 \leq x \leq 0 \\ -3 & : \quad 0 < x < 1 \end{cases}$$

since f is continuous on (-1,1) except at $x = 0$

$\therefore$ f is sectionally continuous.

Definition 4.0.9 *: A function $f(x)$ is said to be sectionally smooth on an interval $[a, b]$, if f is sectionally continuous on $[a, b]$ and $f'(x)$ exists at the points such that f is continuous on $[a, b]$.*

Example 4.0.15 *:* 1) $f(x) = |x|$ *is sectionally continuous on $[-1, 1]$?*

Solution: *f is continuous at $x = 0$, but f is not differentiable at $x = 0$.*
$\therefore$ $|x|$ is not sectionally smooth on $[-1, 1]$.

2) $f(x) = \dfrac{1}{x - 2}$ *is not sectionally continuous on $[-3, 3]$, since $x = 2$ is bad discontinuity of f.*