Calculus I

Mathematics Department

First year – First Course

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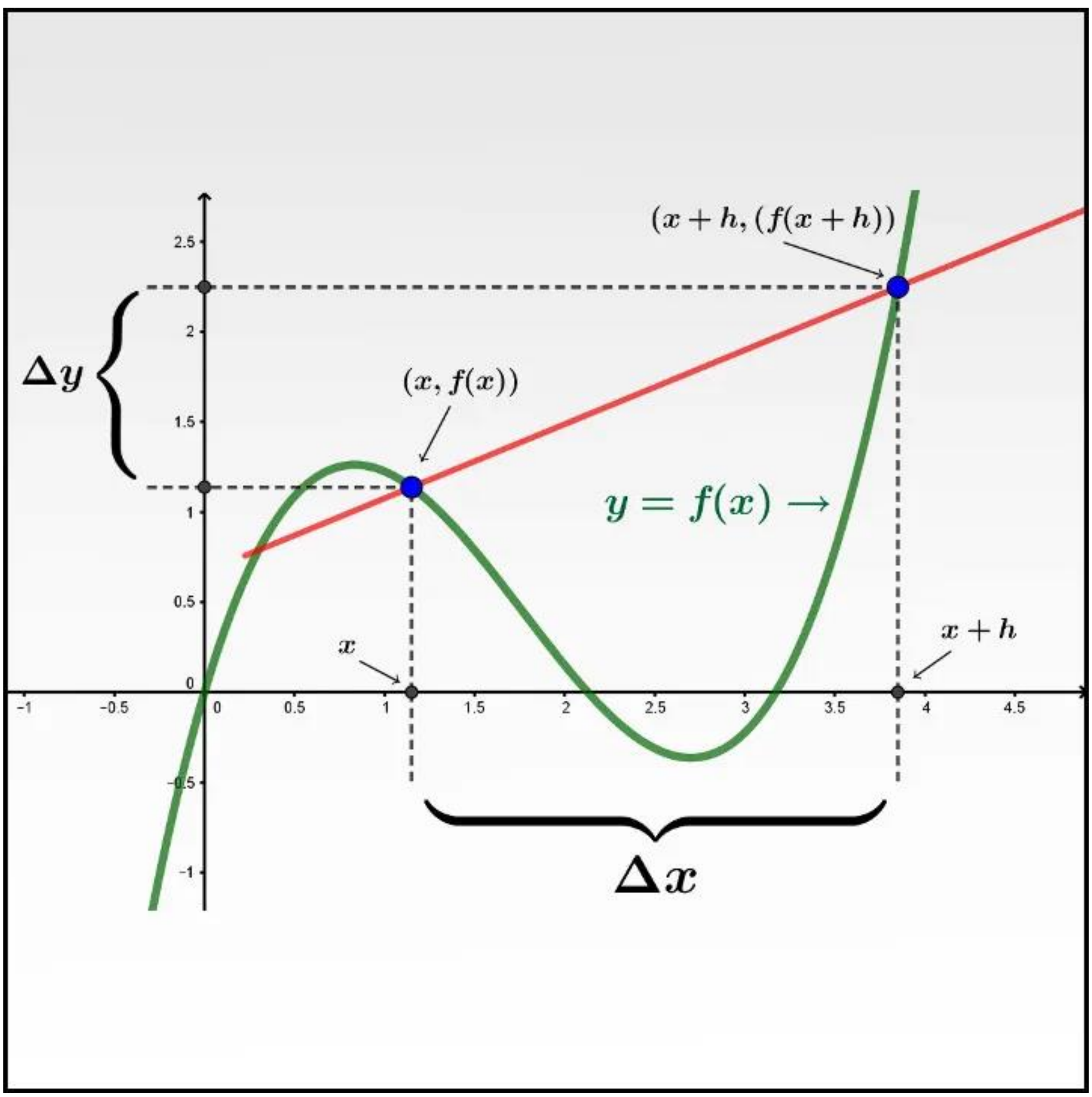

Chapter 1

Functions

In this chapter we review the basic concepts of functions, polynomial functions, rational functions, trigonometric functions, logarithmic functions, exponential functions, hyperbolic functions, algebra of functions, composition of functions and inverses of functions.

1.1 The Concept of a Function

Basically, a *function* f relates each element x of a set, say D_f, with exactly one element y of another set, say R_f. We say that D_f is the *domain* of f and R_f is the *range* of f and express the relationship by the equation $y = f(x)$. It is customary to say that the symbol x is an *independent variable* and the symbol y is the *dependent variable*.

Example 1.1.1 Let $D_f = \{a, b, c\}$, $R_f = \{1, 2, 3\}$ and $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. Sketch the graph of f.

graph

Example 1.1.2 Sketch the graph of $f(x) = |x|$.

Let D_f be the set of all real numbers and R_f be the set of all non-negative real numbers. For each x in D_f, let $y = |x|$ in R_f. In this case, $f(x) = |x|$, the absolute value of x. Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We note that $f(0) = 0, f(1) = 1$ and $f(-1) = 1$.

If the domain D_f and the range R_f of a function f are both subsets of the set of all real numbers, then the *graph* of f is the set of all ordered pairs $(x, f(x))$ such that x is in D_f. This graph may be sketched in the xy-coordinate plane, using $y = f(x)$. The graph of the absolute value function in Example 2 is sketched as follows:

graph

Example 1.1.3 Sketch the graph of

$$f(x) = \sqrt{x-4}.$$

In order that the range of f contain real numbers only, we must impose the restriction that $x \geq 4$. Thus, the domain D_f contains the set of all real numbers x such that $x \geq 4$. The range R_f will consist of all real numbers y such that $y \geq 0$. The graph of f is sketched below.

graph

Example 1.1.4 A useful function in engineering is the unit step function, u, defined as follows:

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

The graph of $u(x)$ has an upward *jump* at $x = 0$. Its graph is given below.

graph

Example 1.1.5 Sketch the graph of

$$f(x) = \frac{x}{x^2 - 4}.$$

It is clear that D_f consists of all real numbers $x \neq \pm 2$. The graph of f is given below.

graph

We observe several things about the graph of this function. First of all, the graph has three distinct pieces, separated by the dotted vertical lines $x = -2$ and $x = 2$. These vertical lines, $x = \pm 2$, are called the *vertical asymptotes*. Secondly, for large positive and negative values of x, $f(x)$ tends to zero. For this reason, the x-axis, with equation $y = 0$, is called a *horizontal asymptote*.

Let f be a function whose domain D_f and range R_f are sets of real numbers. Then f is said to be *even* if $f(x) = f(-x)$ for all x in D_f. And f is said to be *odd* if $f(-x) = -f(x)$ for all x in D_f. Also, f is said to be *one-to-one* if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Example 1.1.6 Sketch the graph of $f(x) = x^4 - x^2$.

This function f is *even* because for all x we have

$$f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x).$$

The graph of f is symmetric to the y-axis because $(x, f(x))$ and $(-x, f(x))$ are on the graph for every x. *The graph of an even function is always symmetric to the y-axis.* The graph of f is given below.

graph

This function f is not one-to-one because $f(-1) = f(1)$.

Example 1.1.7 Sketch the graph of $g(x) = x^3 - 3x$.

The function g is an *odd* function because for each x,

$$g(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -(x^3 - 3x) = -g(x).$$

The graph of this function g is symmetric to the origin because $(x, g(x))$ and $(-x, -g(x))$ are on the graph for all x. *The graph of an odd function is always symmetric to the origin.* The graph of g is given below.

graph

This function g is not one-to-one because $g(0) = g(\sqrt{3}) = g(-\sqrt{3})$.

It can be shown that every function f can be written as the sum of an even function and an odd function. Let

$$g(x) = \frac{1}{2}(f(x) + f(-x)), h(x) = \frac{1}{2}(f(x) - f(-x)).$$

Then,

$$\begin{aligned} g(-x) &= \frac{1}{2}(f(-x) + f(x)) = g(x) \\ h(-x) &= \frac{1}{2}(f(-x) - f(x)) = -h(x). \end{aligned}$$

Furthermore

$$f(x) = g(x) + h(x).$$

Example 1.1.8 Express f as the sum of an even function and an odd function, where,

$$f(x) = x^4 - 2x^3 + x^2 - 5x + 7.$$

We define

$$\begin{aligned} g(x) &= \frac{1}{2}(f(x) + f(-x)) \\ &= \frac{1}{2}\{(x^4 - 2x^3 + x^2 - 5x + 7) + (x^4 + 2x^3 + x^2 + 5x + 7)\} \\ &= x^4 + x^2 + 7 \end{aligned}$$

and

$$\begin{aligned} h(x) &= \frac{1}{2}(f(x) - f(-x)) \\ &= \frac{1}{2}\{(x^4 - 2x^3 + x^2 - 5x + 7) - (x^4 + 2x^3 + x^2 + 5x + 7)\} \\ &= -2x^3 - 5x. \end{aligned}$$

Then clearly $g(x)$ is even and $h(x)$ is odd.

$$\begin{aligned} g(-x) &= (-x)^4 + (-x)^2 + 7 \\ &= x^4 + x^2 + 7 \\ &= g(x) \\ h(-x) &= -2(-x)^3 - 5(-x) \\ &= 2x^3 + 5x \\ &= -h(x). \end{aligned}$$

We note that

$$\begin{aligned} g(x) + h(x) &= (x^4 + x^2 + 7) + (-2x^3 - 5x) \\ &= x^4 - 2x^3 + x^2 - 5x + 7 \\ &= f(x). \end{aligned}$$

It is not always easy to tell whether a function is one-to-one. The graphical test is that if no horizontal line crosses the graph of f more than once, then f is one-to-one. To show that f is one-to-one mathematically, we need to show that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 1.1.9 Show that $f(x) = x^3$ is a one-to-one function.

Suppose that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 0 &= x_1^3 - x_2^3 \\ &= (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) \qquad \text{(By factoring)} \end{aligned}$$

If $x_1 \neq x_2$, then $x_1^2 + x_1x_2 + x_2^2 = 0$ and

$$\begin{aligned} x_1 &= \frac{-x_2 \pm \sqrt{x_2^2 - 4x_2^2}}{2} \\ &= \frac{-x_2 \pm \sqrt{-3x_2^2}}{2}. \end{aligned}$$

This is only possible if x_1 is not a real number. This contradiction proves that $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$ and, hence, f is one-to-one. The graph of f is given below.

graph

If a function f with domain D_f and range R_f is one-to-one, then f has a unique *inverse* function g with domain R_f and range D_f such that for each x in D_f,

$$g(f(x)) = x$$

and for such y in R_f,

$$f(g(y)) = y.$$

This function g is also written as f^{-1}. It is not always easy to express g explicitly but the following algorithm helps in computing g.

Step 1 Solve the equation $y = f(x)$ for x in terms of y and make sure that there exists exactly one solution for x.

Step 2 Write $x = g(y)$, where $g(y)$ is the unique solution obtained in Step 1.

Step 3 If it is desirable to have x represent the independent variable and y represent the dependent variable, then exchange x and y in Step 2 and write

$$y = g(x).$$

Remark 1 If $y = f(x)$ and $y = g(x) = f^{-1}(x)$ are graphed on the same coordinate axes, then the graph of $y = g(x)$ is a mirror image of the graph of $y = f(x)$ through the line $y = x$.

Example 1.1.10 Determine the inverse of $f(x) = x^3$.

We already know from Example 9 that f is one-to-one and, hence, it has a unique inverse. We use the above algorithm to compute $g = f^{-1}$.

Step 1 We solve $y = x^3$ for x and get $x = y^{1/3}$, which is the unique solution.

Step 2 Then $g(y) = y^{1/3}$ and $g(x) = x^{1/3} = f^{-1}(x)$.

Step 3 We plot $y = x^3$ and $y = x^{1/3}$ on the same coordinate axis and compare their graphs.

graph

A polynomial function p of degree n has the general form

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \ \ a_2 \neq 0.$$

The polynomial functions are some of the simplest functions to compute. For this reason, in calculus we approximate other functions with polynomial functions.

A rational function r has the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions. We will assume that $p(x)$ and $q(x)$ have no common non-constant factors. Then the domain of $r(x)$ is the set of all real numbers x such that $q(x) \neq 0$.

Exercises 1.1

1. Define each of the following in your own words.

   (a) f is a function with domain D_f and range R_f

   (b) f is an even function

   (c) f is an odd function

   (d) The graph of f is symmetric to the y-axis

   (e) The graph of f is symmetric to the origin.

   (f) The function f is one-to-one and has inverse g.

2. Determine the domains of the following functions

(a) $f(x) = \frac{|x|}{x}$ (b) $f(x) = \frac{x^2}{x^3 - 27}$

(c) $f(x) = \sqrt{x^2 - 9}$ (d) $f(x) = \frac{x^2 - 1}{x - 1}$

3. Sketch the graphs of the following functions and determine whether they are even, odd or one-to-one. If they are one-to-one, compute their inverses and plot their inverses on the same set of axes as the functions.

(a) $f(x) = x^2 - 1$ (b) $g(x) = x^3 - 1$

(c) $h(x) = \sqrt{9 - x}, x \geq 9$ (d) $k(x) = x^{2/3}$

4. If $\{(x_1, y_1), (x_2, y_2), \ldots, (x_{n+1}, y_{n+1})\}$ is a list of discrete data points in the plane, then there exists a unique nth degree polynomial that goes through all of them. Joseph Lagrange found a simple way to express this polynomial, called the Lagrange polynomial.

For $n = 2$, $P_2(x) = y_1 \left(\frac{x - x_2}{x_1 - x_2}\right) + y_2 \left(\frac{x - x_1}{x_2 - x_1}\right)$

For $n = 3$, $P_3(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$

$$P_4(x) = y_1 \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + y_2 \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} +$$

$$y_3 \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + y_4 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}$$

Consider the data $\{(-2, 1), (-1, -2), (0, 0), (1, 1), (2, 3)\}$. Compute $P_2(x)$, $P_3(x)$, and $P_4(x)$; plot them and determine which data points they go through. What can you say about $P_n(x)$?

5. A *linear function* has the form $y = mx + b$. The number m is called the slope and the number b is called the y-intercept. The graph of this function goes through the point $(0, b)$ on the y-axis. In each of the following determine the slope, y-intercept and sketch the graph of the given linear function:

   a) $y = 3x - 5$ b) $y = -2x + 4$ c) $y = 4x - 3$

   d) $y = 4$ e) $2y + 5x = 10$

6. A quadratic function has the form $y = ax^2 + bx + c$, where $a \neq 0$. On completing the square, this function can be expressed in the form

$$y = a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}.$$

   The graph of this function is a *parabola* with *vertex* $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$ and line of symmetry axis being the vertical line with equation $x = \frac{-b}{2a}$. The graph opens upward if $a > 0$ and downwards if $a < 0$. In each of the following quadratic functions, determine the vertex, symmetry axis and sketch the graph.

   a) $y = 4x^2 - 8$ b) $y = -4x^2 + 16$ c) $y = x^2 + 4x + 5$

   d) $y = x^2 - 6x + 8$ e) $y = -x^2 + 2x + 5$ f) $y = 2x^2 - 6x + 12$

   g) $y = -2x^2 - 6x + 5$ h) $y = -2x^2 + 6x + 10$ i) $3y + 6x^2 + 10 = 0$

   j) $y = -x^2 + 4x + 6$ k) $y = -x^2 + 4x$ l) $y = 4x^2 - 16x$

7. Sketch the graph of the linear function defined by each linear equation and determine the x-intercept and y-intercept if any.

   a) $3x - y = 3$ b) $2x - y = 10$ c) $x = 4 - 2y$

d) $4x - 3y = 12$ e) $3x + 4y = 12$ f) $4x + 6y = -12$

g) $2x - 3y = 6$ h) $2x + 3y = 12$ i) $3x + 5y = 15$

8. Sketch the graph of each of the following functions:

a) $y = 4|x|$ b) $y = -4|x|$

c) $y = 2|x| + |x - 1|$ d) $y = 3|x| + 2|x - 2| - 4|x + 3|$

e) $y = 2|x + 2| - 3|x + 1|$

9. Sketch the graph of each of the following piecewise functions.

a) $y = \begin{cases} 2 & \text{if } x \geq 0 \\ -2 & \text{if } x < 0 \end{cases}$ b) $y = \begin{cases} x^2 & \text{for } x \leq 0 \\ 2x + 4 & \text{for } x > 0 \end{cases}$

c) $y = \begin{cases} 4x^2 & \text{if } x \geq 0 \\ 3x^3 & x < 0 \end{cases}$ d) $y = \begin{cases} 3x^2 & \text{for } x \leq 1 \\ 4 & \text{for } x > 1 \end{cases}$

e) $y = n - 1$ for $n - 1 \leq x < n$, for each integer n.

f) $y = n$ for $n - 1 < x \leq n$ for each integer n.

10. The *reflection* of the graph of $y = f(x)$ is the graph of $y = -f(x)$. In each of the following, sketch the graph of f and the graph of its reflection on the same axis.

a) $y = x^3$ b) $y = x^2$ c) $y = |x|$

d) $y = x^3 - 4x$ e) $y = x^2 - 2x$ f) $y = |x| + |x - 1|$

g) $y = x^4 - 4x^2$ h) $y = 3x - 6$ i) $y = \begin{cases} x^2 + 1 & \text{for } x \leq 0 \\ x^3 + 1 & \text{if } x < 0 \end{cases}$

11. The graph of $y = f(x)$ is said to be

    (i) Symmetric with respect to the y-axis if (x, y) and $(-x, y)$ are both on the graph of f;

    (ii) Symmetric with respect to the origin if (x, y) and $(-x, -y)$ are both on the graph of f.

    For the functions in problems 10 a) – 10 i), determine the functions whose graphs are (i) Symmetric with respect to y-axis or (ii) Symmetric with respect to the origin.

12. Discuss the symmetry of the graph of each function and determine whether the function is even, odd, or neither.

    a) $f(x) = x^6 + 1$ b) $f(x) = x^4 - 3x^2 + 4$ c) $f(x) = x^3 - x^2$

    d) $f(x) = 2x^3 + 3x$ e) $f(x) = (x - 1)^3$ f) $f(x) = (x + 1)^4$

    g) $f(x) = \sqrt{x^2 + 4}$ h) $f(x) = 4|x| + 2$ i) $f(x) = (x^2 + 1)^3$

    j) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ k) $f(x) = \sqrt{4 - x^2}$ l) $f(x) = x^{1/3}$

1.2 Trigonometric Functions

The trigonometric functions are defined by the points (x, y) on the unit circle with the equation $x^2 + y^2 = 1$.

graph

Consider the points $A(0, 0), B(x, 0), C(x, y)$ where $C(x, y)$ is a point on the unit circle. Let θ, read theta, represent the length of the arc joining the points $D(1, 0)$ and $C(x, y)$. This length is the radian measure of the angle CAB. Then we define the following six trigonometric functions of θ as follows:

$$\sin\theta = \frac{y}{1}, \cos\theta = \frac{x}{1}, \tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta},$$

$$\csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}, \cot\theta = \frac{x}{y} = \frac{1}{\tan\theta}.$$

Since each revolution of the circle has arc length $2\pi, \sin\theta$ and $\cos\theta$ have period 2π. That is,

$$\sin(\theta + 2n\pi) = \sin\theta \text{ and } \cos(\theta + 2n\pi) = \cos\theta,\ n = 0, \pm 1, \pm 2, \ldots$$

The function values of some of the common arguments are given below:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

θ	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
$\sin\theta$	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0
$\cos\theta$	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1

A function f is said to have period p if p is the smallest positive number such that, for all x,

$$f(x + np) = f(x),\ n = 0, \pm 1, \pm 2, \ldots .$$

Since $\csc\theta$ is the reciprocal of $\sin\theta$ and $\sec\theta$ is the reciprocal of $\cos(\theta)$, their periods are also 2π. That is,

$$\csc(\theta + 2n\pi) = \csc(\theta) \text{ and } \sec(\theta + 2n\pi) = \sec\theta,\ n = 0, \pm 1, \pm 2, \ldots .$$

It turns out that $\tan\theta$ and $\cot\theta$ have period π. That is,

$$\tan(\theta + n\pi) = \tan\theta \text{ and } \cot(\theta + n\pi) = \cot\theta,\ n = 0, \pm 1, \pm 2, \ldots .$$

Geometrically, it is easy to see that $\cos\theta$ and $\sec\theta$ are the only even trigonometric functions. The functions $\sin\theta, \cos\theta, \tan\theta$ and $\cot\theta$ are all odd functions. The functions $\sin\theta$ and $\cos\theta$ are defined for all real numbers. The functions $\csc\theta$ and $\cot\theta$ are not defined for integer multiples of π, and $\sec\theta$ and $\tan\theta$ are not defined for odd integer multiples of $\pi/2$. The graphs of the six trigonometric functions are sketched as follows:

graph

The dotted vertical lines represent the vertical asymptotes.

There are many useful trigonometric identities and reduction formulas. For future reference, these are listed here.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin^2\theta = 1 - \cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta \qquad \tan^2\theta = \sec^2\theta - 1 \qquad \sec^2\theta - \tan^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta \qquad \cot^2\theta = \csc^2\theta - 1 \qquad \csc^2\theta - \cot^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1 \qquad \cos 2\theta = 1 + 2\sin^2\theta$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y, \quad \cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y, \quad \cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y} \qquad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$$

$$\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\sin(\pi \pm \theta) = \mp \sin\theta$$

$$\cos(\pi \pm \theta) = -\cos\theta$$

$$\tan(\pi \pm \theta) = \pm \tan\theta$$

$$\cot(\pi \pm \theta) = \pm \cot\theta$$

$$\sec(\pi \pm \theta) = -\sec\theta$$

$$\csc(\pi \pm \theta) = \mp \csc\theta$$

In applications of calculus to engineering problems, the graphs of $y = A\sin(bx+c)$ and $y = A\cos(bx+c)$ play a significant role. The first problem has to do with converting expressions of the form $A\sin bx + B\cos bx$ to one of the above forms. Let us begin first with an example.

Example 1.2.1 Express $y = 3\sin(2x) - 4\cos(2x)$ in the form $y = A\sin(2x \pm \theta)$ or $y = A\cos(2x \pm \theta)$.

First of all, we make a right triangle with sides of length 3 and 4 and compute the length of the hypotenuse, which is 5. We label one of the acute angles as θ and compute $\sin\theta, \cos\theta$ and $\tan\theta$. In our case,

$$\sin\theta = \frac{3}{5} \quad , \quad \cos\theta = \frac{4}{5} \quad , \text{and, } \tan\theta = \frac{3}{4}.$$

graph

Then,

$$\begin{aligned} y &= 3\sin 2x - 4\cos 2x \\ &= 5\left[(\sin(2x))\left(\frac{3}{5}\right) - (\cos(2x))\frac{4}{5}\right] \\ &= 5[\sin(2x)\sin\theta - \cos(2x)\cos\theta] \\ &= -5[\cos(2x)\cos\theta - \sin(2x)\sin\theta] \\ &= -5[\cos(2x+\theta)] \end{aligned}$$

Thus, the problem is reduced to sketching a cosine function, ???

$$y = -5\cos(2x+\theta).$$

We can compute the radian measure of θ from any of the equations

$$\sin\theta = \frac{3}{5},\ \cos\theta = \frac{4}{5} \text{ or } \tan\theta = \frac{3}{4}.$$

Example 1.2.2 Sketch the graph of $y = 5\cos(2x+1)$.

In order to sketch the graph, we first compute all of the zeros, relative maxima, and relative minima. We can see that the maximum values will be 5 and minimum values are -5. For this reason the number 5 is called the amplitude of the graph. We know that the cosine function has zeros at odd integer multiples of $\pi/2$. Let

$$2x_n + 1 = (2n+1)\frac{\pi}{2},\ x_n = (2n+1)\frac{\pi}{4} - \frac{1}{2},\ n = 0, \pm 1, \pm 2 \ldots .$$

The max and min values of a cosine function occur halfway between the consecutive zeros. With this information, we are able to sketch the graph of the given function. The period is π, phase shift is $\frac{1}{2}$ and frequency is $\frac{1}{\pi}$.

graph

For the functions of the form $y = A\sin(\omega t \pm d)$ or $y = A\cos(\omega t \pm d)$ we make the following definitions:

$$\textit{period} = \frac{2\pi}{\omega},\ \textit{frequency} = \frac{1}{\text{period}} = \frac{\omega}{2\pi},$$

$$\textit{amplitude} = |A|, \text{ and } \textit{phase shift} = \frac{d}{\omega}.$$

The motion of a particle that follows the curves $A\sin(\omega t \pm d)$ or $A\cos(\omega t \pm d)$ is called *simple harmonic motion*.

Exercises 1.2

1. Determine the amplitude, frequency, period and phase shift for each of the following functions. Sketch their graphs.

   (a) $y = 2\sin(3t - 2)$ (b) $y = -2\cos(2t - 1)$
   (c) $y = 3\sin 2t + 4\cos 2t$ (d) $y = 4\sin 2t - 3\cos 2t$
   (e) $y = \dfrac{\sin x}{x}$

2. Sketch the graphs of each of the following:

   (a) $y = \tan(3x)$ (b) $y = \cot(5x)$ (c) $y = x\sin x$
   (d) $y = \sin(1/x)$ (e) $y = x\sin(1/x)$

3. Express the following products as the sum or difference of functions.

   (a) $\sin(3x)\cos(5x)$ (b) $\cos(2x)\cos(4x)$ (c) $\cos(2x)\sin(4x)$
   (d) $\sin(3x)\sin(5x)$ (e) $\sin(4x)\cos(4x)$

4. Express each of the following as a product of functions:

   (a) $\sin(x+h) - \sin x$ (b) $\cos(x+h) - \cos x$ (c) $\sin(5x) - \sin(3x)$
   (d) $\cos(4x) - \cos(2x)$ (e) $\sin(4x) + \sin(2x)$ (f) $\cos(5x) + \cos(3x)$

5. Consider the graph of $y = \sin x$, $\dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2}$. Take the sample points

$$\left\{\left(-\frac{\pi}{2}, -1\right), \left(-\frac{\pi}{6}, -\frac{\pi}{2}\right), (0,0), \left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 1\right)\right\}.$$

Compute the fourth degree Lagrange Polynomial that approximates and agrees with $y = \sin x$ at these data points. This polynomial has the form

$$\begin{aligned} P_5(x) &= y_1 \frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} + \\ & \quad y_2 \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} + \cdots \\ &+ y_5 \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}. \end{aligned}$$

6. Sketch the graphs of the following functions and compute the amplitude, period, frequency and phase shift, as applicable.

   a) $y = 3\sin t$ b) $y = 4\cos t$ c) $y = 2\sin(3t)$

   d) $y = -4\cos(2t)$ e) $y = -3\sin(4t)$ f) $y = 2\sin\left(t + \frac{\pi}{6}\right)$

   g) $y = -2\sin\left(t - \frac{\pi}{6}\right)$ h) $y = 3\cos(2t + \pi)$ i) $y = -3\cos(2t - \pi)$

   j) $y = 2\sin(4t + \pi)$ k) $y = -2\cos(6t - \pi)$ l) $y = 3\sin(6t + \pi)$

7. Sketch the graphs of the following functions over two periods.

   a) $y = 2\sec x$ b) $y = -3\tan x$ c) $y = 2\cot x$

   d) $y = 3\csc x$ e) $y = \tan(\pi x)$ f) $y = \tan\left(2x + \frac{\pi}{3}\right)$

   g) $y = 2\cot\left(3x + \frac{\pi}{2}\right)$ h) $y = 3\sec\left(2x + \frac{\pi}{3}\right)$ i) $y = 2\sin\left(\pi x + \frac{\pi}{6}\right)$

8. Prove each of the following identities:

   a) $\cos 3t = 3\cos t + 4\cos^3 t$ b) $\sin(3t) = 3\sin x - 4\sin^3 x$

   c) $\sin^4 t - \cos^4 t = -\cos 2t$ d) $\dfrac{\sin^3 t - \cos^3 t}{\sin t - \cos t} = 1 + \sin 2t$

   e) $\cos 4t\cos 7t - \sin 7t\sin 4t = \cos 11t$ f) $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

9. If $f(x) = \cos x$, prove that

$$\frac{f(x+h) - f(x)}{h} = \cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right).$$

10. If $f(x) = \sin x$, prove that

$$\frac{f(x+h) - f(x)}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right).$$

11. If $f(x) = \cos x$, prove that

$$\frac{f(x) - f(t)}{x - t} = \cos t \left(\frac{\cos(x - t) - 1}{x - t} \right) - \sin t \left(\frac{\sin(x - t)}{x - t} \right).$$

12. If $f(x) = \sin x$, prove that

$$\frac{f(x) - f(t)}{x - t} = \sin t \left(\frac{\cos(x - t) - 1}{x - t} \right) + \cos t \left(\frac{\sin(x - t)}{x - t} \right).$$

13. Prove that

$$\cos(2t) = \frac{1 - \tan^2 t}{1 + \tan^2 t}.$$

14. Prove that if $y = \tan \left(\frac{x}{2} \right)$, then

(a) $\cos x = \dfrac{1 - u^2}{1 + u^2}$ (b) $\sin x = \dfrac{2u}{1 + u^2}$

1.3 Inverse Trigonometric Functions

None of the trigonometric functions are one-to-one since they are periodic. In order to define inverses, it is customary to restrict the domains in which the functions are one-to-one as follows.

1. $y = \sin x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, is one-to-one and covers the range $-1 \leq y \leq 1$. Its inverse function is denoted $\arcsin x$, and we define $y = \arcsin x,\ -1 \leq x \leq 1$, if and only if, $x = \sin y,\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

   graph

2. $y = \cos x,\ 0 \leq x \leq \pi$, is one-to-one and covers the range $-1 \leq y \leq 1$. Its inverse function is denoted $\arccos x$, and we define $y = \arccos x,\ -1 \leq x \leq 1$, if and only if, $x = \cos y,\ 0 \leq y \leq \pi$.

   graph

3. $y = \tan x,\ \frac{-\pi}{2} < x < \frac{\pi}{2}$, is one-to-one and covers the range $-\infty < y < \infty$ Its inverse function is denoted $\arctan x$, and we define $y = \arctan x,\ -\infty < x < \infty$, if and only if, $x = \tan y,\ \frac{-\pi}{2} < y < \frac{\pi}{2}$.

   graph

4. $y = \cot x,\ 0, x < \pi$, is one-to-one and covers the range $-\infty < y < \infty$. Its inverse function is denoted arccot x, and we define $y = \operatorname{arccot} x,\ -\infty < x < \infty$, if and only if $x = \cot y,\ 0 < y < \pi$.

   graph

5. $y = \sec x,\ 0 \le x \le \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < x \le \pi$ is one-to-one and covers the range $-\infty < y \le -1$ or $1 \le y < \infty$. Its inverse function is denoted arcsec x, and we define $y = \text{arcsec } x,\ -\infty < x \le -1$ or $1 \le x < \infty$, if and only if, $x = \sec y,\ 0 \le y < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < y \le \pi$.

graph

6. $y = \csc x,\ \dfrac{-\pi}{2} \le x < 0$ or $0 < x \le \dfrac{\pi}{2}$, is one-to-one and covers the range $-\infty < y \le -1$ or $1 \le y < \infty$. Its inverse is denoted $\text{arccsc}\, x$ and we define $y = \text{arccsc}\, x,\ -\infty < x \le -1$ or $1 \le x < \infty$, if and only if, $x = \csc y,\ \dfrac{-\pi}{2} \le y < 0$ or $0 < y \le \dfrac{\pi}{2}$.

Example 1.3.1 Show that each of the following equations is valid.

(a) $\arcsin x + \arccos x = \dfrac{\pi}{2}$

(b) $\arctan x + \text{arccot}\, x = \dfrac{\pi}{2}$

(c) $\text{arcsec}\, x + \text{arccsc}\, x = \dfrac{\pi}{2}$

To verify equation (a), we let $\arcsin x = \theta$.

graph

Then $x = \sin\theta$ and $\cos\left(\dfrac{\pi}{2} - \theta\right) = x$, as shown in the triangle. It follows that

$$\frac{\pi}{2} - \theta = \arccos x, \quad \frac{\pi}{2} = \theta + \arccos x = \arcsin x + \arccos x.$$

The equations in parts (b) and (c) are verified in a similar way.

Example 1.3.2 If $\theta = \arcsin x$, then compute $\cos\theta, \tan\theta, \cot\theta, \sec\theta$ and $\csc\theta$.

If θ is $-\dfrac{\pi}{2}$, 0, or $\dfrac{\pi}{2}$, then computations are easy.

graph

Suppose that $-\dfrac{\pi}{2} < x < 0$ or $0 < x < \dfrac{\pi}{2}$. Then, from the triangle, we get

$$\cos\theta = \sqrt{1-x^2}, \quad \tan\theta = \frac{x}{\sqrt{1-x^2}}, \quad \cot\theta = \frac{\sqrt{1-x^2}}{x},$$
$$\sec\theta = \frac{1}{\sqrt{1-x^2}} \text{ and } \csc\theta = \frac{1}{x}.$$

Example 1.3.3 Make the given substitutions to simplify the given radical expression and compute all trigonometric functions of θ.

(a) $\sqrt{4-x^2}$, $x = 2\sin\theta$ (b) $\sqrt{x^2-9}$, $x = 3\sec\theta$

(c) $(4+x^2)^{3/2}$, $x = 2\tan\theta$

(a) For part (a), $\sin\theta = \dfrac{x}{2}$ and we use the given triangle:

graph

Then

$$\cos\theta = \frac{\sqrt{4-x^2}}{2}, \quad \tan\theta = \frac{x}{\sqrt{4-x^2}}, \quad \cot\theta = \frac{\sqrt{4-x^2}}{x},$$
$$\sec\theta = \frac{2}{\sqrt{4-x^2}}, \quad \csc\theta = \frac{2}{x}.$$

Furthermore, $\sqrt{4-x^2} = 2\cos\theta$ and the radical sign is eliminated.

(b) For part (b), $\sec\theta = \dfrac{x}{3}$ and we use the given triangle:

graph

Then,

$$\sin\theta = \frac{\sqrt{x^2-4}}{x}, \quad \cos\theta = \frac{3}{x}, \quad \tan\theta = \frac{\sqrt{x^2-4}}{3}$$

$$\cot\theta = \frac{3}{\sqrt{x^2-9}}, \quad \csc\theta = \frac{x}{\sqrt{x^2-9}}.$$

Furthermore, $\sqrt{x^2-9} = 3\tan\theta$ and the radical sign is eliminated.

(c) For part (c), $\tan\theta = \dfrac{x}{2}$ and we use the given triangle:

graph

Then,

$$\sin\theta = \frac{x}{\sqrt{x^2+4}}, \quad \cos\theta = \frac{2}{\sqrt{x^2+4}}, \quad \cot\theta = \frac{2}{x},$$

$$\sec\theta = \frac{\sqrt{x^2+4}}{2}, \quad \csc\theta = \frac{\sqrt{x^2+4}}{x}.$$

Furthermore, $\sqrt{x^2+4} = 2\sec\theta$ and hence

$$(4+x)^{3/2} = (2\sec\theta)^3 = 8\sec^3\theta.$$

Remark 2 The three substitutions given in Example 15 are very useful in calculus. In general, we use the following substitutions for the given radicals:

(a) $\sqrt{a^2 - x^2},\ x = a\sin\theta$ (b) $\sqrt{x^2 - a^2},\ x = a\sec\theta$

(c) $\sqrt{a^2 + x^2},\ x = a\tan\theta.$

Exercises 1.3

1. Evaluate each of the following:

   (a) $3\arcsin\left(\frac{1}{2}\right) + 2\arccos\left(\frac{\sqrt{3}}{2}\right)$

   (b) $4\arctan\left(\frac{1}{\sqrt{3}}\right) + 5\text{arccot}\left(\frac{1}{\sqrt{3}}\right)$

   (c) $2\text{arcsec}\,(-2) + 3\arccos\left(-\frac{2}{\sqrt{3}}\right)$

   (d) $\cos(2\arccos(x))$

   (e) $\sin(2\arccos(x))$

2. Simplify each of the following expressions by eliminating the radical by using an appropriate trigonometric substitution.

   (a) $\frac{x}{\sqrt{9 - x^2}}$ (b) $\frac{3 + x}{\sqrt{16 + x^2}}$ (c) $\frac{x - 2}{x\sqrt{x^2 - 25}}$

   (d) $\frac{1 + x}{\sqrt{x^2 + 2x + 2}}$ (e) $\frac{2 - 2x}{\sqrt{x^2 - 2x - 3}}$

   (Hint: In parts (d) and (e), complete squares first.)

3. Some famous polynomials are the so-called Chebyshev polynomials, defined by

$$T_n(x) = \cos(n\arccos x),\ -1 \le x \le 1,\ n = 0, 1, 2, \ldots .$$

(a) Prove the recurrence relation for Chebyshev polynomials:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ for each } n \geq 1.$$

(b) Show that $T_0(x) = 1,\ T_1(x) = x$ and generate $T_2(x), T_3(x), T_4(x)$ and $T_5(x)$ using the recurrence relation in part (a).

(c) Determine the zeros of $T_n(x)$ and determine where $T_n(x)$ has its absolute maximum or minimum values, $n = 1, 2, 3, 4, ?$.
(Hint: Let $\theta = \arccos x, x = \cos\theta$. Then $T_n(x) = \cos(n\theta), T_{n+1}(x) = \cos(n\theta + \theta), T_{n-1}(x) = \cos(n\theta - \theta)$. Use the expansion formulas and then make substitutions in part (a)).

4. Show that for all integers m and n,

$$T_n(x)T_m(x) = \frac{1}{2}\left[T_{m+n}(x) + T_{|m-n|}(x)\right]$$

(Hint: use the expansion formulas as in problem 3.)

5. Find the exact value of y in each of the following

a) $y = \arccos\left(-\frac{1}{2}\right)$ b) $y = \arcsin\left(\frac{\sqrt{3}}{2}\right)$ c) $y = \arctan(-\sqrt{3})$

d) $y = \operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$ e) $y = \operatorname{arcsec}(-\sqrt{2})$ f) $y = \operatorname{arccsc}(-\sqrt{2})$

g) $y = \operatorname{arcsec}\left(-\frac{2}{\sqrt{3}}\right)$ h) $y = \operatorname{arccsc}\left(-\frac{2}{\sqrt{3}}\right)$ i) $y = \operatorname{arcsec}(-2)$

j) $y = \operatorname{arccsc}(-2)$ k) $y = \arctan\left(\frac{-1}{\sqrt{3}}\right)$ l) $y = \operatorname{arccot}(-\sqrt{3})$

6. Solve the following equations for x in radians (all possible answers).

a) $2\sin^4 x = \sin^2 x$ b) $2\cos^2 x - \cos x - 1 = 0$

c) $\sin^2 x + 2\sin x + 1 = 0$ d) $4\sin^2 x + 4\sin x + 1 = 0$

e) $2\sin^2 x + 5\sin x + 2 = 0$ f) $\cot^3 x - 3\cot x = 0$

g) $\sin 2x = \cos x$ h) $\cos 2x = \cos x$

i) $\cos^2\left(\frac{x}{2}\right) = \cos x$ j) $\tan x + \cot x = 1$

7. If $\arctan t = x$, compute $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x$ and $\csc x$ in terms of t.

8. If $\arcsin t = x$, compute $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x$ and $\csc x$ in terms of t.

9. If $\operatorname{arcsec} t = x$, compute $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x$ and $\csc x$ in terms of t.

10. If $\arccos t = x$, compute $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x$ and $\csc x$ in terms of t.

Remark 3 Chebyshev polynomials are used extensively in approximating functions due to their properties that minimize errors. These polynomials are called equal ripple polynomials, since their maxima and minima alternate between 1 and -1.

1.4 Logarithmic, Exponential and Hyperbolic Functions

Most logarithmic tables have tables for $\log_{10} x, \log_e x, e^x$ and e^{-x} because of their universal applications to scientific problems. The key relationship between logarithmic functions and exponential functions, using the same base, is that each one is an inverse of the other. For example, for base 10, we have

$$N = 10^x \text{ if and only if } x = \log_{10} N.$$

We get two very interesting relations, namely

$$x = \log_{10}(10^x) \text{ and } N = 10^{(\log_{10} N)}.$$

For base e, we get

$$x = \log_e(e^x) \text{ and } y = e^{(\log_e y)}.$$

If $b > 0$ and $b \neq 1$, then b is an admissible base for a logarithm. For such an admissible base b, we get

$$x = \log_b(b^x) \text{ and } y = b^{(\log_b y)}.$$

The Logarithmic function with base b, $b > 0$, $b \neq 1$, satisfies the following important properties:

1. $\log_b(b) = 1$, $\log_b(1) = 0$, and $\log_b(b^x) = x$ for all real x.

2. $\log_b(xy) = \log_b x + \log_b y$, $x > 0, y > 0$.

3. $\log_b(x/y) = \log_b x - \log_b y$, $x > 0, y > 0$.

4. $\log_b(x^y) = y \log_b x$, $x > 0, x \neq 1$, for all real y.

5. $(\log_b x)(\log_a b) = \log_a xb > 0, a > 0, b \neq 1, a \neq 1$. Note that $\log_b x = \dfrac{\log_a x}{\log_a b}$.

This last equation (5) allows us to compute logarithms with respect to any base b in terms of logarithms in a given base a.

The corresponding laws of exponents with respect to an admissible base $b, b > 0, b \neq 1$ are as follows:

1. $b^0 = 1$, $b^1 = b$, and $b^{(\log_b x)} = x$ for $x > 0$.

2. $b^x \times b^y = b^{x+y}$

3. $\dfrac{b^x}{b^y} = b^{x-y}$

4. $(b^x)^y = b^{(xy)}$

Notation: If $b = e$, then we will express

$$\log_b(x) \text{ as } \ln(x) \text{ or } \log(x).$$

The notation $\exp(x) = e^x$ can be used when confusion may arise.

The graph of $y = \log x$ and $y = e^x$ are reflections of each other through the line $y = x$.

graph

In applications of calculus to science and engineering, the following six functions, called *hyperbolic functions*, are very useful.

1. $\sinh(x) = \frac{1}{2}\,(e^x - e^{-x})$ for all real x, read as *hyperbolic sine* of x.

2. $\cosh(x) = \frac{1}{2}\,(e^x + e^{-x})$, for all real x, read as *hyperbolic cosine* of x.

3. $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, for all real x, read as *hyperbolic tangent* of x.

4. $\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, $x \neq 0$, read as *hyperbolic cotangent* of x.

5. $\operatorname{sech}(x) = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$, for all real x, read as *hyperbolic secant* of x.

6. $\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$, $x \neq 0$, read as *hyperbolic cosecant* of x.

The graphs of these functions are sketched as follows:

graph

Example 1.4.1 Eliminate quotients and exponents in the following equation by taking the natural logarithm of both sides.

$$y = \frac{(x+1)^3(2x-3)^{3/4}}{(1+7x)^{1/3}(2x+3)^{3/2}}$$

$$\begin{aligned}
\ln(y) &= \ln\left[\frac{(x+1)^3(2x-3)^{3/4}}{(1+7x)^{1/3}(2x+3)^{3/2]}}\right] \\
&= \ln[(x+1)^3(2x-3)^{3/4}] - \ln[(1+7x)^{1/3}(2x+3)^{3/2}] \\
&= \ln(x+1)^3 + \ln(2x-3)^{3/4} - \{\ln(1+7x)^{1/3} + \ln(2x+3)^{3/2}\} \\
&= 3\ln(x+1) + \frac{3}{4}\ \ln(2x-3) - \frac{1}{3}\ \ln(1+7x) - \frac{3}{2}\ \ln(2x+3)
\end{aligned}$$

Example 1.4.2 Solve the following equation for x:

$$\log_3(x^4) + \log_3 x^3 - 2\log_3 x^{1/2} = 5.$$

Using logarithm properties, we get

$$\begin{aligned}
4\log_3 x + 3\log_3 x - \log_3 x &= 5 \\
6\log_3 x &= 5 \\
\log_3 x &= \frac{5}{6} \\
x &= (3)^{5/6}.
\end{aligned}$$

Example 1.4.3 Solve the following equation for x:

$$\frac{e^x}{1+e^x} = \frac{1}{3}.$$

On multiplying through, we get

$$\begin{aligned}
3e^x = 1 + e^x \text{ or } 2e^x = 1, e^x\ &= \frac{1}{2} \\
x =\ &\ln(1/2) = -\ln(2).
\end{aligned}$$

Example 1.4.4 Prove that for all real x, $cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\
&= \frac{1}{4}[e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})] \\
&= \frac{1}{4}[4] \\
&= 1
\end{aligned}$$

Example 1.4.5 Prove that

(a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

(b) $\sinh 2x = 2 \sinh x \cosh y$.

Equation (b) follows from equation (a) by letting $x = y$. So, we work with equation (a).

$$
\begin{aligned}
\text{(a)}\quad \sinh x \cosh y + \cosh x \sinh y &= \frac{1}{2}(e^x - e^{-x}) \cdot \frac{1}{2}(e^y + e^{-y}) \\
&\quad + \frac{1}{2}(e^x + e^{-x}) \cdot \frac{1}{2}(e^y - e^{-y}) \\
&= \frac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) \\
&\quad + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\
&= \frac{1}{4}[2(e^{x+y} - e^{-(x+y)}] \\
&= \frac{1}{2}(e^{(x+y)} - e^{-(x+y)}) \\
&= \sinh(x+y).
\end{aligned}
$$

Example 1.4.6 Find the inverses of the following functions:

(a) $\sinh x$ (b) $\cosh x$ (c) $\tanh x$

(a) Let $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Then

$$
\begin{aligned}
2e^x y = 2e^x \left(\frac{1}{2}(e^x - e^{-x})\right) &= e^{2x} - 1 \\
e^{2x} - 2ye^x - 1 &= 0 \\
(e^x)^2 - (2y)e^x - 1 &= 0 \\
e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} &= y \pm \sqrt{y^2+1}
\end{aligned}
$$

Since $e^x > 0$ for all x, $e^x = y + \sqrt{1+y^2}$.

On taking natural logarithms of both sides, we get

$$x = \ln(y + \sqrt{1 + y^2}).$$

The inverse function of $\sinh x$, denoted arcsinh x, is defined by

$$\boxed{\operatorname{arcsinh} x = \ln(x + \sqrt{1 + x^2})}$$

(b) As in part (a), we let $y = \cosh x$ and

$$\begin{aligned}
2e^x y &= 2e^x \cdot \frac{1}{2}(e^x + e^{-x}) = e^{2x} + 1 \\
e^{2x} &- (2y)e^x + 1 = 0 \\
e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\
e^x &= y \pm \sqrt{y^2 - 1}.
\end{aligned}$$

We observe that $\cosh x$ is an even function and hence it is not one-to-one. Since $\cosh(-x) = \cosh(x)$, we will solve for the larger x. On taking natural logarithms of both sides, we get

$$x_1 = \ln(y + \sqrt{y^2 - 1}) \text{ or } x_2 = \ln(y - \sqrt{y^2 - 1}).$$

We observe that

$$\begin{aligned}
x_2 = \ln(y - \sqrt{y^2 - 1}) &= \ln\left[\frac{(y - \sqrt{y^2 - 1})(y + \sqrt{y^2 - 1})}{y + \sqrt{y^2 - 1}}\right] \\
&= \ln\left(\frac{1}{y + \sqrt{y^2 - 1}}\right) \\
&= -\ln(y + \sqrt{y^2 - 1}) = -x_1.
\end{aligned}$$

Thus, we can define, as the principal branch,

$$\boxed{\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}), \ x \geq 1}$$

(c) We begin with $y = \tanh x$ and clear denominators to get

$$\begin{aligned}
y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad , \quad |y| < 1 \\
e^x[(e^x + e^{-x})y] &= e^x[(e^x - e^{-x})] \quad , \quad |y| < 1 \\
(e^{2x} + 1)y &= e^{2x} - 1 \quad , \quad |y| < 1 \\
e^{2x}(y - 1) &= -(1 + y) \quad , \quad |y| < 1 \\
e^{2x} &= -\frac{(1 + y)}{y - 1} \quad , \quad |y| < 1 \\
e^{2x} &= \frac{1 + y}{1 - y} \quad , \quad |y| < 1 \\
2x &= \ln\left(\frac{1 + y}{1 - y}\right) \quad , \quad |y| < 1 \\
x &= \frac{1}{2}\ln\left(\frac{1 + y}{1 - y}\right) \quad , |y| < 1.
\end{aligned}$$

Therefore, the inverse of the function $\tanh x$, denoted $\text{arctanh} x$, is defined by

$$\boxed{\text{arctanh}\,, x = \frac{1}{2}\ \ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1.}$$

Exercises 1.4

1. Evaluate each of the following

(a) $\log_{10}(0.001)$ (b) $\log_2(1/64)$ (c) $\ln(e^{0.001})$

(d) $\log_{10}\left(\frac{(100)^{1/3}(0.01)^2}{(.0001)^{2/3}}\right)^{0.1}$ (e) $e^{\ln(e^{-2})}$

2. Prove each of the following identities

(a) $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

(b) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(c) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

(d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

3. Simplify the radical expression by using the given substitution.

(a) $\sqrt{a^2+x^2},\ x = a\sinh t$ (b) $\sqrt{x^2-a^2},\ x = a\cosh t$

(c) $\sqrt{a^2-x^2},\ x = a\tanh t$

4. Find the inverses of the following functions:

(a) $\coth x$ (b) sech x (c) csch x

5. If $\cosh x = \frac{3}{2}$, find $\sinh x$ and $\tanh x$.

6. Prove that $\sinh(3t) = 3\sinh t + 4\sinh^3 t$ (Hint: Expand $\sinh(2t+t)$.)

7. Sketch the graph of each of the following functions.

a) $y = 10^x$ b) $y = 2^x$ c) $y = 10^{-x}$ d) $y = 2^{-x}$

e) $y = e^x$ f) $y = e^{-x^2}$ g) $y = xe^{-x^2}$ i) $y = e^{-x}$

j) $y = \sinh x$ k) $y = \cosh x$ l) $y = \tanh x$ m) $y = \coth x$

n) $y = \operatorname{sech} x$ o) $y = \operatorname{csch} x$

8. Sketch the graph of each of the following functions.

a) $y = \log_{10} x$ b) $y = \log_2 x$ c) $y = \ln x$ d) $y = \log_3 x$

e) $y = \operatorname{arcsinh} x$ f) $y = \operatorname{arccosh} x$ g) $y = \operatorname{arctanh} x$

9. Compute the given logarithms in terms $\log_{10} 2$ and $\log_{10} 3$.

a) $\log_{10} 36$ b) $\log_{10}\left(\frac{27}{16}\right)$ c) $\log_{10}\left(\frac{20}{9}\right)$

d) $\log_{10}(600)$ e) $\log_{10}\left(\frac{30}{16}\right)$ f) $\log_{10}\left(\frac{6^{10}}{(20)^5}\right)$

10. Solve each of the following equations for the independent variable.

a) $\ln x - \ln(x+1) = \ln(4)$ b) $2\log_{10}(x-3) = \log_{10}(x+5) + \log_{10} 4$

c) $\log_{10} t^2 = (\log_{10} t)^2$ d) $e^{2x} - 4e^x + 3 = 0$

e) $e^x + 6e^{-x} = 5$ f) $2\sinh x + \cosh x = 4$

Chapter 2

Limits and Continuity

2.1 Intuitive treatment and definitions

2.1.1 Introductory Examples

The concepts of limit and continuity are very closely related. An intuitive understanding of these concepts can be obtained through the following examples.

Example 2.1.1 Consider the function $f(x) = x^2$ as x tends to 2.

As x tends to 2 from the right or from the left, $f(x)$ tends to 4. The value of f at 2 is 4. The graph of f is in one piece and there are no holes or jumps in the graph. We say that f is continuous at 2 because $f(x)$ tends to $f(2)$ as x tends to 2.

graph

The statement that $f(x)$ tends to 4 as x tends to 2 *from the right* is expressed in symbols as

$$\lim_{x \to 2^+} f(x) = 4$$

and is read, "the limit of $f(x)$, as x goes to 2 *from the right*, equals 4."

The statement that $f(x)$ tends to 4 as x tends to 2 *from the left* is written

$$\lim_{x \to 2^-} f(x) = 4$$

and is read, "the limit of $f(x)$, as x goes to 2 from the left, equals 4."

The statement that $f(x)$ tends to 4 as x tends to 2 either from the right or from the left, is written

$$\lim_{x \to 2} f(x) = 4$$

and is read, "the limit of $f(x)$, as x goes to 2, equals 4."

The statement that $f(x)$ is continuous at $x = 2$ is expressed by the equation

$$\lim_{x \to 2} f(x) = f(2).$$

Example 2.1.2 Consider the unit step function as x tends to 0.

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

graph

The function, $u(x)$ tends to 1 as x tends to 0 from the right side. So, we write

$$\lim_{x \to 0^+} u(x) = 1 = u(0).$$

The limit of $u(x)$ as x tends to 0 from the left equals 0. Hence,

$$\lim_{x \to 0^-} u(x) = 0 \neq u(0).$$

Since

$$\lim_{x \to 0^+} u(x) = u(0),$$

we say that $u(x)$ is continuous at 0 from the right. Since

$$\lim_{x \to 0^-} u(x) \neq u(0),$$

we say that $u(x)$ is not continuous at 0 from the left. In this case the jump at 0 is 1 and is defined by

$$\begin{aligned}\text{jump }(u(x),0) &= \lim_{x\to 0^+} u(x) - \lim_{x\to 0^-} u(x)\\ &= 1.\end{aligned}$$

Observe that the graph of $u(x)$ has two pieces that are not joined together. Every horizontal line with equation $y = c,\ 0 < c < 1$, separates the two pieces of the graph without intersecting the graph of $u(x)$. This kind of jump discontinuity at a point is called "finite jump" discontinuity.

Example 2.1.3 Consider the signum function, sign(x), defined by

$$\text{sign }(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

If $x > 0$, then sign$(x) = 1$. If $x < 0$, then sign$(x) = -1$. In this case,

$$\begin{aligned}\lim_{x\to 0^+} \text{sign}(x) &= 1\\ \lim_{x\to 0^-} \text{sign}(x) &= -1\\ \text{jump }(\text{sign}(x),0) &= 2.\end{aligned}$$

Since sign(x) is not defined at $x = 0$, it is not continuous at 0.

Example 2.1.4 Consider $f(\theta) = \dfrac{\sin\theta}{\theta}$ as θ tends to 0.

graph

The point $C(\cos\theta, \sin\theta)$ on the unit circle defines $\sin\theta$ as the vertical length BC. The radian measure of the angle θ is the arc length DC. It is clear that the vertical length BC and arc length DC get closer to each other as θ tends to 0 from above. Thus,

graph

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1.$$

For negative θ, $\sin \theta$ and θ are both negative.

$$\lim_{\theta \to 0^+} \frac{\sin(-\theta)}{-\theta} = \lim_{\theta \to 0^+} \frac{-\sin \theta}{-\theta} = 1.$$

Hence,

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

This limit can be verified by numerical computation for small θ.

Example 2.1.5 Consider $f(x) = \dfrac{1}{x}$ as x tends to 0 and as x tends to $\pm\infty$.

graph

It is intuitively clear that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$
$$\lim_{x \to +\infty} \frac{1}{x} = 0$$
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

The function f is not continuous at $x = 0$ because it is not defined for $x = 0$. This discontinuity is not removable because the limits from the left and from the right, at $x = 0$, are not equal. The horizontal and vertical axes divide the graph of f in two separate pieces. The vertical axis is called the *vertical asymptote* of the graph of f. The horizontal axis is called the *horizontal asymptote* of the graph of f. We say that f has an essential discontinuity at $x = 0$.

Example 2.1.6 Consider $f(x) = \sin(1/x)$ as x tends to 0.

graph

The period of the sine function is 2π. As observed in Example 5, $1/x$ becomes very large as x becomes small. For this reason, many cycles of the sine wave pass from the value -1 to the value $+1$ and a rapid oscillation occurs near zero. None of the following limits exist:

$$\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0^-} \sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0} \sin\left(\frac{1}{x}\right).$$

It is not possible to define the function f at 0 to make it continuous. This kind of discontinuity is called an "oscillation" type of discontinuity.

Example 2.1.7 Consider $f(x) = x\sin\left(\frac{1}{x}\right)$ as x tends to 0.

graph

In this example, $\sin\left(\frac{1}{x}\right)$, oscillates as in Example 6, but the amplitude

$|x|$ tends to zero as x tends to 0. In this case,

$$\lim_{x\to 0^+} x \sin\left(\frac{1}{x}\right) = 0$$

$$\lim_{x\to 0^-} x \sin\left(\frac{1}{x}\right) = 0$$

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

The discontinuity at $x = 0$ is removable. We define $f(0) = 0$ to make f continuous at $x = 0$.

Example 2.1.8 Consider $f(x) = \dfrac{x-2}{x^2-4}$ as x tends to ± 2.

This is an example of a rational function that yields the *indeterminate* form 0/0 when x is replaced by 2. When this kind of situation occurs in rational functions, it is necessary to cancel the common factors of the numerator and the denominator to determine the appropriate limit if it exists. In this example, $x-2$ is the common factor and the reduced form is obtained through cancellation.

graph

$$\begin{aligned} f(x) = \frac{x-2}{x^2-4} &= \frac{x-2}{(x-2)(x+2)} \\ &= \frac{1}{x+2}. \end{aligned}$$

In order to get the limits as x tends to 2, we used the reduced form to get 1/4. The discontinuity at $x = 2$ is removed if we define $f(2) = 1/4$. This function still has the essential discontinuity at $x = -2$.

Example 2.1.9 Consider $f(x) = \dfrac{\sqrt{x}-\sqrt{3}}{x^2-9}$ as x tends to 3.

In this case f is not a rational function; still, the problem at $x = 3$ is caused by the common factor $(\sqrt{x}-\sqrt{3})$.

graph

$$\begin{aligned} f(x) &= \frac{\sqrt{x}-\sqrt{3}}{x^2-9} \\ &= \frac{(\sqrt{x}-\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})} \\ &= \frac{1}{(x+3)(\sqrt{x}+\sqrt{3})}. \end{aligned}$$

As x tends to 3, the reduced form of f tends to $1/(12\sqrt{3})$. Thus,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3} f(x) = \frac{1}{12\sqrt{3}}.$$

The discontinuity of f at $x = 3$ is removed by defining $f(3) = \dfrac{1}{12\sqrt{3}}$. The other discontinuities of f at $x = -3$ and $x = -\sqrt{3}$ are essential discontinuities and cannot be removed.

Even though calculus began intuitively, formal and precise definitions of limit and continuity became necessary. These precise definitions have become the foundations of calculus and its applications to the sciences. Let us assume that a function f is defined in some open interval, (a, b), except possibly at one point c, such that $a < c < b$. Then we make the following definitions using the Greek symbols: ϵ, read "epsilon" and δ, read, "delta."

2.1.2 Limit: Formal Definitions

Definition 2.1.1 The limit of $f(x)$ as x goes to c from the right is L, if and only if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \text{whenever, } c < x < c + \delta.$$

The statement that the limit of $f(x)$ as x goes to c from the right is L, is expressed by the equation

$$\lim_{x \to c^+} f(x) = L.$$

graph

Definition 2.1.2 The limit of $f(x)$ as x goes to c from the left is L, if and only if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \text{whenever, } c - \delta < x < c.$$

The statement that the limit of $f(x)$ as x goes to c from the left is L, is written as

$$\lim_{x \to c^-} f(x) = L.$$

graph

Definition 2.1.3 The (two-sided) limit of $f(x)$ as x goes to c is L, if and only if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \text{whenever } 0 < |x - c| < \delta.$$

graph

The equation

$$\lim_{x \to c} f(x) = L$$

is read "the (two-sided) limit of $f(x)$ as x goes to c equals L."

2.1.3 Continuity: Formal Definitions

Definition 2.1.4 The function f is said to be continuous at c from the right if $f(c)$ is defined, and

$$\lim_{x \to c^+} f(x) = f(c).$$

Definition 2.1.5 The function f is said to be continuous at c from the left if $f(c)$ is defined, and

$$\lim_{x \to c^-} f(x) = f(c).$$

Definition 2.1.6 The function f is said to be (two-sided) continuous at c if $f(c)$ is defined, and

$$\lim_{x \to c} f(x) = f(c).$$

Remark 4 The continuity definition requires that the following conditions be met if f is to be continuous at c:

(i) $f(c)$ is defined as a finite real number,

(ii) $\lim_{x \to c^-} f(x)$ exists and equals $f(c)$,

(iii) $\lim_{x \to c^+} f(x)$ exists and equals $f(c)$,

(iv) $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$.

When a function f is not continuous at c, one, or more, of these conditions are not met.

Remark 5 All polynomials, $\sin x, \cos x, e^x, \sinh x, \cosh x, b^x, b \neq 1$ are continuous for all real values of x. All logarithmic functions, $\log_b x, b > 0, b \neq 1$ are continuous for all $x > 0$. Each rational function, $p(x)/q(x)$, is continuous where $q(x) \neq 0$. Each of the functions $\tan x, \cot x, \sec x, \csc x, \tanh x, \coth x$, sech x, and csch x is continuous at each point of its domain.

Definition 2.1.7 (Algebra of functions) Let f and g be two functions that have a common domain, say D. Then we define the following for all x in D:

1. $(f+g)(x) = f(x) + g(x)$ (sum of f and g)

2. $(f-g)(x) = f(x) - g(x)$ (difference of f and g)

3. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, if $g(x) \neq 0$ (quotient of f and g)

4. $(gf)(x) = g(x)f(x)$ (product of f and g)

   If the range of f is a subset of the domain of g, then we define the composition, $g \circ f$, of f followed by g, as follows:

5. $(g \circ f)(x) = g(f(x))$

Remark 6 The following theorems on limits and continuity follow from the definitions of limit and continuity.

Theorem 2.1.1 *Suppose that for some real numbers L and M, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then*

(i) $\lim_{x\to c} k = k$, *where k is a constant function.*

(ii) $\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(iii) $\lim_{x\to c} (f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

(iv) $\lim_{x \to c} (f(x)g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$

(v) $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, *if* $\lim_{x \to c} g(x) \neq 0$

Proof.
Part (i) Let $f(x) = k$ for all x and $\epsilon > 0$ be given. Then

$$|f(x) - k| = |k - k| = 0 < \epsilon$$

for all x. This completes the proof of Part (i).

For Parts (ii)–(v) let $\epsilon > 0$ be given and let

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M.$$

By definition there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{3} \quad \text{whenever} \quad 0 < |x - c| < \delta_1 \tag{1}$$

$$|g(x) - M| < \frac{\epsilon}{3} \quad \text{whenever} \quad 0 < |x - c| < \delta_2 \tag{2}$$

Part (ii) Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x - c| < \delta$ implies that

$$0 < |x - c| < \delta_1 \quad \text{and} \quad |f(x) - L| < \frac{\epsilon}{3} \quad \text{(by (1))} \tag{3}$$

$$0 < |x - c| < \delta_2 \quad \text{and} \quad |g(x) - M| < \frac{\epsilon}{3} \quad \text{(by (2))} \tag{4}$$

Hence, if $0 < |x - c| < \delta$, then

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} \qquad \text{(by (3) and (4))} \\ &< \epsilon. \end{aligned}$$

This completes the proof of Part (ii).

Part (iii) Let δ be defined as in Part (ii). Then $0 < |x-c| < \delta$ implies that

$$\begin{aligned} |(f(x)-g(x))-(L-M)| &= |(f(x)-L)+(g(x)-M)| \\ &\le |f(x)-L|+|g(x)-M| \\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

This completes the proof of Part (iii).

Part (iv) Let $\epsilon > 0$ be given. Let

$$\epsilon_1 = \min\left(1, \frac{\epsilon}{1+|L|+|M|}\right).$$

Then $\epsilon_1 > 0$ and, by definition, there exist δ_1 and δ_2 such that

$$|f(x)-L| < \epsilon_1 \quad \text{whenever} \quad 0 < |x-c| < \delta_1 \tag{5}$$
$$|g(x)-M| < \epsilon_1 \quad \text{whenever} \quad 0 < |x-c| < \delta_2 \tag{6}$$

Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x-c| < \delta$ implies that

$$0 < |x-c| < \delta_1 \quad \text{and} \quad |f(x)-L| < \epsilon_1 \qquad \text{(by (5))} \tag{7}$$
$$0 < |x-c| < \delta_2 \quad \text{and} \quad |g(x)-M| < \epsilon_1 \qquad \text{(by (6))} \tag{8}$$

Also,

$$\begin{aligned} |f(x)g(x)-LM| &= |(f(x)-L+L)(g(x)-M+M)-LM| \\ &= |(f(x)-L)(g(x)-M)+(f(x)-L)M+L(g(x)-M)| \\ &\le |f(x)-L|\ |g(x)-M|+|f(x)+L|\ |M|+|L|\ |g(x)-M| \\ &< \epsilon_1^2+|M|\epsilon_1+|L|\epsilon_1 \\ &\le \epsilon_1+|M|\epsilon_1+|L|\epsilon_1 \\ &= (1+|M|+|N|)\epsilon_1 \\ &\le \epsilon. \end{aligned}$$

This completes the proof of Part (iv).

Part (v) Suppose that $M > 0$ and $\lim_{x\to c} g(x) = M$. Then we show that

$$\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}.$$

Since $M/2 > 0$, there exists some $\delta_1 > 0$ such that

$$\begin{aligned} |g(x) - M| &< \frac{M}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1, \\ -\frac{M}{2} + M < g(x) &< \frac{3M}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1, \\ 0 < \frac{M}{2} < g(x) &< \frac{3M}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1, \\ \frac{1}{|g(x)|} &< \frac{2}{M} \quad \text{whenever} \quad 0 < |x - c| < \delta_1. \end{aligned}$$

Let $\epsilon > 0$ be given. Let $\epsilon_1 = M^2\epsilon/2$. Then $\epsilon_1 > 0$ and there exists some $\delta > 0$ such that $\delta < \delta_1$ and

$$\begin{aligned} |g(x) - M| < \epsilon_1 \quad &\text{whenever} \quad 0 < |x - c| < \delta < \delta_1, \\ \left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{g(x)M}\right| &= \frac{|g(x) - M|}{|g(x)|M} \\ &= \frac{1}{M} \cdot \frac{1}{|g(x)|}\ |g(x) - M| \\ &< \frac{1}{M} \cdot \frac{2}{M} \cdot \epsilon_1 \\ &= \frac{2\epsilon_1}{M^2} \\ &= \epsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta. \end{aligned}$$

This completes the proof of the statement

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M} \quad \text{whenever} \quad M > 0.$$

The case for $M < 0$ can be proven in a similar manner. Now, we can use Part (iv) to prove Part (v) as follows:

$$\begin{aligned} \lim_{x \to c} \frac{f(x)}{g(x)} &= \lim_{x \to c} \left(f(x) \cdot \frac{1}{g(x)} \right) \\ &= \lim_{x \to c} f(x) \cdot \lim_{x \to c} \left(\frac{1}{g(x)} \right) \\ &= L \cdot \frac{1}{M} \\ &= \frac{L}{M}. \end{aligned}$$

This completes the proof of Theorem 2.1.1.

Theorem 2.1.2 *If f and g are two functions that are continuous on a common domain D, then the sum, $f+g$, the difference, $f-g$ and the product, fg, are continuous on D. Also, f/g is continuous at each point x in D such that $g(x) \neq 0$.*

Proof. If f and g are continuous at c, then $f(c)$ and $g(c)$ are real numbers and

$$\lim_{x\to c} f(x) = f(c), \ \lim_{x\to c} g(x) = g(c).$$

By Theorem 2.1.1, we get

$$\begin{aligned}
\lim_{x\to c}(f(x)+g(x)) &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c) \\
\lim_{x\to c}(f(x)-g(x)) &= \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = f(c)-g(c) \\
\lim_{x\to c}(f(x)g(x)) &= \left(\lim_{x\to c} f(x)\right) \lim_{x\to c}(g(x)) = f(c)g(c) \\
\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)}, \text{ if } g(c) \neq 0.
\end{aligned}$$

This completes the proof of Theorem 2.1.2.

2.1.4 Continuity Examples

Example 2.1.10 Show that the constant function $f(x) = 4$ is continuous at every real number c. Show that for every constant k, $f(x) = k$ is continuous at every real number c.

First of all, if $f(x) = 4$, then $f(c) = 4$. We need to show that

$$\lim_{x\to c} 4 = 4.$$

graph

For each $\epsilon > 0$, let $\delta = 1$. Then

$$|f(x) - f(c)| = |4-4| = 0 < \epsilon$$

for all x such that $|x - c| < 1$. Secondly, for each $\epsilon > 0$, let $\delta = 1$. Then

$$|f(x) - f(c)| = |k - k| = 0 < \epsilon$$

for all x such that $|x - c| < 1$. This completes the required proof.

Example 2.1.11 Show that $f(x) = 3x - 4$ is continuous at $x = 3$.

Let $\epsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - f(3)| &= |(3x - 4) - (5)| \\ &= |3x - 9| \\ &= 3|x - 3| \\ &< \epsilon \end{aligned}$$

whenever $|x - 3| < \dfrac{\epsilon}{3}$.

We define $\delta = \dfrac{\epsilon}{3}$. Then, it follows that

$$\lim_{x \to 3} f(x) = f(3)$$

and, hence, f is continuous at $x = 3$.

Example 2.1.12 Show that $f(x) = x^3$ is continuous at $x = 2$.

Since $f(2) = 8$, we need to prove that

$$\lim_{x \to 2} x^3 = 8 = 2^3.$$

graph

Let $\epsilon > 0$ be given. Let us concentrate our attention on the open interval

$(1, 3)$ that contains $x = 2$ at its mid-point. Then

$$\begin{aligned}
|f(x) - f(2)| &= |x^3 - 8| = |(x-2)(x^2+2x+4)| \\
&= |x-2|\ |x^2+2x+4| \\
&\le |x-2|(|x|^2+2|x|+4) \quad \text{(Triangle Inequality } |u+v| \le |u|+|v|\text{)} \\
&\le |x-2|(9+18+4) \\
&= 31|x-2| \\
&< \epsilon
\end{aligned}$$

Provided

$$|x-2| < \frac{\epsilon}{31}.$$

Since we are concentrating on the interval $(1, 3)$ for which $|x-2| < 1$, we need to define δ to be the minimum of 1 and $\frac{\epsilon}{31}$. Thus, if we define $\delta = \min\{1, \epsilon/31\}$, then

$$|f(x) - f(2)| < \epsilon$$

whenever $|x-2| < \delta$. By definition, $f(x)$ is continuous at $x = 2$.

Example 2.1.13 Show that every polynomial $P(x)$ is continuous at every c.

From algebra, we recall that, by the Remainder Theorem,

$$P(x) = (x-c)Q(x) + P(c).$$

Thus,

$$|P(x) - P(c)| = |x-c||Q(x)|$$

where $Q(x)$ is a polynomial of degree one less than the degree of $P(x)$. As in Example 12, $|Q(x)|$ is bounded on the closed interval $[c-1, c+1]$. For example, if

$$Q(x) = q_0x^{n-1} + q_1x^{n-2} + \cdots + q_{n-2}x + q_{n-1}$$

$|Q(x)| \le |q_0|\ |x|^{n-1} + |q_1|\ |x|^{n-2} + \cdots + |q_{n-2}|\ |x| + |q_{n-1}|$.

Let $m = \max\{|x| : c-1 \le x \le c+1\}$. Then

$$|Q(x)| \le |q_0|m^{n-1} + |q_1|m^{n-2} + \cdots + q_{n-2}m + |q_{n-1}| = M,$$

for some M. Then

$$|P(x) - P(c)| = |x - c|\ |Q(x)| \leq M|x - c| < \epsilon$$

whenever $|x - c| < \dfrac{\epsilon}{M}$. As in Example 12, we define $\delta = \min\left\{1, \dfrac{\epsilon}{M}\right\}$. Then $|P(x) - P(c)| < \epsilon$, whenever $|x - c| < \delta$. Hence,

$$\lim_{x \to c} P(x) = P(c)$$

and by definition $P(x)$ is continuous at each number c.

Example 2.1.14 Show that $f(x) = \dfrac{1}{x}$ is continuous at every real number $c > 0$.

We need to show that

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}.$$

Let $\epsilon > 0$ be given. Let us concentrate on the interval $|x - c| \leq \dfrac{c}{2}$; that is, $\dfrac{c}{2} \leq x \leq \dfrac{3c}{2}$. Clearly, $x \neq 0$ in this interval. Then

$$\begin{aligned}
|f(x) - f(c)| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\
&= \left|\frac{c - x}{cx}\right| \\
&= |x - c| \cdot \frac{1}{c} \cdot \frac{1}{|x|} \\
&< |x - c| \cdot \frac{1}{c} \cdot \frac{2}{c} \\
&= \frac{2}{c^2}\ |x - c| \\
&< \epsilon
\end{aligned}$$

whenever $|x - c| < \dfrac{c^2\epsilon}{2}$.

We define $\delta = \min\left\{\dfrac{c}{2}, \dfrac{c^2\epsilon}{2}\right\}$. Then for all x such that $|x - c| < \delta$,

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon.$$

Hence,

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

and the function $f(x) = \dfrac{1}{x}$ is continuous at each $c > 0$.

A similar argument can be used for $c < 0$. The function $f(x) = \dfrac{1}{x}$ is continuous for all $x \neq 0$.

Example 2.1.15 Suppose that the domain of a function g contains an open interval containing c, and the range of g contains an open interval containing $g(c)$. Suppose further that the domain of f contains the range of g. Show that if g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

We need to show that

$$\lim_{x \to c} f(g(x)) = f(g(c)).$$

Let $\epsilon > 0$ be given. Since f is continuous at $g(c)$, there exists $\delta_1 > 0$ such that

1. $|f(y) - f(g(c))| < \epsilon$, whenever, $|y - g(c)| < \delta_1$.

   Since g is continuous at c, and $\delta_1 > 0$, there exists $\delta > 0$ such that

2. $|g(x) - g(c)| < \delta_1$, whenever, $|x - c| < \delta$.

On replacing y by $g(x)$ in equation (1), we get

$$|f(g(x)) - f(g(c))| < \epsilon, \text{ whenever, } |x - c| < \delta.$$

By definition, it follows that

$$\lim_{x \to c} f(g(x)) = f(g(c))$$

and the composition $f \circ g$ is continuous at c.

Example 2.1.16 Suppose that two functions f and g have a common domain that contains one open interval containing c. Suppose further that f and g are continuous at c. Then show that

(i) $f + g$ is continuous at c,

(ii) $f - g$ is continuous at c,

(iii) kf is continuous at c for every constant $k \neq 0$,

(iv) $f \cdot g$ is continuous at c.

Part (i) We need to prove that

$$\lim_{x \to c} [f(x) + g(x)] = f(c) + g(c).$$

Let $\epsilon > 0$ be given. Then $\frac{\epsilon}{2} > 0$. Since f is continuous at c and $\frac{\epsilon}{2} > 0$, there exists some $\delta_1 > 0$ such that

(1) $$|f(x) - f(c)| < \frac{\epsilon}{2}, \text{ whenever, } |x - c| \leq \delta_1.$$

Also, since g is continuous at c and $\frac{\epsilon}{2} > 0$, there exists some $\delta_2 > 0$ such that

(2) $$|g(x) - g(c)| < \frac{\epsilon}{2}, \text{ whenever, } |x - c| < \frac{\delta}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$. Let $|x - c| < \delta$. Then $|x - c| < \delta_1$ and $|x - c| < \delta_2$. For this choice of x, we get

$$\begin{aligned}
&|\{f(x) + g(x)\} - \{f(c) + g(c)\}| \\
&\quad = |\{f(x) - f(c)\} + \{g(x) - g(c)\}| \\
&\quad \leq |f(x) - f(c)| + |g(x) - g(c)| \qquad \text{(by triangle inequality)} \\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\quad = \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x \to 0} (f(x) + g(x)) = f(c) + g(c)$$

and $f + g$ is continuous at c. This proves part (i).

Part (ii) For Part (ii) we chose $\epsilon, \epsilon/2, \delta_1, \delta_2$ and δ exactly as in Part (i). Suppose $|x - c| < \delta$. Then $|x - c| < \delta_1$ and $|x - c| < \delta_2$. For these choices of x we get

$$\begin{aligned}
&|\{f(x) - g(x)\} - \{f(c) - g(c)\}| \\
&\qquad = |\{f(x) - f(c)\} - \{g(x) - g(c)\}| \\
&\qquad \leq |f(x) - f(c)| + |g(x) - g(c)| \qquad \text{(by triangle inequality)} \\
&\qquad < \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\qquad = \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x \to c} (f(x) - g(x)) = f(c) - g(c)$$

and, hence, $f - g$ is continuous at c.

Part (iii) For Part (iii) let $\epsilon > 0$ be given. Since $k \neq 0$, $\dfrac{\epsilon}{|k|} > 0$. Since f is continuous at c, there exists some $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{\epsilon}{|k|}, \quad \text{whenever, } |x - c| < \delta.$$

If $|x - c| < \delta$, then

$$\begin{aligned}
|kf(x) - kf(c)| &= |k(f(x) - f(c))| \\
&= |k| \, |(f(x) - f(c)| \\
&< |k| \cdot \frac{\epsilon}{|k|} \\
&= \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x \to c} kf(x) = kf(c)$$

and, hence, kf is continuous at c.

Part (iv) We need to show that

$$\lim_{x \to c} (f(x)g(x)) = f(c)g(c).$$

Let $\epsilon > 0$ be given. Without loss of generality we may assume that $\epsilon < 1$. Let $\epsilon_1 = \dfrac{\epsilon}{2(1+|f(c)|+|g(c)|)}$. Then $\epsilon_1 > 0, \epsilon_1 < 1$ and $\epsilon_1(1+|f|+|g(c)|) = \dfrac{\epsilon}{2} < \epsilon$. Since f is continuous at c and $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$|f(x) - f(c)| < \epsilon_1 \qquad \text{whenever, } |x-c| < \delta_1.$$

Also, since g is continuous at c and $\epsilon_1 > 0$, there exists $\delta_2 > 0$ such that

$$|g(x) - g(c)| < \epsilon_1 \quad \text{whenever, } |x-c| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and $|x - c| < \delta$. For these choices of x, we get

$$\begin{aligned}
&|f(x)g(x) - f(c)g(c)| \\
&= |(\overline{f(x) - f(c)} + f(c))(\overline{g(x) - g(c)} + g(c)) - f(c)g(c)| \\
&= |(\overline{f(x) - f(c)})(g(x) - g(c)) + (\overline{f(x) - f(c)})g(c) + f(c)(\overline{g(x) - g(c)})| \\
&\leq |f(x) - f(c)|\ |g(x) - g(c)| + |f(x) - f(c)|\ |g(c)| + |f(c)|\ |g(x) - g(c)| \\
&< \epsilon_1 \cdot \epsilon_1 + \epsilon_1 |g(c)| + \epsilon_1 |f(c)| \\
&< \epsilon_1(1 + |g(c)| + |f(c)|) \quad , \text{ (since } \epsilon_1 < 1) \\
&< \epsilon.
\end{aligned}$$

It follows that

$$\lim_{x \to c} f(x)g(x) = f(c)g(c)$$

and, hence, the product $f \cdot g$ is continuous at c.

Example 2.1.17 Show that the quotient f/g is continuous at c if f and g are continuous at c and $g(c) \neq 0$.

First of all, let us observe that the function $1/g$ is a composition of $g(x)$ and $1/x$ and hence $1/g$ is continuous at c by virtue of the arguments in Examples 14 and 15. By the argument in Example 16, the product $f(1/g) = f/g$ is continuous at c, as required in Example 17.

Example 2.1.18 Show that a rational function of the form $p(x)/q(x)$ is continuous for all c such that $g(c) \neq 0$.

In Example 13, we showed that each polynomial function is continuous at every real number c. Therefore, $p(x)$ is continuous at every c and $q(x)$ is continuous at every c. By virtue of the argument in Example 17, the quotient $p(x)/q(x)$ is continuous for all c such that $q(c) \neq 0$.

Example 2.1.19 Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing c and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L.$$

Then, show that,

$$\lim_{x\to c} g(x) = L.$$

Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0,\ \delta_2 > 0$, and $\delta = \min\{\delta_1, \delta_2\}$ such that

$$|f(x) - L| < \frac{\epsilon}{2} \text{ whenever } 0 < |x-c| < \delta_1$$
$$|h(x) - L) < \frac{\epsilon}{2} \text{ whenever } 0 < |x-c| < \delta_2.$$

If $0 < |x-c| < \delta_1$, then $0 < |x-c| < \delta_1,\ 0 < |x-c| < \delta_2$ and, hence,

$$-\frac{\epsilon}{2} < f(x) - L < g(x) - L < h(x) - L < \frac{\epsilon}{2}.$$

It follows that

$$|g(x) - L| < \frac{\epsilon}{2} < \epsilon \text{ whenever } 0 < |x-c| < \delta,$$

and

$$\lim_{x\to c} g(x) = L.$$

Example 2.1.20 Show that $f(x) = |x|$ is continuous at 0.

We need to show that

$$\lim_{x\to 0} |x| = 0.$$

Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Then $|x - 0| < \epsilon$ implies that $|x| < \epsilon$ Hence,

$$\lim_{x\to 0} |x| = 0$$

Example 2.1.21 Show that

(i) $\lim_{\theta \to 0} \sin\theta = 0$ (ii) $\lim_{\theta \to 0} \cos\theta = 1$

(iii) $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ (iv) $\lim_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta} = 0$

graph

Part (i) By definition, the point $C(\cos\theta, \sin\theta)$, where θ is the length of the arc CD, lies on the unit circle. It is clear that the length $BC = \sin\theta$ is less than θ, the arclength of the arc CD, for small positive θ. Hence,

$$-\theta \le \sin\theta \le \theta$$

and

$$\lim_{\theta \to 0^+} \sin\theta = 0.$$

For small negative θ, we get

$$\theta \le \sin\theta \le -\theta$$

and

$$\lim_{\theta \to 0^-} \sin\theta = 0.$$

Therefore,

$$\lim_{\theta \to 0} \sin\theta = 0.$$

Part (ii) It is clear that the point B approaches D as θ tends to zero. Therefore,

$$\lim_{\theta \to 0} \cos\theta = 1.$$

Part (iii) Consider the inequality

Area of triangle $ABC \le$ Area of sector $ADC \le$ Area of triangle ADE

$$\frac{1}{2}\cos\theta\sin\theta \le \frac{1}{2}\,\theta \le \frac{1}{2}\,\frac{\sin\theta}{\cos\theta}.$$

Assume that θ is small but positive. Multiply each part of the inequality by $2/\sin\theta$ to get

$$\cos\theta \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}.$$

On taking limits and using the squeeze theorem, we get

$$\lim_{\theta\to 0^+} \frac{\theta}{\sin\theta} = 1.$$

By taking reciprocals, we get

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

Since

$$\frac{\sin(-\theta)}{-\theta} = \frac{\sin\theta}{\theta},$$

$$\lim_{\theta - 0^-} \frac{\sin\theta}{\theta} = 1.$$

Therefore,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Part (iv)

$$\begin{aligned}
\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{\theta(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta} \cdot \frac{1}{(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta} \\
&= 1 \cdot \frac{0}{2} \\
&= 0.
\end{aligned}$$

Example 2.1.22 Show that

(i) $\sin\theta$ and $\cos\theta$ are continuous for all real θ.

(ii) $\tan\theta$ and $\sec\theta$ are continuous for all $\theta \neq 2n\pi \pm \dfrac{\pi}{2}, n$ integer.

(iii) $\cot\theta$ and $\csc\theta$ are continuous for all $\theta \neq n\pi, n$ integer.

Part (i) First, we show that for all real c,

$$\lim_{\theta\to c} \sin\theta = \sin c \text{ or equivalently } \lim_{\theta\to c} |\sin\theta - \sin c| = 0.$$

We observe that

$$\begin{aligned} 0 \leq |\sin\theta - \sin c| &= \left|2\cos\frac{\theta+c}{2}\ \sin\frac{\theta-c}{2}\right| \\ &\leq \left|2\sin\frac{(\theta-c)}{2}\right| \\ &= |(\theta-c)|\ \left|\frac{\sin\frac{(\theta-c)}{2}}{\frac{(\theta-c)}{2}}\right| \end{aligned}$$

Therefore, by squeeze theorem,

$$0 \leq \lim_{\theta - c} |\sin\theta - \sin c| \leq 0\cdot 1 = 0.$$

It follows that for all real $c, \sin\theta$ is continuous at c.
Next, we show that

$$\lim_{x\to c} \cos x = \cos c \text{ or equivalently } \lim_{x\to c} |\cos x - \cos c| = 0.$$

We observe that

$$\begin{aligned} 0 \leq |\cos x - \cos c| &= \left|-2\sin\frac{x+c}{2}\ \sin\frac{(x-c)}{2}\right| \\ &\leq |\theta - c|\ \left|\frac{\sin\left(\frac{x-c}{2}\right)}{\left(\frac{x-c}{2}\right)}\right|\ ; \qquad \left(\left|\sin\ \frac{x+c}{2}\right| \leq 1\right) \end{aligned}$$

Therefore,

$$0 \leq \lim_{x\to c} |\cos x - \cos c| \leq 0\cdot 1 = 0$$

and $\cos x$ is continuous at c.

Part (ii) Since for all $\theta \neq 2n\pi \pm \frac{\pi}{2}, n$ integer,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \ \sec\theta = \frac{1}{\cos\theta}$$

it follows that $\tan\theta$ and $\sec\theta$ are continuous functions.

Part (iii) Both $\cot\theta$ and $\csc\theta$ are continuous as quotients of two continuous functions where the denominators are not zero for $n \neq n\pi, n$ integer.

Exercises 2.1 Evaluate each of the following limits.

1. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^3-1}$
2. $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{x}$
3. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\sin 7x}$
4. $\lim\limits_{x\to 2^+} \dfrac{1}{x^2-4}$
5. $\lim\limits_{x\to 2^-} \dfrac{1}{x^2-4}$
6. $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$
7. $\lim\limits_{x\to 2^+} \dfrac{x-2}{|x-2|}$
8. $\lim\limits_{x\to 2^-} \dfrac{x-2}{|x-2|}$
9. $\lim\limits_{x\to 2} \dfrac{x-2}{|x-2|}$
10. $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$
11. $\lim\limits_{x\to 3} \dfrac{x^2-9}{x+3}$
12. $\lim\limits_{x\to \frac{\pi}{2}} \tan x$
13. $\lim\limits_{x\to \frac{\pi}{2}^+} \tan x$
14. $\lim\limits_{x\to 0^-} \csc x$
15. $\lim\limits_{x\to 0^+} \csc x$
16. $\lim\limits_{x\to 0^+} \cot x$
17. $\lim\limits_{x\to 0^-} \cot x$
18. $\lim\limits_{x\to \frac{\pi}{2}^+} \sec x$
19. $\lim\limits_{x\to \frac{\pi}{2}} \sec x$
20. $\lim\limits_{x\to 0} \dfrac{\sin 2x + \sin 3x}{x}$
21. $\lim\limits_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4}$
22. $\lim\limits_{x\to 4^+} \dfrac{\sqrt{x}-2}{x-4}$

23 $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

24. $\lim\limits_{x\to 3} \dfrac{x^4-81}{x^2-9}$

Sketch the graph of each of the following functions. Determine all the discontinuities of these functions and classify them as (a) removable type, (b) finite jump type, (c) essential type, (d) oscillation type, or other types.

25. $f(x) = 2\dfrac{x-1}{|x-1|} - \dfrac{x-2}{|x-2|}$

26. $f(x) = \dfrac{x}{x^2-9}$

27. $f(x) = \begin{cases} 2x & \text{for } x \leq 0 \\ x^2+1 & \text{for } x > 0 \end{cases}$

28. $f(x) = \begin{cases} \sin x & \text{if } x \leq 0 \\ \sin\left(\frac{2}{x}\right) & \text{if } x > 0 \end{cases}$

29. $f(x) = \dfrac{x-1}{(x-2)(x-3)}$

30. $f(x) = \begin{cases} |x-1| & \text{if } x \leq 1 \\ |x-2| & \text{if } x > 1 \end{cases}$

Recall the unit step function $u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$

Sketch the graph of each of the following functions and determine the left hand limit and the right hand limit at each point of discontinuity of f and g.

31. $f(x) = 2u(x-3) - u(x-4)$

32. $f(x) = -2u(x-1) + 4u(x-5)$

33. $f(x) = u(x-1) + 2u(x+1) - 3u(x-2)$

34. $f(x) = \sin x \left[u\left(x + \frac{\pi}{2}\right) - u\left(x - \frac{\pi}{2}\right)\right]$

35. $g(x) = (\tan x)\left[u\left(x + \frac{\pi}{2}\right) - u\left(x - \frac{\pi}{2}\right)\right]$

36. $f(x) = [u(x) - u(x-\pi)]\cos x$

2.2 Linear Function Approximations

One simple application of limits is to approximate a function $f(x)$, in a small neighborhood of a point c, by a line. The approximating line is called the tangent line. We begin with a review of the equations of a line.

A vertical line has an equation of the form $x = c$. A vertical line has no slope. A horizontal line has an equation of the form $y = c$. A horizontal line has slope zero. A line that is neither horizontal nor vertical is called an oblique line.

Suppose that an oblique line passes through two points, say (x_1, y_1) and (x_2, y_2). Then the slope of this line is define as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

If (x, y) is any arbitrary point on the above oblique line, then

$$m = \frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2}.$$

By equating the two forms of the slope m we get an equation of the line:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

On multiplying through, we get the "two point" form of the equation of the line, namely,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}\,(x - x_1) \quad \text{or} \quad y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}\,(x - x_2).$$

Example 2.2.1 Find the equations of the lines passing through the following pairs of points:

(i) $(4, 2)$ and $(6, 2)$
(ii) $(1, 3)$ and $(1, 5)$
(iii) $(3, 4)$ and $(5, -2)$
(iv) $(0, 2)$ and $(4, 0)$.

Part (i) Since the y-coordinates of both points are the same, the line is horizontal and has the equation $y = 2$. This line has slope 0.

Part (ii) Since the x-coordinates of both points are equal, the line is vertical and has the equation $x = 1$.

Part (iii) The slope of the line is given by

$$m = \frac{-2 - 4}{5 - 3} = -3.$$

The equation of this line is

$$y - 4 = -3(x - 3) \quad \text{or} \quad y + 2 = -3(x - 5).$$

On solving for y, we get the equation of the line as

$$y = -3x + 13.$$

This line goes through the point $(0, 13)$. The number 13 is called the y-intercept. The above equation is called the *slope-intercept form* of the line.

Example 2.2.2 Determine the equations of the lines satisfying the given conditions:

(i) slope $= 3$, passes through $(2, 4)$

(ii) slope $= -2$, passes through $(1, -3)$

(iii) slope $= m$, passes through (x_1, y_1)

(iv) passes through $(3, 0)$ and $(0, 4)$

(v) passes through $(a, 0)$ and $(0, b)$

Part (i) If (x, y) is on the line, then we equate the slopes and simplify:

$$3 = \frac{y-4}{x-2} \quad \text{or} \quad y - 4 = 3(x-2).$$

Part (ii) If (x, y) is on the line, then we equate slopes and simplify:

$$-2 = \frac{y+3}{x-1} \quad \text{or} \quad y + 3 = -2(x-1).$$

Part (iii) On equating slopes and clearing fractions, we get

$$m = \frac{y-y_1}{x-x_1} \quad \text{or} \quad y - y_1 = m(x - x_1).$$

This form of the line is called the “point-slope” form of the line.

Part (iv) Using the two forms of the line we get

$$\frac{y-0}{x-3} = \frac{4-0}{0-3} \quad \text{or} \quad y = -\frac{4}{3}(x-3).$$

If we divide by 4 we get

$$\frac{x}{3} + \frac{y}{4} = 1.$$

The number 3 is called the x-intercept and the number 4 is called the y-intercept of the line. This form of the equation is called the "two-intercept" form of the line.

Part (v) As in Part (iv), the "two-intercept" form of the line has the equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

In order to approximate a function f at the point c, we first define the slope m of the line that is tangent to the graph of f at the point $(c, f(c))$.

graph

$$m = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Then the equation of the tangent line is

$$y - f(c) = m(x - c),$$

written in the point-slope form. The point $(c, f(c))$ is called the point of tangency. This tangent line is called the linear approximation of f about $x = c$.

Example 2.2.3 Find the equation of the line tangent to the graph of $f(x) = x^2$ at the point $(2, 4)$.

The slope m of the tangent line at $(3,9)$ is

$$\begin{aligned} m &= \lim_{x\to 3} \frac{x^2-9}{x-3} \\ &= \lim_{x\to 3} (x+3) \\ &= 6. \end{aligned}$$

The equation of the tangent line at $(3,9)$ is

$$y-9 = 6(x-3).$$

Example 2.2.4 Obtain the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at the point $(9,3)$.

The slope m of the tangent line is given by

$$\begin{aligned} m &= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \\ &= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} \\ &= \frac{1}{6}. \end{aligned}$$

The equation of the tangent line is

$$y-3 = \frac{1}{6}(x-9).$$

Example 2.2.5 Derive the equation of the line tangent to the graph of $f(x) = \sin x$ at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

The slope m of the tangent line is given by

$$\begin{aligned}
m &= \lim_{x\to\frac{\pi}{6}} \frac{\sin x - \sin\left(\frac{\pi}{6}\right)}{x-\frac{\pi}{6}} \\
&= \lim_{x\to\frac{\pi}{6}} \frac{2\cos\left(\frac{x+\pi/6}{2}\right)\sin\left(\frac{x-\pi/6}{2}\right)}{(x-\pi/6)} \\
&= \cos(\pi/6)\cdot \lim_{x\to\frac{\pi}{6}} \frac{\sin\left(\frac{x-\pi/6}{2}\right)}{\left(\frac{x-\pi/6}{2}\right)} \\
&= \cos(\pi/6) \\
&= \frac{\sqrt{3}}{2}.
\end{aligned}$$

The equation of the tangent line is

$$y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right).$$

Example 2.2.6 Derive the formulas for the slope and the equation of the line tangent to the graph of $f(x) = \sin x$ at $(c, \sin c)$.

As in Example 27, replacing $\pi/6$ by c, we get

$$\begin{aligned}
m &= \lim_{x\to c} \frac{\sin x - \sin c}{x-c} \\
&= \lim_{x\to c} \frac{2\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)}{x-c} \\
&= \lim_{x\to c} \cos\left(\frac{x+c}{2}\right)\cdot \lim_{x\to c} \frac{\sin\left(\frac{x-c}{2}\right)}{\left(\frac{x-c}{2}\right)} \\
&= \cos c.
\end{aligned}$$

Therefore the slope of the line tangent to the graph of $f(x) = \sin x$ at $(c, \sin c)$ is $\cos c$.

The equation of the tangent line is

$$y - \sin c = (\cos c)(x - c).$$

Example 2.2.7 Derive the formulas for the slope, m, and the equation of the line tangent to the graph of $f(x) = \cos x$ at $(c, \cos c)$. Then determine the slope and the equation of the tangent line at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.

As in Example 28, we replace the sine function with the cosine function,

$$\begin{aligned} m &= \lim_{x\to c} \frac{\cos x - \cos c}{x - c} \\ &= \lim_{x\to c} \frac{-2\sin\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)}{x - c} \\ &= \lim_{x\to c} \sin\left(\frac{x+c}{2}\right) \lim_{x\to c} \frac{\sin\left(\frac{x-c}{2}\right)}{\left(\frac{x-c}{2}\right)} \\ &= -\sin(c). \end{aligned}$$

The equation of the tangent line is

$$y - \cos c = -\sin c(x - c).$$

For $c = \frac{\pi}{3}$, slope $= -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and the equation of the tangent line

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right).$$

Example 2.2.8 Derive the formulas for the slope, m, and the equation of the line tangent to the graph of $f(x) = x^n$ at the point (c, c^n), where n is a natural number. Then get the slope and the equation of the tangent line for $c = 2, n = 4$.

By definition, the slope m is given by

$$m = \lim_{x\to c} \frac{x^n - c^n}{x - c}.$$

To compute this limit for the general natural number n, it is convenient to let $x = c + h$. Then

$$\begin{aligned} m &= \lim_{h\to 0} \frac{(c+h)^n - c^n}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\left(c^n + nc^{n-1}h + \frac{n(n-1)}{2!}\, c^{n-2}h^2 + \cdots + h^n\right) - c^n\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[nc^{n-1}h + \frac{n(n-1)}{2!}\, c^{n-2}h^2 + \cdots + h^n\right] \\ &= \lim_{h\to 0} \left[nc^{n-1} + \frac{n(n-1)}{2!}\, c^{n-2}h + \cdots + h^{n-1}\right] \\ &= nc^{n-1}. \end{aligned}$$

Therefore, the equation of the tangent line through (c, c^n) is

$$y - c^n = nc^{n-1}(x - c).$$

For $n = 4$ and $c = 2$, we find the slope, m, and equation for the tangent line to the graph of $f(x) = x^4$ at $c = 2$:

$$\begin{aligned} &m = 4c^3 = 32 \\ &y - 2^4 = 32(x-2) \quad \text{or} \quad y - 16 = 32(x-2). \end{aligned}$$

Definition 2.2.1 Suppose that a function f is defined on a closed interval $[a, b]$ and $a < c < b$. Then c is called a *critical point* of f if the slope of the line tangent to the graph of f at $(c, f(c))$ is zero or undefined. The *slope function* of f at c is defined by

$$\begin{aligned} \text{slope } (f(x), c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c}. \end{aligned}$$

Example 2.2.9 Determine the slope functions and critical points of the following functions:

(i) $f(x) = \sin x,\ 0 \le x \le 2\pi$

(ii) $f(x) = \cos x,\ 0 \le x \le 2\pi$

(iii) $f(x) = |x|,\ -1 \le x \le 1$

(iv) $f(x) = x^3 - 4x,\ -2 \le x \le 2$

Part (i) In Example 28, we derived the slope function formula for $\sin x$, namely

$$\text{slope } (\sin x, c) = \cos c.$$

Since $\cos c$ is defined for all c, the non-end point critical points on $[0, 2\pi]$ are $\pi/2$ and $3\pi/2$ where the cosine has a zero value. These critical points correspond to the maximum and minimum values of $\sin x$.

Part (ii) In Example 29, we derived the slope function formula for $\cos x$, namely

$$\text{slope } (\cos x, c) = -\sin c.$$

The critical points are obtained by solving the following equation for c:

$$\begin{aligned} -\sin c &= 0, \quad 0 \le c \le 2\pi \\ c &= 0, \pi, 2\pi. \end{aligned}$$

These values of c correspond to the maximum value of $\cos x$ at $c = 0$ and 2π, and the minimum value of $\cos x$ at $c = \pi$.

Part (iii)

$$\begin{aligned} \text{slope } (|x|, c) &= \lim_{x \to c} \frac{|x| - |c|}{x - c} \\ &= \lim_{x \to c} \frac{|x| - |c|}{x - c} \cdot \frac{|x| + |c|}{|x| + |c|} \\ &= \lim_{x \to c} \frac{x^2 - c^2}{(x - c)(|x| + |c|)} \\ &= \lim_{x \to c} \frac{x + c}{|x| + |c|} \\ &= \frac{2c}{2|c|} \\ &= \frac{c}{|c|} \\ &= \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \\ \text{undefined} & \text{if } c = 0 \end{cases} \end{aligned}$$

The only critical point is $c = 0$, where the slope function is undefined. This critical point corresponds to the minimum value of $|x|$ at $c = 0$. The slope function is undefined because the tangent line does not exist at $c = 0$. There is a sharp corner at $c = 0$.

Part (iv) The slope function for $f(x) = x^3 - 4x$ is obtained as follows:

$$\begin{aligned}
\text{slope } (f(x), c) &= \lim_{h\to 0} \frac{1}{h} \left[((c+h)^3 - 4(c+h)) - (c^3 - 4c)\right] \\
&= \lim_{h\to 0} \frac{1}{h} \left[c^3 + 3c^2h + 3ch^2 + h^3 - 4c - 4h - c^3 + 4c\right] \\
&= \lim_{h\to 0} \frac{1}{h} \left[3c^2h + 3ch^2 + h^3 - 4h\right] \\
&= \lim_{h\to 0} \left[3c^2 + 3ch + h^2 - 4\right] \\
&= 3c^2 - 4
\end{aligned}$$

graph

The critical points are obtained by solving the following equation for c:

$$\begin{aligned}
3c^2 - 4 &= 0 \\
c &= \pm \frac{2}{\sqrt{3}}
\end{aligned}$$

At $c = \frac{-2}{\sqrt{3}}$, f has a local maximum value of $\frac{16}{3\sqrt{3}}$ and at $c = \frac{2}{\sqrt{3}}$, f has a local minimum value of $\frac{-16}{3\sqrt{3}}$. The end point $(-2, 0)$ has a local end-point minimum and the end point $(2, 0)$ has a local end-point maximum.

Remark 7 The zeros and the critical points of a function are helpful in sketching the graph of a function.

Exercises 2.2

1. Express the equations of the lines satisfying the given information in the form $y = mx + b$.

   (a) Line passing through $(2, 4)$ and $(5, -2)$

   (b) Line passing through $(1, 1)$ and $(3, 4)$

   (c) Line with slope 3 which passes through $(2, 1)$

   (d) Line with slope 3 and y-intercept 4

   (e) Line with slope 2 and x-intercept 3

   (f) Line with x-intercept 2 and y-intercept 4.

2. Two oblique lines are parallel if they have the same slope. Two oblique lines are perpendicular if the product of their slopes is -1. Using this information, solve the following problems:

   (a) Find the equation of a line that is parallel to the line with equation $y = 3x - 2$ which passes through $(1, 4)$.

   (b) Solve problem (a) when "parallel" is changed to "perpendicular."

   (c) Find the equation of a line with y-intercept 4 which is parallel to $y = -3x + 1$.

   (d) Solve problem (c) when "parallel" is changed to "perpendicular."

   (e) Find the equation of a line that passes through $(1, 1)$ and is

      (i) parallel to the line with equation $2x - 3y = 6$.

      (ii) perpendicular to the line with equation $3x + 2y = 6$

3. For each of the following functions $f(x)$ and values c,

   (i) derive the slope function, slope $(f(x), c)$ for arbitrary c;

   (ii) determine the equations of the tangent line and normal line (perpendicular to tangent line) at the point $(c, f(c))$ for the given c;

   (iii) determine all of the critical points $(c, f(c))$.

      (a) $f(x) = x^2 - 2x, \quad c = 3$

      (b) $f(x) = x^3 \quad , \quad c = 1$

(c) $f(x) = \sin(2x), \quad c = \dfrac{\pi}{12}$

(d) $f(x) = \cos(3x), \quad c = \dfrac{\pi}{9}$

(e) $f(x) = x^4 - 4x^2, \quad c = -2, 0, 2, -\sqrt{2}, \sqrt{2}.$

2.3 Limits and Sequences

We begin with the definitions of sets, sequences, and the completeness property, and state some important results. If x is an element of a set S, we write $x \in S$, read "x is in S." If x is not an element of S, then we write $x \notin S$, read "x is not in S."

Definition 2.3.1 If A and B are two sets of real numbers, then we define

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

and

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}.$$

We read "$A \cap B$" as the "intersection of A and B." We read "$A \cup B$" as the "union of A and B." If $A \cap B$ is the empty set, $\emptyset$, then we write $A \cap B = \emptyset$.

Definition 2.3.2 Let A be a set of real numbers. Then a number m is said to be an *upper bound* of A if $x \leq m$ for all $x \in A$. The number m is said to be a *least upper bound* of A, written lub(A) if and only if,

(i) m is an upper bound of A, and,

(ii) if $q < m$, then there is some $x \in A$ such that $q < x \leq m$.

Definition 2.3.3 Let B be a set of real numbers. Then a number ℓ is said to be a *lower bound* of B if $\ell \leq y$ for each $y \in B$. This number ℓ is said to be the *greatest lower bound* of B, written, glb(b), if and only if,

(i) ℓ is a lower bound of B, and,

(ii) if $\ell < p$, then there is some element $y \in B$ such that $\ell \leq y < p$.

Definition 2.3.4 A real number p is said to be a *limit point* of a set S if and only if every open interval that contains p also contains an element q of S such that $q \neq p$.

Example 2.3.1 Suppose $A = [1, 10]$ and $B = [5, 15]$.

Then $A \cap B = [5, 10]$, $A \cup B = [1, 15]$, $\text{glb}(A) = 1$, $\text{lub}(A) = 10$, $\text{glb}(B) = 5$ and $\text{lub}(B) = 15$. Each element of A is a limit point of A and each element of B is a limit point of B.

Example 2.3.2 Let $S = \left\{ \dfrac{1}{n} : n \text{ is a natural number} \right\}$.

Then no element of S is a limit point of S. The number 0 is the only limit point of S. Also, $\text{glb}(S) = 0$ and $\text{lub}(S) = 1$.

Completeness Property: The *completeness property* of the set R of all real numbers states that if A is a non-empty set of real numbers and A has an upper bound, then A has a least upper bound which is a real number.

Theorem 2.3.1 *If B is a non-empty set of real numbers and B has a lower bound, then B has a greatest lower bound which is a real number.*

Proof. Let m denote a lower bound for B. Then $m \leq x$ for every $x \in B$. Let $A = \{-x : x \in B\}$. then $-x \leq -m$ for every $x \in B$. Hence, A is a non-empty set that has an upper bound $-m$. By the completeness property, A has a least upper bound $\text{lub}(A)$. Then, -$\text{lub}(A) = \text{glb}(B)$ and the proof is complete.

Theorem 2.3.2 *If x_1 and x_2 are real numbers such that $x_1 < x_2$, then $x_1 < \frac{1}{2}(x_1 + x_2) < x_2$.*

Proof. We observe that

$$x_1 \leq \frac{1}{2}(x_1 + x_2) < x_2 \leftrightarrow 2x_1 < x_1 + x_2 < 2x_2$$
$$\leftrightarrow x_1 < x_2 < x_2 + (x_2 - x_1).$$

This completes the proof.

Theorem 2.3.3 *Suppose that A is a non-empty set of real numbers and $m = \text{lub}(A)$. If $m \notin A$, then m is a limit point of A.*

Proof. Let an open interval (a, b) contain m. That is, $a < m < b$. By the definition of a least upper bound, **a** is not an upper bound for A. Therefore, there exists some element q of A such that $a < q < m < b$. Thus, every open interval (a, b) that contains m must contain a point of A other than m. It follows that m is a limit point of A.

Theorem 2.3.4 (Dedekind-Cut Property). *The set R of all real numbers is not the union of two non-empty sets A and B such that*

(i) if $x \in A$ and $y \in B$, then $x < y$,

(ii) A contains no limit point of B, and,

(iii) B contains no limit point of A.

Proof. Suppose that $R = A \cup B$ where A and B are non-empty sets that satisfy conditions (i), (ii) and (iii). Since A and B are non-empty, there exist real numbers a and b such that $a \in A$ and $b \in B$. By property (i), **a** is a lower bound for B and b is an upper bound for A. By the completeness property and theorem 2.3.1, A has a least upper bound, say m, and B has a greatest lower bound, say M. If $m \notin A$, then m is a limit point of A. Since B contains no limit point of A, $m \in A$. Similarly, $M \in B$. It follows that $m < M$ by condition (i). However, by Theorem 2.3.2,

$$m < \frac{1}{2}(m + M) < M.$$

The number $\frac{1}{2}(m + M)$ is neither in A nor in B. This is a contradiction, because $R = A \cup B$. This completes the proof.

Definition 2.3.5 An *empty set* is considered to be a *finite set*. A *non-empty set* S is said to be *finite* if there exists a natural number n and a one-to-one function that maps S onto the set $\{1, 2, 3, \ldots, n\}$. Then we say that S has n elements. If S is not a finite set, then S is said to be an *infinite set*. We say that an infinite set has an infinite number of elements. Two sets are said to have the same number of elements if there exists a one-to-one correspondence between them.

Example 2.3.3 Let $A = \{a, b, c\}, B = \{1, 2, 3\}, C = \{1, 2, 3, \dots\}$, and $D = \{0, 1, -1, 2, -2, \dots\}$.

In this example, A and B are finite sets and contain three elements each. The sets C and D are infinite sets and have the same number of elements. A one-to-one correspondence f between n, C and D can be defined as $f : C \to D$ such that

$$f(1) = 0,\ f(2n) = n \text{ and } f(2n+1) = -n \text{ for } n = 1, 2, 3, \dots .$$

Definition 2.3.6 A set that has the same number of elements as $C = \{1, 2, 3, \dots\}$ is said to be *countable*. An infinite set that is not countable is said to be *uncountable*.

Remark 8 The set of all rational numbers is *countable* but the set of all real numbers is *uncountable*.

Definition 2.3.7 A *sequence* is a function, say f, whose domain is the set of all natural numbers. It is customary to use the notation $f(n) = a_n, n = 1, 2, 3, \dots$. We express the sequence as a list without braces to avoid confusion with the set notation:

$$a_1, a_2, a_3, \dots , a_n, \dots \quad \text{or, simply,} \quad \{a_n\}_{n=1}^{\infty}.$$

The number a_n is called the nth term of the sequence. The sequence is said to *converge* to the limit **a** if for every $\epsilon > 0$, there exists some natural number, say N, such that $|a_m - a| < \epsilon$ for all $m \geq N$. We express this convergence by writing

$$\lim_{n \to \infty} a_n = a.$$

If a sequence does not converge to a limit, it is said to diverge or be divergent.

Example 2.3.4 For each natural number n, let

$$a_n = (-1)^n,\ b_n = 2^{-n},\ c_n = 2^n,\ d_n = \frac{(-1)^n}{n}.$$

The sequence $\{a_n\}$ does not converge because its terms oscillate between -1 and 1. The sequence $\{b_n\}$ converges to 0. The sequence $\{c_n\}$ diverges to ∞. The sequence $\{d_n\}$ converges to 0.

Definition 2.3.8 A sequence $\{a_n\}_{n=1}^{\infty}$ diverges to ∞ if, for every natural number N, there exists some m such that

$$a_{m+j} \geq N \text{ for all } j = 1, 2, 3, \cdots .$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is said to diverge to $-\infty$ if, for every natural number N, there exists some m such that

$$a_{m+j} \leq -N \text{ , for all } j = 1, 2, 3, \ldots .$$

Theorem 2.3.5 *If p is a limit point of a non-empty set A, then every open interval that contains p must contain an infinite subset of A.*

Proof. Let some open interval (a, b) contain p. Suppose that there are only two finite subsets $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$ of distinct elements of A such that

$$a < a_1 < a_2 < \cdots < a_n < p < b_m < b_{m-1} < \cdots < b_1 < b.$$

Then the open interval (a_n, b_m) contains p but no other points of A distinct from p. Hence p is not a limit point of A. The contradiction proves the theorem.

Theorem 2.3.6 *If p is a limit point of a non-empty set A, then there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points p_n of A, that converges to p.*

Proof. Let $a_1 = p - \dfrac{1}{2}$, $b_1 = p + \dfrac{1}{2}$. Choose a point p_1 of A such that $p_1 \neq p$ and $a_1 < p_1 < p < b_1$ or $a_1 < p < p_1 < b_1$. If $a_1 < p_1 < p < b_1$, then define $a_2 = \max\left\{p_1, p - \dfrac{1}{2^2}\right\}$ and $b_2 = p + \dfrac{1}{2^2}$. Otherwise, define $a_2 = p - \dfrac{1}{2^2}$ and $b_2 = \min\left\{p_1, p + \dfrac{1}{2^2}\right\}$. Then the open interval (a_2, b_2) contains p but not p_1 and $b_2 - a_2 \leq \dfrac{1}{2}$. We repeat this process indefinitely to select the sequence $\{p_n\}$, of distinct points p_n of A, that converges to p. The fact that $\{p_n\}$ is an infinite sequence is guaranteed by Theorem 2.3.5. This completes the proof.

Theorem 2.3.7 *Every bounded infinite set A has at least one limit point p and there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points of A, that converges to p.*

Proof. We will show that A has a limit point. Since A is bounded, there exists an open interval (a, b) that contains all points of A. Then either $\left(a, \frac{1}{2}(a+b)\right)$ contains an infinite subset of A or $\left(\frac{1}{2}(a+b), b\right)$ contains an infinite subset of A. Pick one of the two intervals that contains an infinite subset of A. Let this interval be denoted (a_1, b_1). We continue this process repeatedly to get an open interval (a_n, b_n) that contains an infinite subset of A and $|b_n - a_n| = \dfrac{|b-a|}{2^n}$. Then the lub of the set $\{a_n, a_2, \dots\}$ and glb of the set $\{b_1, b_2, \dots\}$ are equal to some real number p. It follows that p is a limit point of A. By Theorem 2.3.6, there exists a sequence $\{p_n\}$, of distinct points of A, that converges to p. This completes the proof.

Definition 2.3.9 A set is said to be a *closed set* if it contains all of its limit points. The complement of a closed set is said to be an *open set.* (Recall that the complement of A is $\{x \in R : x \notin A\}$.)

Theorem 2.3.8 *The interval $[a, b]$ is a closed and bounded set. Its complement $(-\infty, a) \cup (b, \infty)$ is an open set.*

Proof. Let $p \in (-\infty, a) \cup (b, \infty)$. Then $-\infty < p < a$ or $b < p < \infty$. The intervals $\left(p - \frac{1}{2}, \frac{1}{2}(a+p)\right)$ or $\left(\frac{1}{2}(b+p), p + \frac{1}{2}\right)$ contain no limit point of $[a, b]$. Thus $[a, b]$ must contain its limit points, because they are not in the complement.

Theorem 2.3.9 *If a non-empty set A has no upper bound, then there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points of A, that diverges to ∞. Furthermore, every subsequence of $\{p_n\}_{n=1}^{\infty}$ diverges to ∞*

Proof. Since 1 is not an upper bound of A, there exists an element p_1 of A such that $1 < p_1$. Let $a_1 = \max\{2, p_1\}$. Choose a point, say p_2, of A such that $a_1 < p_2$. By repeating this process indefinitely, we get the sequence $\{p_n\}$ such that $p_n > n$ and $p_1 < p_2 < p_3 < \dots$. Clearly, the sequence $\{p_n\}_{n=1}^{\infty}$ diverges to ∞. It is easy to see that every subsequence of $\{p_n\}_{n=1}^{\infty}$ also diverges to ∞.

Theorem 2.3.10 *If a non-empty set B has no lower bound, then there exists a sequence $\{q_n\}_{n=1}^{\infty}$, of distinct points of B, that diverges to $-\infty$. Furthermore, every subsequence of $\{q_n\}_{n=1}^{\infty}$ diverges to $-\infty$.*

Proof. Let $A = \{-x : x \in B\}$. Then A has no upper bound. By Theorem 2.3.9, there exists a sequence $\{p_n\}_{n=1}^{\infty}$, of distinct points of A, that diverges to ∞. Let $q_n = -p_n$. Then $\{q_n\}_{n=1}^{\infty}$ is a sequence that meets the requirements of the Theorem 2.3.10. Also, every subsequence of $\{q_n\}_{n=1}^{\infty}$ diverges to $-\infty$.

Theorem 2.3.11 *Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points of a closed set S that converges to a point p of S. If f is a function that is continuous on S, then the sequence $\{f(p_n)\}_{n=1}^{\infty}$ converges to $f(p)$. That is, continuous functions preserve convergence of sequences on closed sets.*

Proof. Let $\epsilon > 0$ be given. Since f is continuous at p, there exists a $\delta > 0$ such that

$$|f(x) - f(p)| < \epsilon \quad \text{whenever} \quad |x - p| < \delta, \quad \text{and } x \in S.$$

The open interval $(p - \delta, p + \delta)$ contains the limit point p of S. The sequence $\{p_n\}_{n=1}^{\infty}$ converges to p. There exists some natural numbers N such that for all $n \geq N$,

$$p - \delta < p_n < p + \delta.$$

Then

$$|f(p_n) - f(p)| < \epsilon \quad \text{whenever } n \geq N.$$

By definition, $\{f(p_n)\}_{n=1}^{\infty}$ converges to $f(p)$. We write this statement in the following notation:

$$\lim_{n\to\infty} f(p_n) = f\left(\lim_{n\to\infty} p_n\right).$$

That is, continuous functions allow the interchange of taking the limit and applying the function. This completes the proof of the theorem.

Corollary 1 *If S is a closed and bounded interval $[a, b]$, then Theorem 2.3.11 is valid for $[a, b]$.*

Theorem 2.3.12 *Let a function f be defined and continuous on a closed and bounded set S. Let $R_f = \{f(x) : x \in S\}$. Then R_f is bounded.*

Proof. Suppose that R_f has no upper bound. Then there exists a sequence $\{f(x_n)\}_{n=1}^{\infty}$, of distinct points of R_f, that diverges to ∞. The set $A = \{x_1, x_2, \dots\}$ is an infinite subset of S. By Theorem 2.3.7, the set A has some limit point, say p. Since S is closed, $p \in S$. There exists a sequence

$\{p_n\}_{n=1}^{\infty}$, of distinct points of A that converges to p. By the continuity of f, $\{f(p_n)\}_{n=1}^{\infty}$ converges to $f(p)$. Without loss of generality, we may assume that $\{f(p_n)\}_{n=1}^{\infty}$ is a subsequence of $\{f(x_n)\}_{n=1}^{\infty}$. Hence $\{f(p_n)\}_{n=1}^{\infty}$ diverges to ∞, and $f(p) = \infty$. This is a contradiction, because $f(p)$ is a real number. This completes the proof of the theorem.

Theorem 2.3.13 *Let a function f be defined and continuous on a closed and bounded set S. Let $R_f = \{f(x) : x \in S\}$. Then R_f is a closed set.*

Proof. Let q be a limit point of R_f. Then there exists a sequence $\{f(x_n)\}_{n=1}^{\infty}$, of distinct points of R_f, that converges to q. As in Theorem 2.3.12, the set $A = \{x_1, x_2, \dots\}$ has a limit point $p, p \in S$, and there exists a subsequence $\{p_n\}_{n=1}^{\infty}$, of $\{x_n\}_{n=1}^{\infty}$ that converges to p. Since f is defined and continuous on S,

$$q = \lim_{n\to\infty} f(p_n) = f\left(\lim_{n\to\infty} p_n\right) = f(p).$$

Therefore, $q \in R_f$ and R_f is a closed set. This completes the proof of the theorem.

Theorem 2.3.14 *Let a function f be defined and continuous on a closed and bounded set S. Then there exist two numbers c_1 and c_2 in S such that for all $x \in S$,*

$$f(c_1) \le f(x) \le f(c_2).$$

Proof. By Theorems 2.3.12 and 2.3.13, the range, R_f, of f is a closed and bounded set. Let

$$m = \text{glb}(R_f) \text{ and } M = \text{lub}(R_f).$$

Since R_f is a closed set, m and M are in R_f. Hence, there exist two numbers, say c_1 and c_2, in S such that

$$m = f(c_1) \text{ and } M = f(c_2).$$

This completes the proof of the theorem.

Definition 2.3.10 A set S of real numbers is said to be compact, if and only if S is closed and bounded.

Theorem 2.3.15 *A continuous function maps compact subsets of its domain onto compact subsets of its range.*

Proof. Theorems 2.3.13 and 2.3.14 together prove Theorem 2.1.15.

Definition 2.3.11 Suppose that a function f is defined and continuous on a compact set S. A number m is said to be an *absolute minimum* of f on S if $m \leq f(x)$ for all $x \in S$ and $m = f(c)$ for some c in S.

A number M is said to be an *absolute maximum* of f on S if $M \geq f(x)$ for all $x \in S$ and $M = f(d)$ for some d in S.

Theorem 2.3.16 *Suppose that a function f is continuous on a compact set S. Then there exist two points c_1 and c_2 in S such that $f(c_1)$ is the absolute minimum and $f(c_2)$ is the absolute maximum of f on S.*

Proof. Theorem 2.3.14 proves Theorem 2.3.16.

Exercises 2.3

1. Find lub(A), glb(A) and determine all of the limit points of A.

   (a) $A = \{x : 1 \leq x^2 \leq 2\}$
   (b) $A = \{x : x \sin(1/x),\ x > 0\}$
   (c) $A = \{x^{2/3} : -8 < x < 8\}$
   (d) $A = \{x : 2 < x^3 < 5\}$
   (e) $A = \{x : x \text{ is a rational number and } 2 < x^3 < 5\}$

2. Determine whether or not the following sequences converge. Find the limit of the convergent sequences.

   (a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$
   (b) $\left\{\dfrac{n}{n^2}\right\}_{n=1}^{\infty}$
   (c) $\left\{(-1)^n \dfrac{n}{3n+1}\right\}_{n=1}^{\infty}$
   (d) $\left\{\dfrac{n^2}{n+1}\right\}_{n=1}^{\infty}$
   (e) $\{1 + (-1)^n\}_{n=1}^{\infty}$

3. Show that the Dedekind-Cut Property is equivalent to the completeness property.

4. Show that a convergent sequence cannot have more than one limit point.

5. Show that the following principle of mathematical induction is valid: If $1 \in S$, and $k+1 \in S$ whenever $k \in S$, then S contains the set of all natural numbers. (Hint: Let $A = \{n : n \notin S\}$. A is bounded from below by 2. Let $m = \text{glb}(A)$. Then $k = m - 1 \in S$ but $k + 1 = m \notin S$. This is a contradiction.)

6. Prove that every rational number is a limit point of the set of all rational numbers.

7. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

   (i) $\{a_n\}_{n=1}^{\infty}$ is said to be *increasing* if $a_n < a_{n+1}$, for all n.

   (ii) $\{a_n\}_{n=1}^{\infty}$ is said to be *non-decreasing* if $a_n \leq a_{n+1}$ for all n.

   (iii) $\{a_n\}_{n=1}^{\infty}$ is said to be *non-increasing* if $a_n \geq a_{n+1}$ for all n.

   (iv) $\{a_n\}_{n=1}^{\infty}$ is said to be *decreasing* if $a_n > a_{n+1}$ for all n.

   (v) $\{a_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is increasing, non-decreasing, non-increasing or decreasing.

   (a) Determine which sequences in Exercise 2 are monotone.

   (b) Show that every bounded monotone sequence converges to some point.

   (c) A sequence $\{b_m\}_{m=1}^{\infty}$ is said to be a *subsequence* of the $\{a_n\}_{n=1}^{\infty}$ if and only if every b_m is equal to some a_n, and if

   $$b_{m_1} = a_{n_1} \quad \text{and} \quad b_{m_2} = a_{n_2} \quad \text{and} \quad n_1 < n_2, \text{ then } m_1 < m_2.$$

   That is, a subsequence preserves the order of the parent sequence. Show that if $\{a_n\}_{n=1}^{\infty}$ converges to p, then every subsequence of $\{a_n\}_{n=1}^{\infty}$ also converges to p

   (d) Show that a divergent sequence may contain one or more convergent sequences.

(e) In problems 2(c) and 2(e), find two convergent subsequences of each. Do the parent sequences also converge?

8. (Cauchy Criterion) A sequence $\{a_n\}_{n=1}^{\infty}$ is said to satisfy a *Cauchy Criterion*, or be a *Cauchy sequence*, if and only if for every $\epsilon > 0$, there exists some natural number N such that $(a_n - a_m) < \epsilon$ whenever $n \geq N$ and $m \geq N$. Show that a sequence $\{a_n\}_{n=1}^{\infty}$ converges if and only if it is a Cauchy sequence. (Hint: (i) If $\{a_n\}$ converges to p, then for every $\epsilon > 0$ there exists some N such that if $n \geq N$, then $|a_n - p| < \epsilon/2$. If $m \geq N$ and $n \geq N$, then

$$\begin{aligned} |a_n - a_m| &= |(a_n - p) + (p - a_m)| \\ &\leq |a_n - p| + |a_m - p| \qquad \text{(why?)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So, if $\{a_n\}$ converges, then it is Cauchy.

(ii) Suppose $\{a_n\}$ is Cauchy. Let $\epsilon > 0$. Then there exists $N > 0$ such that

$$|a_n - a_m| < \epsilon \quad \text{whenever} \quad n \geq N \text{ and } m \geq N.$$

In particular,

$$|a_n - a_N| < \epsilon \quad \text{whenever} \quad n \geq N.$$

Argue that the sequence $\{a_n\}$ is bounded. Unless an element is repeated infinitely many times, the set consisting of elements of the sequence has a limit point. Either way, it has a convergent subsequence that converges, say to p. Then show that the Cauchy Criterion forces the parent sequence $\{a_n\}$ to converge to p also.)

9. Show that the set of all rational numbers is countable. (Hint: First show that the positive rationals are countable. List them in reduced form without repeating according to denominators, as follows:

$$\begin{aligned} &\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \cdots. \\ &\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{2}{2}, \cdots. \\ &\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{10}{3}, \cdots. \end{aligned}$$

Count them as shown, one-by-one. That is, list them as follows:

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, 2, 3, \frac{5}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}.$$

Next, insert the negative rational right after its absolute value, as follows:

$$\left\{0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \cdots\right\}.$$

Now assign the even natural numbers to the positive rationals and the odd natural numbers to the remaining rationals.)

10. A non-empty set S has the property that if $x \in S$, then there is some open interval (a, b) such that $x \in (a, b) \subset S$. Show that the complement of S is closed and hence S is open.

11. Consider the sequence $\left\{a_n = \dfrac{\pi}{2} + \dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$. Determine the convergent or divergent properties of the following sequences:

    (a) $\{\sin(a_n)\}_{n=1}^{\infty}$
    (b) $\{\cos(a_n)\}_{n=1}^{\infty}$
    (c) $\{\tan(a_n)\}_{n=1}^{\infty}$
    (d) $\{\cot(a_n)\}_{n=1}^{\infty}$
    (e) $\{\sec(a_n)\}_{n=1}^{\infty}$
    (f) $\{\csc(a_n)\}_{n=1}^{\infty}$

12. Let

    (a) $f(x) = x^2, -2 \leq x \leq 2$
    (b) $g(x) = x^3, -2 \leq x \leq 2$
    (c) $h(x) = \sqrt{x}, 0 \leq x \leq 4$
    (d) $p(x) = x^{1/3}, -8 \leq x \leq 8$

    Find the absolute maximum and absolute minimum of each of the functions f, g, h, and p. Determine the points at which the absolute maximum and absolute minimum are reached.

13. A function f is said to have a *fixed point* p if $f(p) = p$. Determine all of the fixed points of the functions f, g, h, and p in Exercise 12.

14. Determine the range of each of the functions in Exercise 12, and show that it is a closed and bounded set.

2.4 Properties of Continuous Functions

We recall that if two functions f and g are defined and continuous on a common domain D, then $f + g,\ f - g,\ af + bg,\ g \cdot f$ are all continuous on D, for all real numbers a and b. Also, the quotient f/g is continuous for all x in D where $g(x) \neq 0$. In section 2.3 we proved the following:

(i) Continuous functions preserve convergence of sequences.

(ii) Continuous functions map compact sets onto compact sets.

(iii) If a function f is continuous on a closed and bounded interval $[a, b]$, then $\{f(x) : x \in [a, b]\} \subseteq [m, M]$, where m and M are absolute minimum and absolute maximum of f, on $[a, b]$, respectively.

Theorem 2.4.1 *Suppose that a function f is defined and continuous on some open interval (a, b) and $a < c < b$.*

(i) If $f(c) > 0$, then there exists some $\delta > 0$ such that $f(x) > 0$ whenever $c - \delta < x < c + \delta$.

(ii) If $f(c) < 0$, then there exists some $\delta > 0$ such that $f(x) < 0$ whenever $c - \delta < x < c + \delta$.

Proof. Let $\epsilon = \dfrac{1}{2}\ |f(c)|$. For both cases (i) and (ii), $\epsilon > 0$. Since f is continuous at c and $\epsilon > 0$, there exists some $\delta > 0$ such that $a < (c - \delta) < c < (c + \delta) < b$ and

$$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta.$$

We observe that

$$\begin{aligned}|f(x)-f(c)|<\epsilon \leftrightarrow |f(x)-f(c)| &< \frac{1}{2}\,|f(c)|\\ &\leftrightarrow -\frac{1}{2}\,|f(c)| < f(x)-f(c) < \frac{1}{2}\,|f(c)|\\ &\leftrightarrow f(c)-\frac{1}{2}\,|f(c)| < f(x) < f(c)+\frac{1}{2}\,|f(c)|.\end{aligned}$$

We note also that the numbers $f(c)-\frac{1}{2}\,|f(c)|$ and $f(c)+\frac{1}{2}\,|f(c)|$ have the same sign as $f(c)$. Therefore, for all x such that $|x-c|<\delta$, we have $f(x)>0$ in part (i) and $f(x)<0$ in part (ii) as required. This completes the proof.

Theorem 2.4.2 *Suppose that a function f is defined and continuous on some closed and bounded interval $[a,b]$ such that either*

(i) $f(a)<0<f(b)$ *or* *(ii)* $f(b)<0<f(a)$.

Then there exists some c such that $a<c<b$ and $f(c)=0$.

Proof. *Part (i)* Let A $\{x : x \in [a,b] \text{ and } f(x)<0\}$. Then A is non-empty because it contains a. Since A is a subset of $[a,b]$, A is bounded. Let $c_1 = \text{lub}(A)$. We claim that $f(c_1)=0$. Suppose $f(c_1)\neq 0$. Then $f(c_1)>0$ or $f(c_1)<0$. By Theorem 2.4.1, there exists $\delta>0$ such that $f(x)$ has the same sign as $f(c_1)$ for all x such that $c_1-\delta<x<c_1+\delta$.

If $f(c_1)<0$, then $f(x)<0$ for all x such that $c_1<x<c_1+\delta$ and hence $c_1\neq \text{lub}(A)$. If $f(c_1)>0$, then $f(x)>0$ for all x such that $c_1-\delta<x<c_1$ and hence $c_1\neq\text{lub}(A)$. This contradiction proves that $f(c_1)=0$.

Part (ii) is proved by a similar argument.

Example 2.4.1 Show that Theorem 2.4.2 guarantees the validity of the following method of bisection for finding zeros of a continuous function f:

Bisection Method: We wish to solve $f(x)=0$ for x.

Step 1. Locate two points such that $f(a)f(b)<0$.

Step 2. Determine the sign of $f\left(\frac{1}{2}(a+b)\right)$.

(i) If $f\left(\frac{1}{2}(a+b)\right) = 0$, stop the procedure; $\frac{1}{2}(a+b)$ is a zero of f.

(ii) If $f\left(\frac{1}{2}(a+b)\right) \cdot f(a) < 0$, then let $a_1 = a, b_1 = \frac{1}{2}(a+b)$.

(iii) If $f\left(\frac{1}{2}(a+b)\right) \cdot f(b) < 0$, then let $a_1 = \frac{1}{2}(a+b),\ b_1 = b$.

Then $f(a_1) \cdot f(b_1) < 0$, and $|b_1 - a_1| = \frac{1}{2}(b-a)$.

Step 3. Repeat Step 2 and continue the loop between Step 2 and Step 3 until

$$|b_n - a_n|/2^n < \text{ Tolerance Error.}$$

Then stop.

This method is slow but it approximates the number c guaranteed by Theorem 2.4.2. This method is used to get close enough to the zero. The switchover to the faster Newton's Method that will be discussed in the next section.

Theorem 2.4.3 (Intermediate Value Theorem). *Suppose that a function is defined and continuous on a closed and bounded interval $[a, b]$. Suppose further that there exists some real number k such that either (i) $f(a) < k < f(b)$ or (ii) $f(b) < k < f(a)$. Then there exists some c such that $a < c < b$ and $f(c) = k$.*

Proof. Let $g(x) = f(x) - k$. Then g is continuous on $[a, b]$ and either (i) $g(a) < 0 < g(b)$ or (ii) $g(b) < 0 < g(a)$. By Theorem 2.4.2, there exists some c such that $a < c < b$ and $g(c) = 0$. Then

$$0 = g(c) = f(c) - k$$

and

$$f(c) = k$$

as required. This completes the proof.

Theorem 2.4.4 *Suppose that a function f is defined and continuous on a closed and bounded interval $[a, b]$. Then there exist real numbers m and M such that*

$$[m, M] = \{f(x) : a \leq x \leq b\}.$$

That is, a continuous function f maps a closed and bounded interval $[a, b]$ onto a closed and bounded interval $[m, M]$.

Proof. By Theorem 2.3.14, there exist two numbers c_1 and c_2 in $[a, b]$ such that for all $x \in [a, b]$,

$$m = f(c_1) \leq f(x) \leq f(c_2) = M.$$

By the Intermediate Value Theorem (2.4.3), every real value between m and M is in the range of f contained in the interval with end points c_1 and c_2. Therefore,

$$[m, M] = \{f(x) : a \leq x \leq b\}.$$

Recall that m = absolute minimum and M = absolute maximum of f on $[a, b]$. This completes the proof of the theorem.

Theorem 2.4.5 *Suppose that a function f is continuous on an interval $[a, b]$ and f has an inverse on $[a, b]$. Then f is either strictly increasing on $[a, b]$ or strictly decreasing on $[a, b]$.*

Proof. Since f has in inverse on $[a, b]$, f is a one-to-one function on $[a, b]$. So, $f(a) \neq f(b)$. Suppose that $f(a) < f(b)$. Let

$$A = \{x : f \text{ is strictly increasing on } [a, x] \text{ and } a \leq x \leq b\}.$$

Let c be the least upper bound of A. If $c = b$, then f is strictly increasing on $[a, b]$ and the proof is complete. If $c = a$, then there exists some d such that $a < d < b$ and $f(d) < f(a) < f(b)$. By the intermediate value theorem there must exist some x such that $d < x < b$ and $f(x) = f(a)$. This contradicts the fact that f is one-to-one. Then $a < c < b$ and there exists some d such that $c < d < b$ and $f(a) < f(d) < f(c)$. By the intermediate value theorem there exists some x such that $a < x < c$ and $f(x) = f(c)$ and f is not one-to-one. It follows that c must equal b and f is strictly increasing on $[a, b]$. Similarly, if $f(a) > f(b)$, f will be strictly decreasing on $[a, b]$. This completes the proof of the theorem.

Theorem 2.4.6 *Suppose that a function f is continuous on $[a, b]$ and f is one-to-one on $[a, b]$. Then the inverse of f exists and is continuous on $J = \{f(x) : a \leq x \leq b\}$.*

Proof. By Theorem 2.4.4, $J = [m, M]$ where m and M are the absolute minimum and the absolute maximum of f on $[a, b]$. Also, there exist numbers c_1 and c_2 on $[a, b]$ such that $f(c_1) = m$ and $f(c_2) = M$. Since f is either strictly increasing or strictly decreasing on $[a, b]$, either $a = c_1$ and $b = c_2$ or $a = c_2$ and $b = c_1$. Consider the case where f is strictly increasing and $a = c_1, b = c_2$. Let $m < d < M$ and $d = f(c)$. Then $a < c < b$. We show that f^{-1} is continuous at d. Let $\epsilon > 0$ be such that $a < c - \epsilon < c < c + \epsilon 2b$. Let $d_1 = f(c - \epsilon), d_2 = f(c + \epsilon)$. Since f is strictly increasing, $d_1 < d < d_2$. Let $\delta = \min(d - d_1, d_2 - d)$. It follows that if $0 < |y - d| < \delta$, then $|f^{-1}(y) - f^{-1}(d)| < \epsilon$ and f^{-1} is continuous at d. Similarly, we can prove the one-sided continuity of f^{-1} at m and M. A similar argument will prove the continuity of f^{-1} if f is strictly decreasing on $[a, b]$.

Theorem 2.4.7 *Suppose that a function f is continuous on an interval I and f is one-to-one on I. Then the inverse of f exists and is continuous on I.*

Proof. Let $J = \{f(x) : x \text{ is in } I\}$. By the intermediate value theorem J is also an interval. Let d be an interior point of J. Then there exists a closed interval $[m, M]$ contained in I and $m < d < M$. Let $c_1 = f^{-1}(m)$, $c_2 = f^{-1}(b)$, $a = \min\{c_1, c_2\}$ Since the theorem is valid on $[a, b]$, f^{-1} is continuous at d. The end points can be treated in a similar way. This completes the proof of the theorem. (See the proof of Theorem 2.4.6).

Theorem 2.4.8 (Fixed Point Theorem). *Let f satisfy the conditions of Theorem 2.4.4. Suppose further that $a \leq m \leq M \leq b$, where m and M are the absolute minimum and absolute maximum, respectively, of f on $[a, b]$. Then there exists some $p \in [a, b]$ such that $f(p) = p$. That is, f has a fixed point p on $[a, b]$.*

Proof. If $f(a) = a$, then a is a fixed point. If $f(b) = b$, then b is a fixed point. Suppose that neither a nor b is a fixed point of f. Then we define

$$g(x) = f(x) - x$$

for all $x \in [a, b]$.

We observe that $g(b) < 0 < g(a)$. By the Intermediate Value Theorem (2.4.3) there exists some p such that $a < p < b$ and $g(p) = 0$. Then

$$0 = g(p) = f(p) - p$$

and hence,

$$f(p) = p$$

and p is a fixed point of f on $[a, b]$. This completes the proof.

Remark 9 The Fixed Point Theorem (2.4.5) is the basis of the fixed point iteration methods that are used to locate zeros of continuous functions. We illustrate this concept by using Newton's Method as an example.

Example 2.4.2 Consider $f(x) = x^3 + 4x - 10$.

Since $f(1) = -5$ and $f(2) = 6$, by the Intermediate Value Theorem (2.4.3) there is some c such that $1 < c < 2$ and $f(c) = 0$. We construct a function g whose fixed points agree with the zeros of f. In Newton's Method we used the following general formula:

$$g(x) = x - \frac{f(x)}{\text{slope}(f(x), x)}.$$

Note that if $f(x) = 0$, then $g(x) = x$, provided slope $(f(x), x) \neq 0$. We first compute

$$\begin{aligned}
\text{Slope}(f(x), x) &= \lim_{h \to 0} \frac{1}{h} [f(x+h) - f(x)] \\
&= \lim_{h \to 0} \frac{1}{h} [\{(x+h)^3 + 4(x+h) - 10\} - \{x^3 + 4x - 10\}] \\
&= \lim_{h \to 0} \frac{1}{h} [3x^2 h + 3xh^2 + h^3 + 4h] \\
&= \lim_{h \to 0} [3x^2 + 3xh + h^2 + 4] \\
&= 3x^2 + 4.
\end{aligned}$$

We note that $3x^2 + 4$ is never zero. So, Newton's Method is defined.

The fixed point iteration is defined by the equation

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{\text{slope}(f(x), x_n)}$$

or

$$\boxed{x_{n+1} = x_n - \frac{x_n^3 + 4x_n - 10}{3x_n^2 + 4}.}$$

Geometrically, we draw a tangent line at the point $(x_n, f(x_n))$ and label the x-coordinate of its point of intersection with the x-axis as x_{n+1}.

graph

Tangent line: $y - f(x_n) = m(x - x_n)$

$$0 - f(x_n) = m(x_{n+1} - x_n)$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{m}} \quad ,$$

where $m = \text{slope } (f(x), x_n) = 3x_n^2 + 4$.

To begin the iteration we required a guess x_0. This guess is generally obtained by using a few steps of the Bisection Method described in Example 36. Let $x_0 = 1.5$. Next, we need a stopping rule. Let us say that we will stop when a few digits of x_n do not change anymore. Let us stop when

$$|x_{n+1} - x_n| < 10^{-4}.$$

We will leave the computation of $x_1, x_2, x_3, \ldots$ as an exercise.

Remark 10 Newton's Method is fast and quite robust as long as the initial guess is chosen close enough to the intended zeros.

Example 2.4.3 Consider the same equation ($x^3 + 4x - 10 = 0$) as in the preceding example.

We solve for x in some way, such as,

$$x = \left(\frac{10}{4 + x}\right)^{1/2} = g(x).$$

In this case the new equation is good enough for positive roots. We then define

$$x_{n+1} = g(x_n), x_0 = 1.5$$

and stop when

$$|x_{n+1} - x_n| < 10^{-4}.$$

We leave the computations of $x_1, x_2, x_3 \dots$ as an exercise. Try to compare the number of iterations needed to get the same accuracy as Newton's Method in the previous example.

Exercises 2.4

1. Perform the required iterations in the last two examples to approximate the roots of the equation $x^3 + 4x - 10 = 0$.

2. Let $f(x) = x - \cos x$. Then slope $(f(x), x) = 1 + \sin x > 0$ on $\left[0, \frac{\pi}{2}\right]$. Approximate the zeros of $f(x)$ on $\left[0, \frac{\pi}{2}\right]$ by Newton's Method:

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}\ , x_0 = 0.8$$

and stop when

$$|x_{n+1} - x_n| < 10^{-4}.$$

3. Let $f(x) = x - 0.8 - 0.4 \sin x$ on $\left[0, \frac{\pi}{2}\right]$. then slope $(f(x), x) = 1 - 0.4 \cos x > 0$ on $\left[0, \frac{\pi}{2}\right]$. Approximate the zero of f using Newton's Iteration

$$x_{n+1} = x_n - \frac{x_n - 0.8 - 0.4 \sin(x_n)}{1 - 0.4 \cos(x_n)}\ , x_0 = 0.5$$

4. To avoid computing the slope function f, the Secant Method of iteration uses the slope of the line going through the previous two points

$(x_n, f(x_n))$ and $(x_{n+1}, f(x_{n+1}))$ to define x_{n+2} as follows: Given x_0 and x_1, we define

$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})}{\left(\frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n}\right)}$$

$$\boxed{x_{n+2} = x_{n+1} - \frac{f(x_{n+1})(x_{n+1} - x_n)}{f(x_{n+1} - f(x_n)}}$$

This method is slower than Newton's Method, but faster than the Bisection. The big advantage is that we do not need to compute the slope function for f. The stopping rule can be the same as in Newton's Method. Use the secant Method for Exercises 2 and 3 with $x_0 = 0.5, x_1 = 0.7$ and $|x_{n+1} - x_n| < 10^{-4}$. Compare the number of iterations needed with Newton's Method.

5. Use the Bisection Method to compute the zero of $x^3+4x-10$ on $[1, 2]$ and compare the number of iterations needed for the stopping rule $|x_{n+1} - x_n| < 10^{-4}$.

6. A set S is said to be *connected* if S is not the union of two non-empty sets A and B such that A contains no limit point of B and B contains no limit point of A. Show that every closed and bounded interval $[a, b]$ is connected.

   (Hint: Assume that $[a, b]$ is not connected and $[a, b] = A \cup B$, $a \in A, B \neq \emptyset$ as described in the problem. Let $m = \text{lub}(A)$, $M = \text{glb}(B)$. Argue that $m \in A$ and $m \in B$. Then $\frac{1}{2}(m + M) \notin (A \cup B)$. The contradiction proves the result.

7. Show that the Intermediate Value Theorem (2.4.3) guarantees that continuous functions map connected sets onto connected sets. (Hint: Let S be connected and f be continuous on S. Let $R_f = \{f(x) : x \in S\}$. Suppose $R_f = A \cup B, A \neq \emptyset, B \neq \emptyset$, such that A contains no limit point of B and B contains no limit point of A. Let $U = \{x \in S : f(x) \in A\}$, $V = \{x \in S : f(x) \in B\}$. Then $S = U \cup V, U \neq \emptyset$ and $V \neq \emptyset$. Since S is connected, either U contains a limit point of V or V contains a limit point of U. Suppose $p \in V$ and p is a limit point of U. Then choose a

sequence $\{u_n\}$ that converges to $p, u_n \in U$. By continuity, $\{f(u_n)\}$ converges to $f(p)$. But $f(u_n) \in A$ and $f(p) \in B$. This is a contradiction.)

8. Find all of the fixed points of the following:

(a) $f(x) = x^2, \quad -4 \leq x \leq 4$

(b) $f(x) = x^3, \quad -2 \leq x \leq 2$

(c) $f(x) = x^2 + 3x + 1$

(d) $f(x) = x^3 - 3x, \quad -4 \leq x \leq 4$

(e) $f(x) = \sin x$

9. Determine which of the following sets are
(i) bounded, (ii) closed, (iii) connected.

(a) $N = \{1, 2, 3, \dots ,\}$

(b) $Q = \{x : x \text{ is rational number}\}$

(c) $R = \{x : x \text{ is a real number}\}$

(d) $B_1 = \{\sin x : -\pi \leq x \leq \pi\}$

(e) $B_2 = \{\sin x : -\pi < x < \pi\}$

(f) $B_3 = \left\{\sin x : \frac{-\pi}{2} < x < \frac{\pi}{2}\right\}$

(g) $B_4 = \left\{\tan x : \frac{-\pi}{2} < x < \frac{\pi}{2}\right\}$

(h) $C_1 = [(-1, 0) \cup (0, 1]$

(i) $C_2 = \left\{f(x) : -\pi \leq x \leq \pi, f(x) = \frac{\sin x}{x}, x \neq 0; f(0) = 2\right\}$

(j) $C_3 = \left\{g(x) : -\pi \leq x \leq \pi, g(x) = \frac{1 - \cos x}{x}, g(0) = 1\right\}$

10. Suppose f is continuous on the set of all real numbers. Let the open interval (c, d) be contained in the range of f. Let

$$A = \{x : c < f(x) < d\}.$$

Show that A is an open set.

(Hint: Let $p \in A$. Then $f(p) \in (c, d)$. Choose $\epsilon > 0$ such that $c < p - \epsilon < p + \epsilon < d$. Since f is continuous at p, there is $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever $|x - p| < \delta$. This means that the open interval $(p - \delta, p + \delta)$ is contained in A. By definition, A is open. This proves that the inverse of a continuous function maps an open set onto an open set.)

2.5 Limits and Infinity

The convergence of a sequence $\{a_n\}_{n=1}^{\infty}$ depends on the limit of a_n as n tends to ∞.

Definition 2.5.1 Suppose that a function f is defined on an open interval (a, b) and $a < c < b$. Then we define the following limits:

(i) $\lim_{x \to c^-} f(x) = +\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) > M$ whenever $c - \delta < x < c$.

(ii) $\lim_{x \to c^+} f(x) = +\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) > M$ whenever $c < x < c + \delta$.

(iii) $\lim_{x \to c} f(x) = +\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$.

(iv) $\lim_{x \to c} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) < -M$ whenever $0 < |x - c| < \delta$.

(v) $\lim_{x \to c^+} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) < -M$ whenever $c < x < c + \delta$.

(vi) $\lim_{x\to c^-} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $\delta > 0$ such that $f(x) < -M$ whenever $c - \delta < x < c$.

Definition 2.5.2 Suppose that a function f is defined for all real numbers.

(i) $\lim_{x\to+\infty} f(x) = L$

if and only if for every $\epsilon > 0$ there exists some $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > M$.

(ii) $\lim_{x\to-\infty} f(x) = L$

if and only if for every $\epsilon > 0$ there exists some $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x < -M$.

(iii) $\lim_{x\to+\infty} f(x) = \infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) > M$ whenever $x > N$.

(iv) $\lim_{x\to+\infty} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) < -M$ whenever $x > M$.

(v) $\lim_{x\to-\infty} f(x) = \infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) > M$ whenever $x < -N$.

(vi) $\lim_{x\to-\infty} f(x) = -\infty$

if and only if for every $M > 0$ there exists some $N > 0$ such that $f(x) < -M$ whenever $x < -N$.

Definition 2.5.3 The vertical line $x = c$ is called a *vertical asymptote* to the graph of f if and only if either

(i) $\lim_{x\to c} f(x) = \infty$ or $-\infty$; or

(ii) $\lim_{x\to c^-} f(x) = \infty$ or $-\infty$; or both.

Definition 2.5.4 The horizontal line $y = L$ is a *horizontal asymptote* to the graph of f if and only if

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L, \text{ or both.}$$

Example 2.5.1 Compute the following limits:

(i) $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$ (ii) $\displaystyle\lim_{x\to\infty} \frac{\cos x}{x}$

(iii) $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{3x^3+10}$ (iv) $\displaystyle\lim_{x\to-\infty} \frac{x^3-2}{3x^3+2x-3}$

(v) $\displaystyle\lim_{x\to-\infty} \frac{3x^3+4x-7}{2x^2+5x+2}$ (vi) $\displaystyle\lim_{x\to-\infty} \frac{-x^4+3x-10}{2x^2+3x-5}$

(i) We observe that $-1 \le \sin x \le 1$ and hence

$$0 = \lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0.$$

Hence, $y = 0$ is the horizontal asymptote and

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(ii) $-1 \le \cos x \le 1$ and, by a similar argument as in part (i),

$$\lim_{x\to\infty} \frac{\cos x}{x} = 0.$$

(iii) We divide the numerator and denominator by x^2 and then take the limit as follows:

$$\lim_{x\to\infty} \frac{x^2+1}{3x^3+10} = \lim_{x\to\infty} \frac{1+1/x^2}{3x+10/x^2} = 0.$$

(iv) We divide the numerator and denominator by x^3 and then take the limit as follows:

$$\lim_{x\to-\infty} \frac{x^3-2}{3x^3+2x-3} = \lim_{x\to-\infty} \frac{1-2/x^3}{3+2/x^2-3/x^3} = \frac{1}{3}.$$

(v) We divide the numerator and denominator by x^2 and then take the limit as follows:

$$\lim_{x\to-\infty} \frac{3x^3+4x-7}{2x^2+5x-2} = \lim_{x\to-\infty} \frac{3x+4/x-7/x^2}{2+5/x+2/x^2} = -\infty.$$

(vi) We divide the numerator and denominator by x^2 and then take the limit as follows:

$$\lim_{x\to-\infty} \frac{-x^4+3x-10}{2x^2+3x-5} = \lim_{x\to-\infty} \frac{-x^2+3/x-10/x^2}{2+3/x-5/x^2} = -\infty.$$

Example 2.5.2

(i) $\lim\limits_{n\to\infty} \dfrac{(-1)^n+1}{n} = 0$

(ii)
$$\begin{aligned}\lim_{n\to\infty} \left\{\frac{n^2}{n+3} - \frac{n^2}{n+4}\right\} &= \lim_{n\to\infty} \frac{n^3+4n^2-n^3-3n^2}{n^2+7n+12} \\ &= \lim_{n\to\infty} \frac{n^2}{n^2+7n+12} \\ &= \lim_{n\to\infty} \frac{1}{1+7/n+12/n^2} \\ &= 1\end{aligned}$$

(iii)
$$\begin{aligned}\lim_{n\to\infty} (\sqrt{n+4}-n) &= \lim_{n\to\infty} \frac{(\sqrt{n+4}-\sqrt{n})(\sqrt{n+4}+\sqrt{n})}{(\sqrt{n+4}+\sqrt{n})} \\ &= \lim_{n\to\infty} \frac{4}{(\sqrt{n+4}+\sqrt{n})} \\ &= 0\end{aligned}$$

(vi) $\lim_{n\to\infty} \left\{ \frac{n^2}{1+n^2} \sin\left(\frac{n\pi}{2}\right) \right\}$ does not exist because it oscillates:

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2m \\ 1 & \text{if } n = 2m+1 \\ -1 & \text{if } n = 2m+3 \end{cases}$$

(v) $\lim_{n\to\infty} \frac{3^n}{4+3^n} = \lim_{h\to\infty} \frac{1}{4 \cdot e^{-n} + 1} = 1$

(vi) $\lim_{n\to\infty} \{\cos(n\pi)\} = \lim_{n\to\infty} (-1)^n$ does not exist.

Exercises 2.5 Evaluate the following limits:

1. $\lim_{x\to 2} \frac{x}{x^2-4}$
2. $\lim_{x\to 2^+} \frac{x}{x^2-4}$
3. $\lim_{x\to 1^-} \frac{x}{x^2-1}$
4. $\lim_{x\to \frac{\pi}{2}^-} \tan(x)$
5. $\lim_{x\to \frac{\pi}{2}^+} \sec x$
6. $\lim_{x\to 0^+} \cot x$
7. $\lim_{x\to 0^-} \csc x$
8. $\lim_{x\to\infty} \frac{3x^2-7x+5}{4x^2+5x-7}$
9. $\lim_{x\to -\infty} \frac{x^2+4}{4x^3+3x-5}$
10. $\lim_{x\to\infty} \frac{-x^4+2x-1}{x^2+3x+2}$
11. $\lim_{x\to\infty} \frac{\cos(n\pi)}{n^2}$
12. $\lim_{x\to\infty} \frac{1+(-1)^n}{n^3}$
13. $\lim_{x\to\infty} \frac{\sin(n)}{n}$
14. $\lim_{x\to\infty} \frac{1-\cos n}{n}$
15. $\lim_{x\to\infty} \frac{\cos\left(\frac{n\pi}{2}\right)}{n}$
16. $\lim_{x\to\infty} \tan\left(\frac{n\pi}{n}\right)$

Chapter 3

Differentiation

In Definition 2.2.2, we defined the slope function of a function f at c by

$$\begin{aligned}\text{slope}(f(x), c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.\end{aligned}$$

The slope $(f(x), c)$ is called the *derivative* of f at c and is denoted $f'(c)$. Thus,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

Link to another file.

3.1 The Derivative

Definition 3.1.1 Let f be defined on a closed interval $[a, b]$ and $a < x < b$. Then the derivative of f at x, denoted $f'(x)$, is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

whenever the limit exists. When $f'(x)$ exists, we say that f is differentiable at x. At the end points a and b, we define one-sided derivatives as follows:

(i) $f'(a^+) = \lim_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = \lim_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}.$

We call $f'(a+)$ the right-hand derivative of f at a.

(ii) $f'(b^-) = \lim_{x \to b^-} \dfrac{f(x) - f(b)}{x - b} = \lim_{h \to 0^-} \dfrac{f(b+h) - f(b)}{h}$.

We call $f'(b)$ the left-hand derivative of f at b.

Example 3.1.1 In Example 28 of Section 2.2, we proved that if $f(x) = \sin x$, then $f'(c) = \text{slope } (\sin x, c) = \cos c$. Thus, $f'(x) = \cos x$ if $f(x) = \sin x$.

Example 3.1.2 In Example 29 of Section 2.2, we proved that if $f(x) = \cos x$, then $f'(c) = -\sin c$. Thus, $f'(x) = -\sin x$ if $f(x) = \cos x$.

Example 3.1.3 In Example 30 of Section 2.2, we proved that if $f(x) = x^n$ for a natural number n, then $f'(c) = nc^{n-1}$. Thus $f'(x) = nx^{n-1}$, when $f(x) = x^n$, for any natural number n.

In order to find derivatives of functions obtained from the basic elementary functions using the operations of addition, subtraction, multiplication and division, we state and prove the following theorem.

Theorem 3.1.1 *If f is differentiable at c, then f is continuous at c. The converse is false.*

Proof. Suppose that f is differentiable at c. Then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

and $f'(c)$ is a real number. So,

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c} \left[\left(\frac{f(x) - f(c)}{x - c} \right) (x - c) + f(c) \right] \\
&= \lim_{x \to 0} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) + f(c) \\
&= f'(c) \cdot 0 + f(c) \\
&= f(c).
\end{aligned}$$

Therefore, if f is differentiable at c, then f is continuous at c.

To prove that the converse is false we consider the function $f(x) = |x|$. This function is continuous at $x = 0$. But

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \left[\frac{|x+h| - |x|}{h}\right] \\ &= \lim_{h\to 0} \frac{(|x+h| - |x|)(|x+h| + |x|)}{h(|x+h| + |x|)} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h(|x+h| + |x|)} \\ &= \lim_{h\to 0} \frac{2x + h}{|x+h|\ |x|} \\ &= \frac{x}{|x|} \\ &= \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ \text{undefined for } x = 0. \end{cases} \end{aligned}$$

Thus, $|x|$ is continuous at 0 but not differentiable at 0. This completes the proof of Theorem 3.1.1.

Theorem 3.1.2 *Suppose that functions f and g are defined on some open interval (a, b) and $f'(x)$ and $g'(x)$ exist at each point x in (a, b). Then*

(i) $(f + g)'(x) = f'(x) + g'(x)$ (The Sum Rule)

(ii) $(f - g)'(x) = f'(x) - g'(x)$ (The Difference Rule)

(iii) $(kf)'(x) = kf'(x)$, *for each constant k.* (The Multiple Rule)

(iv) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ (The Product Rule)

(v) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$, *if $g(x) \neq 0$.* (The Quotient Rule)

Proof.

Part (i) $$(f+g)'(x) = \lim_{h\to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x).$$

Part (ii) $$(f-g)'(x) = \lim_{h\to 0} \frac{[f(x+h)-g(x+h)] - [f(x)-g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$
$$= f'(x) - g'(x).$$

Part (iii) $$(kf)'(x) = \lim_{h\to 0} \frac{kf(x+h)-kf(x)}{h}$$

$$= k \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= kf'(x).$$

Part (iv)

$$\begin{aligned}(f\cdot g)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[(f(x+h)-f(x))g(x+h) + f(x)(g(x+h)-g(x))\right] \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x)\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

Part (v)
$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)\cdot g(x) - g(x+h)f(x)}{g(x+h)g(x)}\right]$$
$$= \frac{1}{(g(x))^2} \lim_{h\to 0}\left[\frac{(f(x+h)-f(x))}{h} g(x) - f(x)\frac{(g(x+h)-g(x))}{h}\right]$$
$$= \frac{1}{(g(x))^2}\cdot[f'(x)g(x) - f(x)g'(x)]$$
$$= \frac{g(x)f'(x) - g(x)g'(x)}{(g(x))^2}, \text{ if } g(x) \neq 0.$$

To emphasize the fact that the derivatives are taken with respect to the independent variable x, we use the following notation, as is customary:

$$f'(x) = \frac{d}{dx}(f(x)).$$

Based on Theorem 3.1.2 and the definition of the derivative, we get the following theorem.

Theorem 3.1.3

(i) $\dfrac{d(k)}{dx} = 0$, *where k is a real constant.*

(ii) $\dfrac{d}{dx}(x^n) = nx^{n-1}$, *for each real number x and natural number n.*

(iii) $\dfrac{d}{dx}(\sin x) = \cos x$, *for all real numbers (radian measure) x.*

(iv) $\dfrac{d}{dx}(\cos x) = -\sin x$, *for all real numbers (radian measure) x.*

(v) $\dfrac{d}{dx}(\tan x) = \sec^2 x$, *for all real numbers $x \neq (2n+1)\dfrac{\pi}{2}$, n = integer.*

(vi) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$, *for all real numbers* $x \neq n\pi$, n = *integer.*

(vii) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$, *for all real numbers* $x \neq (2n+1)\dfrac{\pi}{2}$, n = *integer.*

(viii) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$, *for all real numbers* $x \neq n\pi$, n = *integer.*

Proof.

Part(i)

$$\frac{d(k)}{dx}(k) = \lim_{h\to 0} \frac{k-k}{h}$$
$$= \lim_{h\to 0} \frac{0}{h}$$
$$= 0.$$

Part (ii) For each natural n, we get

$$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \qquad \text{(Binomial Expansion)}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}\, x^{n-2}h^2 + \cdots + h^n - x^n\right]$$
$$= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}\, x^{n-2}h + \cdots + h^{n-1}\right]$$
$$= nx^{n-1}.$$

Part (iii) By definition, we get

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$
$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \lim_{h\to 0} \left[\cos x\, \frac{\sin h}{h} - \sin x\, \left(\frac{1-\cos h}{h}\right)\right]$$
$$= \cos x \cdot 1 - \sin x \cdot 0$$
$$= \cos x$$

since

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \; \lim_{h\to 0} \frac{1-\cos h}{h} = 0. \quad \text{(Why?)}$$

Part (iv) By definition, we get

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\cos x \cos h - \sin x \sin h - \cos x\right] \\
&= \lim_{h\to 0} \left[-\sin x \cdot \frac{\sin h}{h} - \cos x \left(\frac{1-\cos h}{h}\right)\right] \\
&= -\sin x \cdot 1 - \cos x \cdot 0 \qquad \text{(Why?)} \\
&= -\sin x.
\end{aligned}$$

Part (v) Using the quotient rule and parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \qquad \text{(Why?)} \\
&= \sec^2 x, \quad x \neq (2n+1)\frac{\pi}{2}, n = \text{ integer.}
\end{aligned}$$

Part (vi) Using the quotient rule and Parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{(\sin x)(\cos x)' - (\cos x)(\sin x)'}{(\sin x)^2} \\
&= \frac{-\sin^2 x - \cos^2 x}{(\sin x)^2} \qquad \text{(Why?)} \\
&= \frac{-1}{(\sin x)^2} \qquad \text{(why?)} \\
&= -\csc^2 x, \quad x \neq n\pi,\ n = \text{ integer.}
\end{aligned}$$

Part (vii) Using the quotient rule and Parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= \frac{(\cos x)\cdot 0 - 1\cdot(\cos x)'}{(\cos x)^2} \\
&= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \qquad \text{(Why?)} \\
&= \sec x \tan x, \quad x \neq (2n+1)\frac{\pi}{2},\ n = \text{ integer.}
\end{aligned}$$

Part (viii) Using the quotient rule and Parts (iii) and (iv), we get

$$\begin{aligned}
\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\
&= \frac{\sin x\cdot 0 - 1\cdot(\sin x)'}{(\sin x)^2} \\
&= \frac{1}{\sin x}\cdot\frac{-\cos x}{\sin x} \qquad \text{(Why?)} \\
&= -\csc x \cot x,\ x \neq n\pi, n = \text{ integer.}
\end{aligned}$$

This concludes the proof of Theorem 3.1.3.

Example 3.1.4 Compute the following derivatives:

(i) $\dfrac{d}{dx}\,(4x^3 - 3x^2 + 2x + 10)$ (ii) $\dfrac{d}{dx}\,(4\sin x - 3\cos x)$

(iii) $\dfrac{d}{dx}\,(x\sin x + x^2\cos x)$ (iv) $\dfrac{d}{dx}\left(\dfrac{x^3+1}{x^2+4}\right)$

Part (i) Using the sum, difference and constant multiple rules, we get

$$\begin{aligned}\frac{d}{dx}\,(4x^3 - 3x^2 + 2x + 10) &= 4\,\frac{d}{dx}(x^3) - 3\,\frac{d}{dx}\,(x^2) + 2\,\frac{d}{dx} + 0\\ &= 12x^2 - 6x + 2.\end{aligned}$$

Part (ii)
$$\begin{aligned}\frac{d}{dx}\,(4\sin x - 3\cos x) &= 4\,\frac{d}{dx}\,(\sin x) - 3\,\frac{d}{dx}\,(\cos x)\\ &= 4\cos x - 3(-\sin x)\\ &= 4\cos x + 3\sin x.\end{aligned}$$

Part (iii) Using the sum and product rules, we get

$$\begin{aligned}\frac{d}{dx}\,(x\sin x + x^2\cos x) &= \frac{d}{dx}\,(x\sin x) + \frac{d}{dx}\,(x^2\cos x) \quad \text{(Sum Rule)}\\ &= \left[\frac{d}{dx}\,\sin x + x\,\frac{d}{dx}\,(\sin x)\right]\\ &\quad + \left[\frac{d}{dx}\,(x^2)\cos x + x^2\,\frac{d}{dx}\,(\cos x)\right]\\ &= 1\cdot\sin x + x\cos x + 2x\cos x + x^2(-\sin x)\\ &= \sin x + 3x\cos x - x^2\sin x.\end{aligned}$$

Part (iv). Using the sum and quotient rules, we get

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^3+1}{x^2+4}\right) &= \frac{(x^2+4)\frac{d}{dx}\,(x^3+1)-(x^3+1)\,\frac{d}{dx}\,(x^2+4)}{(x^2+4)^2} && \text{(Why?)}\\
&= \frac{(x^2+4)(3x^2)-(x^3+1)=x}{(x^2+4)^2} && \text{(Why?)}\\
&= \frac{3x^4+12x^2-2x^3-2x}{(x^2+4)^2} && \text{(Why?)}\\
&= \frac{3x^4-2x^3+12x^2-2x}{(x^2+4)^2}.
\end{aligned}$$

Exercises 3.1

1. From the definition, prove that $\frac{d}{dx}\,(x^3) = 3x^2$.

2. From the definition, prove that $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$.

Compute the following derivatives:

3. $\frac{d}{dx}\,(x^5 - 4x^2 + 7x - 2)$

4. $\frac{d}{dx}\,(4\sin x + 2\cos x - 3\tan x)$

5. $\frac{d}{dx}\left(\frac{2x+1}{x^2+1}\right)$

6. $\frac{d}{dx}\left(\frac{x^4+2}{3x+1}\right)$

7. $\frac{d}{dx}\,(3x\sin x + 4x^2\cos x)$

8. $\frac{d}{dx}\,(4\tan x - 3\sec x)$

9. $\frac{d}{dx}\,(3\cot x + 5\csc x)$

10. $\frac{d}{dx}\,(x^2\tan x + x\cot x)$

Recall that the equation of the line tangent to the graph of f at $(c, f(c))$ has slope $f'(c)$ and equations.

Tangent Line: $\quad y - f(c) = f'(c)(x - c)$

The normal line has slope $-1/f'(c)$, if $f'(c) \neq 0$ and has the equation:

Normal Line: $$y - f(c) = \frac{-1}{f'(c)}\,(x - c).$$

In each of the following, find the equation of the tangent line and the equation of the normal line for the graph of f at the given c.

11. $f(x) = x^3 + 4x - 12,\ c = 1$
12. $f(x) = \sin x,\ c = \pi/6$
13. $f(x) = \cos x,\ c = \pi/3$
14. $f(x) = \tan x,\ c = \pi/4$
15. $f(x) = \cot x,\ c = \pi/4$
16. $f(x) = \sec x,\ c = \pi/3$
17. $f(x) = \csc x, c = \pi/6$
18. $f(x) = 3\sin x + 4\cos x,\ c = 0.$

Recall that Newton's Method solves $f(x) = 0$ for x by using the fixed point iteration algorithm:

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}\ ,\ x_0 = \text{ given},$$

with the stopping rule, for a given natural number n,

$$|x_{n+1} - x_n| < 10^{-n}.$$

In each of the following, set up Newton's Iteration and perform 3 calculations for a given x_0.

19. $f(x) = 2x - \cos x\ ,\ x_0 = 0.5$
20. $f(x) = x^3 + 2x + 1\ ,\ x_0 = -0.5$
21. $f(x) = x^3 + 3x^2 - 1 = 0,\ x_0 = 0.5$
22. Suppose that $f'(c)$ exists. Compute each of the following limits in terms of $f'(c)$

    (a) $\displaystyle\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$

    (b) $\displaystyle\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$

    (c) $\displaystyle\lim_{h\to 0} \frac{f(c-h) - f(c)}{h}$

    (d) $\displaystyle\lim_{t\to c} \frac{f(c) - f(t)}{t - c}$

    (e) $\displaystyle\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h}$

    (f) $\displaystyle\lim_{h\to 0} \frac{f(c+2h) - f(c-2h)}{h}$

23. Suppose that g is differentiable at c and

$$f(t) = \begin{cases} \frac{g(t)-g(c)}{t-c} & \text{if } t \neq c \\ g'(c) & \text{if } t = c. \end{cases}$$

Show that f is continuous at c.

Suppose that a business produces and markets x units of a commercial item. Let

$$\begin{aligned} C(x) &= \text{ The total cost of producing } x\text{-units.} \\ p(x) &= \text{ The sale price per item when } x\text{-units are on the market.} \\ R(x) &= xp(x) = \text{ The revenue for selling } x\text{-units.} \\ P(x) &= R(x) - C(x) = \text{ The gross profit for selling } x\text{-items.} \\ C'(x) &= \text{ The marginal cost.} \\ R'(x) &= \text{ The marginal revenue.} \\ P'(x) &= \text{ The marginal profit.} \end{aligned}$$

In each of the problems 24–26, use the given functions $C(x)$ and $p(x)$ and compute the revenue, profit, marginal cost, marginal revenue and marginal profit.

24. $C(x) = 100x - (0.2)x^2,\ 0 \leq x \leq 5000,\ p(x) = 10 - x$

25. $C(x) = 5000 + \dfrac{2}{x},\ 1 \leq x \leq 5000,\ p(x) = 20 + \dfrac{1}{x}$

26. $C(x) = 1000 + 4x - 0.1x^2,\ 1 \leq x \leq 2000,\ p(x) = 10 - \dfrac{1}{x}$

In exercises 27–60, compute the derivative of the given function.

27. $f(x) = 4x^3 - 2x^2 + 3x - 10$

28. $f(x) = 2\sin x - 3\cos x + 4$

29. $f(x) = 3\tan x - 4\sec x$

30. $f(x) = 2\cot x + 3\csc x$

31. $f(x) = 2x^2 + 4x + 5$

32. $f(x) = x^{2/3} - 4x^{1/3} + 5$

33. $f(x) = 3x^{-4/3} + 3x^{-2/3} + 10$

34. $f(x) = 2\sqrt{x} + 4$

35. $f(x) = \dfrac{2}{x^2}$

36. $f(x) = \dfrac{4}{x^3} - \dfrac{3}{x^2} + \dfrac{2}{x} + 1$

37. $f(x) = x^4 - 4x^2$

38. $f(x) = (x^2 + 2)(x^2 + 1)$

39. $f(x) = (x + 2)(x - 4)$

40. $f(x) = (x^3 + 1)(x^3 - 1)$

41. $y = (x^2 + 1)\sin x$

42. $y = x^2 \cos x$

43. $y = (x^2 + 1)(x^{10} - 5)$

44. $y = x^2 \tan x$

45. $y = (x^{1/2} + 4)(x^{1/3} - 5)$

46. $y = (2x + \sin x)(x^2 + 4)$

47. $y = x^5 \sin x$

48. $y = x^4(2\sin x - 3\cos x)$

49. $y = x^2 \cot x - 2x + 5$

50. $y = (x + \sin x)(4 + \csc x)$

51. $y = (\sec x + \tan x)(\sin x + \cos x)$

52. $y = x^2(2\cot x - 3\csc x)$

53. $y = \dfrac{x^2 + 1}{x^2 + 4}$

54. $y = \dfrac{1 + \sin x}{1 + \cos x}$

55. $y = \dfrac{x^{1/2} + 1}{3x^{3/2} + 2}$

56. $y = \dfrac{\sin x - \cos x}{\sin x + \cos x}$

57. $y = \dfrac{t^2 + 3t + 2}{t^3 + 1}$

58. $y = \dfrac{x^2 e^x}{1 + e^x}$

59. $y = \dfrac{3 + \sin t \cos t}{4 + \sec t \tan t}$

60. $y = \dfrac{t^2 \sin t}{4 + t^2}$

3.2 The Chain Rule

Suppose we have two functions, u and y, related by the equations:

$$u = g(x) \quad \text{and} \quad y = f(u).$$

Then $y = (f \circ g)(x) = f(g(x))$.

The chain rule deals with the derivative of the composition and may be stated as the following theorem:

Theorem 3.2.1 (The Chain Rule). *Suppose that g is defined in an open interval I containing c, and f is defined in an open interval J containing $g(c)$, such that $g(x)$ is in J for all x in I. If g is differentiable at c, and f is differentiable at $g(c)$, then the composition $(f \circ g)$ is differentiable at c and*

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c).$$

In general, if $u = g(x)$ and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx.}$$

Proof. Let F be defined on J such that

$$F(u) = \begin{cases} \frac{f(u)-f(g(c))}{u-g(c)} & \text{if } u \neq g(c) \\ f'(g(c)) & \text{if } u = g(c) \end{cases}$$

since f is differentiable at $g(c)$,

$$\begin{aligned} \lim_{u \to g(c)} F(u) &= \lim_{u \to g(c)} \frac{f(u) - f(g(c))}{u - g(c)} \\ &= f'(g(c)) \\ &= F(g(c)). \end{aligned}$$

Therefore, F is continuous at $g(c)$. By the definition of F,

$$f(u) - f(g(c)) = F(u)(u - g(c))$$

for all u in J. For each x in I, we let $y = g(x)$ on I. Then

$$\begin{aligned} (f \circ g)'(c) &= \lim_{x \to c} \frac{(f \circ g)(x) - (f \circ g)(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \\ &= \lim_{u \to g(c)} F(u) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= f'(g(c)) \cdot g'(c). \end{aligned}$$

It follows that $f \circ g$ is differentiable at c. The general result follows by replacing c by the independent variable x. This completes the proof of Theorem 3.2.1.

Example 3.2.1 Let $y = u^2 + 1$ and $u = x^3 + 4$. Then

$$\frac{dy}{du} = 2u \text{ and } \frac{du}{dx} = 3x^2.$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 2u \cdot 3x^2 \\ &= \boxed{6x^2(x^3 + 4)}\ . \end{aligned}$$

Using the composition notation, we get

$$y = (x^3 + 4)^2 + 1 = x^6 + 8x^3 + 17$$

and

$$\begin{aligned} \frac{dy}{dx} &= 6x^5 + 24x^2 \\ &= \boxed{6x^2(x^3 + 4)}\ . \end{aligned}$$

Using

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

we see that

$$(f \circ g)(x) = (x^3 + 4)^2 + 1$$

and

$$\begin{aligned} (f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \\ &= 2(x^3 + 4)^1 \cdot (3x^2) \\ &= \boxed{6x^2(x^3 + 4)}\ . \end{aligned}$$

Example 3.2.2 Suppose that $y = \sin(x^2 + 3)$.

We let $u = x^2 + 3$, and $y = \sin u$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (\cos u)(2x) \\ &= (\cos(x^2 + 3)) \cdot (2x). \end{aligned}$$

Example 3.2.3 Suppose that $y = w^2$, $w = \sin u + 3$, and $u = (4x + 1)$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx} \\ &= (2w) \cdot (\cos u) \cdot 4 \\ &= 8w \cos u \\ &= 8[\sin(4x + 1) + 3] \cdot \cos(4x + 1) \cdot 4 \\ &= 8(\sin(4x + 1) + 3) \cdot \cos(4x + 1). \end{aligned}$$

If we express y in terms of x explicitly, then we get

$$y = (\sin(4x + 1) + 3)^2$$

and

$$\begin{aligned} \frac{dy}{dx} &= 2(\sin(4x + 1) + 3)^1 \cdot ((\cos(4x + 1)) \cdot 4 + 0) \\ &= 8(\sin(4x + 1) + 3) \ \cos(4x + 1). \end{aligned}$$

Example 3.2.4 Suppose that $y = (\cos(3x + 1))^5$. Then

$$\begin{aligned} \frac{dy}{dx} &= 5(\cos(3x + 1))^4 \cdot (-\sin(3x + 1)) \cdot 3 \\ &= -15(\cos(3x + 1))^4 \ \sin(3x + 1). \end{aligned}$$

Example 3.2.5 Suppose that $y = \tan^3(2x^2+1)$. Then

$$\begin{aligned}\frac{dy}{dx} &= 3(\tan^2(2x^2+1)) \cdot (\sec^2(2x^2+1)) \cdot 4x \\ &= 12x \cdot \tan^2(2x^2+1) \cdot \sec^2(2x^2+1).\end{aligned}$$

Example 3.2.6 Suppose that $y = \cot\left(\dfrac{x+1}{x^2+1}\right)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \left[-\csc^2\left(\frac{x+1}{x^2+1}\right)\right] \left[\frac{(x^2+1)\cdot 1 - (x+1)^{2x}}{(x^2+1)\cdot 2x}\right] \\ &= \frac{x^2+2x-1}{(x^2+1)^2} \csc^2\left(\frac{x+1}{x^2+1}\right).\end{aligned}$$

Example 3.2.7 Suppose that $y = \sec\left(\dfrac{x^2+1}{x^4+2}\right)^3$.

Since the function y has a composition of several functions, let us define some intermediate functions. Let

$$y = \sec w, \ w = u^3, \quad \text{and} \quad u = \frac{x^2+1}{x^4+2}.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx} \\ &= [\sec(w)\tan(w)] \cdot [3u^2] \cdot \frac{(x^4+2)\cdot 2x - (x^2+1)\cdot 4x^3}{(x^4+2)^2} \\ &= 3u^2(\sec w \tan w) \cdot \frac{4x-4x^3-2x^5}{(x^4+2)^2} \\ &= 3\left(\frac{x^2+1}{x^4+2}\right)^2 \sec\left(\frac{x^2+1}{x^4+2}\right)^3 \tan\left(\frac{x^2+1}{x^4+2}\right)^3 \cdot \frac{4x-4x^3-2x^5}{(x^4+2)^5}.\end{aligned}$$

Example 3.2.8 Suppose that $y = \csc(2x+5)^4$. Then

$$\frac{dy}{dx} = [-\csc(2x+5)^4 \cot(2x+5)^4] \cdot 4(2x+5)^3 \cdot 2$$

$$= -8(2x+5)^3 \csc(2x+5)^4 \cot(2x+5)^4.$$

Exercises 3.2 Evaluate $\dfrac{dy}{dx}$ for each of the following:

1. $y = (2x-5)^{10}$
2. $y = \left(\dfrac{x^2+2}{x^5+4}\right)^3$
3. $y = \sin(3x+5)$
4. $y = \cos(x^3+1)$
5. $y = \tan^5(3x+1)$
6. $y = \sec^2(x^2+1)$
7. $y = \cot^4(2x-4)$
8. $y = \csc^3(3x^2+2)$
9. $y = \left(\dfrac{3x+1}{x^2+2}\right)^5$
10. $y = \left(\dfrac{x^2+1}{x^3+2}\right)^4$
11. $y = \sin(w),\ w = u^3,\ u = (2x-1)$
12. $y = \cos(w),\ w = u^2+1,\ u = (3x+5)$
13. $y = \tan(w),\ w = v^2,\ v = u^3+1,\ u = \left(\dfrac{1}{x}\right)$
14. $y = \sec w,\ w = v^3,\ v = 2u^2-1,\ u = \dfrac{x}{x^2+1}$
15. $y = \csc w,\ w = 3v+2,\ v = (u+1)^3,\ u = (x^2+3)^2$

In exercises 16–30, compute the derivative of the given function.

16. $y = \left(\dfrac{x^3+1}{x^2+4}\right)^3$
17. $y = (x^2-1)^{10}$

18. $y = (x^2 + x + 2)^{100}$

19. $y = (2\sin t - 3\cos t)^3$

20. $y = (x^{2/3} + x^{4/3})^2$

21. $y = (x^{1/2} + 1)^{50}$

22. $y = \sin(3x + 2)$

23. $y = \cos(3x^2 + 1)$

24. $y = \sin(2x)\cos(3x)$

25. $y = \sec 2x + \tan 3x$

26. $y = \sec 2x \tan 3x$

27. $y = (x^2 + 1)^2 \sin 2x$

28. $y = x\sin(1/x^2)$

29. $y = \sin^2(3x) + \sec^2(5x)$

30. $y = \cot(x^2) + \csc(3x)$

In exercises 31–60, assume that

(a) $\dfrac{d}{dx}(e^x) = e^x$ (b) $\dfrac{d}{dx}(e^{-x}) = -e^{-x}$ (c) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

(d) $\dfrac{d}{dx}(b^x) = b^x \ln b$ (e) $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x\ln b}$ for $b > 0$ and $b \neq 1$.

Compute the derivative of the given function.

31. $y = \sinh x$

32. $y = \cosh x$

33. $y = \tanh x$

34. $y = \coth x$

35. $y = \operatorname{sech} x$

36. $y = \operatorname{csch} x$

37. $y = \ln(1 + x)$

38. $y = \ln(1 - x)$

39. $y = \dfrac{1}{2}\ln\left(\dfrac{1-x}{1+x}\right)$

40. $y = \ln\left(x + \sqrt{x^2 + 1}\right)$

41. $y = \ln\left(x + \sqrt{x^2 - 1}\right)$

42. $y = xe^{-x^2}$

43. $y = e^{\sin 3x}$

44. $y = e^{2x}\sin 4x$

45. $y = e^{x^2}(2\sin 3x - 4\cos 5x)$

46. $y = xe^{-x^2} + 4e^{-x}$

47. $y = 4^{x^2}$

48. $y = 10^{(x^2+4)}$

49. $y = 10^{\sin 2x}$

50. $y = 3^{\cos 3x}$

51. $y = \log_{10}(x^2 + 10)$

52. $y = \log_3(x^2 \sin x + x)$

53. $y = \ln(\sin(e^{2x}))$

54. $y = \ln(1 + e^{-x})$

55. $y = \ln(\cos x + 2)$

56. $y = \ln(\ln(x^2 + 4))$

57. $y = \left\{\ln\left(\frac{x^4+3}{x^2+10}\right)\right\}^3$

58. $y = (1 + \sin^2 x)^{3/2}$

59. $y = \ln(\sec 2x + \tan 2x)$

60. $y = \ln(\csc 3x - \cot 3x)$

3.3 Differentiation of Inverse Functions

One of the applications of the chain rule is to compute the derivatives of inverse functions. We state the exact result as the following theorem:

Theorem 3.3.1 *Suppose that a function f has an inverse, f^{-1}, on an open interval I. If $u = f^{-1}(x)$, then*

(i) $\dfrac{du}{dx} = \dfrac{1}{\left(\frac{dx}{du}\right)}$

(ii) $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{f'(u)}$

Proof. By comparison, $x = f(f^{-1}(x)) = x$. Hence, by the chain rule

$$1 = \frac{dx}{dx} = f'(f^{-1}(x)) \cdot (f^{-1})'(x)$$

and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In the $u = f^{-1}(x)$ notation, we have

$$\frac{du}{dx} = \frac{1}{\left(\frac{dx}{du}\right)}.$$

Remark 11 In Examples 76–81, we assume that the inverse trigonometric functions are differentiable.

Example 3.3.1 Let $u = \arcsin x$, $-1 \le x \le 1$, and $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $x = \sin u$ and by the chain rule, we get

$$\begin{aligned} 1 = \frac{dx}{dx} &= \frac{d(\sin u)}{du} \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} \\ \frac{du}{dx} &= \frac{1}{\cos u}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dx}(\arcsin x) &= \frac{1}{\cos u}\ , \ -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ &= \frac{1}{\sqrt{1 - \sin^2 u}} \qquad \text{(Why?)} \\ &= \frac{1}{\sqrt{1 - x^2}}\ , \ -1 < x < 1. \qquad \text{(Why?)} \end{aligned}$$

Thus,

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}\ , \ -1 < x < 1.$$

We note that $x = \pm 1$ are excluded.

Example 3.3.2 Let $u = \arccos x$, $-1 \leq x \leq 1$, and $0 \leq u \leq \pi$. Then $x = \cos u$ and

$$\begin{aligned} 1 = \frac{dx}{dx} &= -\sin u \,\frac{du}{dx} \\ \frac{du}{dx} &= -\frac{1}{\sin u}, \ 0 < u < \pi \\ &= -\frac{1}{\sqrt{1-\cos^2 u}}, \ 0 < u < \pi \qquad \text{(Why?)} \\ &= -\frac{1}{\sqrt{1-x^2}}, \ -1 < x < 1. \qquad \text{(Why?)} \end{aligned}$$

We note again that $x = \pm 1$ are excluded.

Thus,

$$\frac{d}{dx}\,(\arccos x) = \frac{-1}{\sqrt{1-x^2}}, \ -1 < x < 1.$$

Example 3.3.3 Let $u = \arctan x$, $-\infty < x < \infty$, and $-\frac{\pi}{2} < u < \frac{\pi}{2}$. Then,

$$\begin{aligned} x &= \tan u, \ -\frac{\pi}{2} < u < \frac{\pi}{2} \\ 1 = \frac{dx}{dx} &= (\sec^2 u), \ \frac{du}{dx}\ , \ -\frac{\pi}{2} < u < \frac{\pi}{2} \\ \frac{du}{dx} &= \frac{1}{\sec^2 u} \\ &= \frac{1}{1+\tan^2 u}, \ -\frac{\pi}{2} < u < \frac{\pi}{2} \\ &= \frac{1}{1+x^2}, \ -\infty < x < \infty \end{aligned}$$

Therefore,

$$\frac{d}{dx}\,(\arctan x) = \frac{1}{1+x^2}, \ -\infty < x < \infty.$$

Example 3.3.4 Let $u = \operatorname{arcsec} x,\ x \in (-\infty, -1] \cup [1, \infty)$ and $u \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Then,

$$\begin{aligned}
x &= \sec u \\
1 = \frac{dx}{dx} &= \sec u \tan u \cdot \frac{du}{dx}\ , \qquad u \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\
\frac{du}{dx} &= \frac{1}{\sec u \tan u}\ , \qquad u \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \\
&= \frac{1}{|\sec u|\sqrt{\sec^2 u - 1}} \qquad \text{(Why the absolute value?)} \\
&= \frac{1}{|x|\sqrt{x^2 - 1}}, \qquad x \in (-\infty, -1) \cup (1, \infty).
\end{aligned}$$

Thus,

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2 - 1}},\ x \in (-\infty, -1) \cup (1, \infty).$$

Example 3.3.5 Let $u = \operatorname{arccsc} x,\ x \in (-\infty, -1] \cup [1, \infty)$, and $u \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Then,

$$\begin{aligned}
x &= \csc u\ , \quad u \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\
1 = \frac{dx}{dx} &= -\csc u \cot u \cdot \frac{du}{dx}, \quad u \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\
\frac{du}{dx} &= \frac{-1}{\csc u \cot u}, \quad u \in \left(\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right), \quad \text{(Why?)} \\
&= \frac{1}{|\csc u|\sqrt{\csc^2 u - 1}} \qquad \text{(Why?)} \\
&= \frac{1}{|x|\sqrt{x^2 - 1}}, \quad x \in (-\infty, -1) \cup (1, \infty).
\end{aligned}$$

Note that $x = \pm 1$ are excluded.
Thus,

$$\frac{d}{dx}\ (\operatorname{arccsc} x) = \frac{-1}{x\sqrt{x^2 - 1}},\ x \in (-\infty, 1] \cup (1, \infty).$$

Example 3.3.6 Let $u = \text{arccot}\ x,\ x \in (-\infty, 0] \cup [0, \infty)$ and $u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right)$. Then

$$x = \cot u, \quad u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right)$$

and

$$\begin{aligned}
1 = \frac{dx}{dx} &= -\csc^2(u) \cdot \frac{du}{dx}, u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right) \\
\frac{du}{dx} &= \frac{-1}{\csc^2 u}, \quad u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right) \\
&= \frac{-1}{1 + \cot^2 u}, \quad u \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right) \\
&= \frac{-1}{1 + x^2}, \quad x \in (-\infty, 0] \cup [0, \infty).
\end{aligned}$$

Therefore,

$$\frac{d}{dx}\,(\text{arccot} x) = \frac{-1}{1 + x^2}, \quad x \in (-\infty, 0] \cup [0, \infty).$$

The results of these examples are summarized in the following theorem:

Theorem 3.3.2 (The Inverse Trigonometric Functions) *The following differentiation formulas are valid for the inverse trigonometric functions:*

(i) $\frac{d}{dx}\,(\arcsin x) = \frac{1}{\sqrt{1 - x^2}},\ -1 < x < 1.$

(ii) $\frac{d}{dx}\,(\arccos x) = \frac{-1}{\sqrt{1 - x^2}},\ -1 < x < 1.$

(iii) $\frac{d}{dx}\,(\arctan x) = \frac{1}{1 + x^2},\ -\infty < x < \infty.$

(iv) $\frac{d}{dx}\,(\text{arccot}\ x) = \frac{-1}{1 + x^2},\ -\infty < x < \infty.$

(v) $\frac{d}{dx}\,(\text{arcsec}\ x) = \frac{1}{|x|\sqrt{x^2 - 1}},\ -\infty < x < -1 \ \ \textit{or}\ \ 1 < x < \infty.$

(vi) $\dfrac{d}{dx}\,(\text{arccsc}\ x) = \dfrac{-1}{|x|\sqrt{x^2-1}}, \quad -\infty < x < -1 \quad \textit{or} \quad 1 < x < \infty.$

Proof. Proof of Theorem 3.3.2 is outlined in Examples 76–80.

Theorem 3.3.3 (Logarithmic and Exponential Functions)

(i) $\dfrac{d}{dx}\,(\ln x) = \dfrac{1}{x}$ *for all* $x > 0$.

(ii) $\dfrac{d}{dx}\,(e^x) = e^x$ *for all real* x.

(iii) $\dfrac{d}{dx}\,(\log_b x) = \dfrac{1}{x\ln b}$ *for all* $x > 0$ *and* $b \neq 1$.

(iv) $\dfrac{d}{dx}\,(b^x) = b^x(\ln b)$ *for all real* x, $b > 0$ *and* $b \neq 1$.

(v) $\dfrac{d}{dx}\,(u(x)^{v(x)} = (u(x))^{v(x)}\left[v'(x)\ln(u(x)) + v(x)\,\dfrac{u'(x)}{u(x)}\right].$

Proof. Proof of Theorem 3.3.3 is outlined in the proofs of Theorems 5.5.1–5.5.5. We illustrate the proofs of parts (iii), (iv) and (v) here.

Part (iii) By definition for all $x > 0$, $b > 0$ and $b \neq 1$,

$$\log_b x = \frac{\ln x}{\ln b}.$$

Then,

$$\begin{aligned}\frac{d}{dx}\,(\log_b x) &= \frac{d}{dx}\,\left(\left(\frac{1}{\ln b}\right)\ln x\right)\\ &= \left(\frac{1}{\ln b}\right)\cdot\frac{1}{x}\\ &= \frac{1}{x\ln b}.\end{aligned}$$

Part (iv) By definition, for real x, $b > 0$ and $b \neq 1$,

$$b^x = e^{x\ln b}.$$

Therefore,

$$\begin{aligned}
\frac{d}{dx}\,(b^x) &= \frac{d}{dx}\,(e^{x\ln b}) \\
&= e^{x\ln b}, \frac{d}{dx}\,(x\ln b) \qquad \text{(by the chain rule)} \\
&= b^x \ln b. \qquad \text{(Why?)}
\end{aligned}$$

Part (v)

$$\begin{aligned}
\frac{d}{dx}\,(u(x))^{v(x)} &= \frac{d}{dx}\left\{e^{v(x)\ln(u(x))}\right\} \\
&= e^{v(x)\ln(u(x))}\left\{v'(x)\ln(u(x)) + v(x)\,\frac{u'(x)}{u(x)}\right\} \\
&= (u(x))^{v(x)}\left\{v'(x)\ln u(x) + v(x)\,\frac{u'(x)}{u(x)}\right\}
\end{aligned}$$

Example 3.3.7 Let $y = \log_{10}(x^2+1)$. Then

$$\begin{aligned}
\frac{d}{dx}\,(\log_{10}(x^2+1)) &= \frac{d}{dx}\left(\frac{\ln(x^2+1)}{\ln 10}\right) \\
&= \frac{1}{\ln 10}\left(\frac{1}{x^2+1}\cdot 2x\right) \qquad \text{(by the chain rule)} \\
&= \frac{2x}{(x^2+1)\ln 10}.
\end{aligned}$$

Example 3.3.8 Let $y = e^{x^2+1}$. Then, by the chain rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= e^{x^2+1}\cdot 2x \\
&= 2xe^{x^2+1}.
\end{aligned}$$

Example 3.3.9 Let $y = 10^{(x^3+2x+1)}$. By definition and the chain rule, we get

$$\frac{dy}{dx} = 10^{(x^3+2x+1)} \cdot (\ln 10) \cdot (3x^2 + 2).$$

Example 3.3.10

$$\frac{dx}{dx}\,(x^2+1)^{\sin x} = (x^2+1)^{\sin x}\left\{\cos x \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}\right\}$$

$$\begin{aligned}\frac{d}{dx}\,(x^2+1)^{\sin x} &= \frac{d}{dx}\,\left[e^{\sin x \ln(x^2+1)}\right] = 3^{\sin x \ln(x^2+1)} \cdot \left[\cos x \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}\right] \\ &= (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right].\end{aligned}$$

Theorem 3.3.4 (Differentiation of Hyperbolic Functions)

(i) $\frac{d}{dx}\,(\sinh x) = \cosh x$

(ii) $\frac{d}{dx}\,(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}\,(\tanh x) = \text{sech}^2 x$

(iv) $\frac{d}{dx}\,(\text{coth} x) = -\text{csch}^2 x$

(v) $\frac{d}{dx}\,(\text{sech } x) = -\text{sech } x \tanh x$

(vi) $\frac{d}{dx}\,(\text{csch } x) = -\text{csch } x \text{ coth } x.$

Proof.

Part (i)

$$\begin{aligned}\frac{d}{dx}\,(\sinh x) &= \frac{d}{dx}\,\left(\frac{1}{2}\,(e^x - e^{-x})\right) \\ &= \frac{1}{2}\,(e^x - e^{-x}(-1)) \qquad \text{(by the chain rule)} \\ &= \frac{1}{2}\,(e^x + e^{-x}) \\ &= \cosh x.\end{aligned}$$

Part (ii)

$$\begin{aligned}\frac{d}{dx}\,(\cosh x) &= \frac{d}{dx}\,\left(\frac{1}{2}\,(e^x + e^{-x})\right)\\ &= \frac{1}{2}\,(e^x + e^{-x}(-1)) \qquad \text{(by the chain rule)}\\ &= \frac{1}{2}\,(e^x - e^{-x})\\ &= \sinh x.\end{aligned}$$

Part (iii)

$$\begin{aligned}\frac{d}{dx}\,(\tanh x) &= \frac{d}{dx}\,\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)\\ &= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}\\ &= \frac{4}{(e^x + e^{-x})^2}\\ &= \left(\frac{2}{e^x + e^{-x}}\right)^2\\ &= \operatorname{sech}^2 x.\end{aligned}$$

Part (iv)

$$\begin{aligned}\frac{d}{dx}\,(\operatorname{sech}\,x) &= \frac{d}{dx}\,\left(\frac{2}{e^x + e^{-x}}\right)\\ &= \frac{(e^x + e^{-x}) \cdot 0 - 2(e^x - e^{-x})}{(e^x + e^{-x})^2}\\ &= -\frac{2}{e^x + e^{-x}} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}}\\ &= -\operatorname{sech} x \tanh x.\end{aligned}$$

Part (v)

$$\begin{aligned}
\frac{d}{dx}\,(\coth x) &= \frac{d}{dx}\,\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)\;, \quad x \neq 0 \\
&= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2} \quad x \neq 0 \\
&= \frac{-4}{(e^x-e^{-x})^2}\;, \quad x \neq 0 \\
&= -\left(\frac{2}{e^x-e^{-x}}\right)^2\;, \quad x \neq 0 \\
&= -\mathrm{csch}^2 x\;, \quad x \neq 0.
\end{aligned}$$

Part (vi)

$$\begin{aligned}
\frac{d}{dx}\,(\mathrm{csch}\, x) &= \frac{d}{dx}\,\left(\frac{2}{e^x-e^{-x}}\right)\;, \quad x \neq 0 \\
&= \frac{(e^x-e^{-x})\cdot 0 - 2(e^x+e^{-x})}{(e^x-e^{-x})^2}\;, \quad x \neq 0 \\
&= -\frac{2}{e^x-e^{-x}}\cdot\frac{e^x+e^{-x}}{e^x-e^{-x}}\;, \quad x \neq 0 \\
&= -\mathrm{csch}\, x\,\coth x, \quad x \neq 0.
\end{aligned}$$

Theorem 3.3.5 (Inverse Hyperbolic Functions)

(i) $$\frac{d}{dx}\,(\mathrm{arcsinh}\, x) = \frac{1}{\sqrt{1+x^2}}$$

(ii) $$\frac{d}{dx}\,(\mathrm{arccosh}\, x) = \frac{1}{\sqrt{x^2-1}}\;, \quad x > 1$$

(iii) $$\frac{d}{dx}\,(\mathrm{arctanh}\, x) = \frac{1}{1-x^2}\;, \quad |x| < 1$$

Proof.

Part (i)

$$\begin{aligned}
\frac{d}{dx}\,(\operatorname{arcsinh} x) &= \frac{d}{dx}\ \ln(x+\sqrt{1+x^2}) \\
&= \frac{1}{x+\sqrt{1+x^2}}\cdot\left[1+\frac{x}{\sqrt{1+x^2}}\right] \qquad \text{(by chain rule)} \\
&= \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} \\
&= \frac{1}{\sqrt{1+x^2}}.
\end{aligned}$$

Part (ii)

$$\begin{aligned}
\frac{d}{dx}\,(\operatorname{arccosh} x) &= \frac{d}{dx}\ \ln(x+\sqrt{x^2-1})\ , \quad x \ge 1 \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right)\ , \quad x > 0 \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\ , x > 0 \\
&= \frac{1}{\sqrt{x^2-1}}\ , x > 0.
\end{aligned}$$

Part (iii)

$$\begin{aligned}
\frac{d}{dx}\,(\operatorname{arctanh} x) &= \frac{d}{dx}\ \left[\frac{1}{2}\ \ln\left(\frac{1+x}{1-x}\right)\right]\ , \quad |x| < 1 \\
&= \frac{d}{dx}\ \left[\frac{1}{2}\ \ln(1+x)-\ln(1-x)\right]\ , \quad |x| < 1 \\
&= \frac{1}{2}\ \left[\frac{1}{1+x}-\frac{-1}{1-x}\right]\ , \quad |x| < 1 \\
&= \frac{1}{2}\ \left[\frac{1}{1+x}+\frac{1}{1-x}\right]\ , \quad |x| < 1 \\
&= \frac{1}{2}\ \left[\frac{1-x+1+x}{1-x^2}\right]\ , \quad |x| < 1 \\
&= \frac{1}{1-x^2}\ , \quad |x| < 1.
\end{aligned}$$

Exercises 3.3 Compute $\frac{dy}{dx}$ for each of the following:

1. $y = \ln(x^2 + 1)$
2. $y = \ln\left(\frac{1-x}{1+x}\right)$, $-1 < x < 1$
3. $y = \log_2(x)$
4. $y = \log_5(x^3 + 1)$
5. $y = \log_{10}(3x + 1)$
6. $y = \log_{10}(x^2 + 4)$
7. $y = 2e^{-x}$
8. $y = e^{x^2}$
9. $y = \frac{1}{2}\left(e^{x^2} - e^{-x^2}\right)$
10. $y = \frac{1}{2}\left(e^{x^2} + e^{-x^2}\right)$
11. $y = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$
12. $y = \frac{2}{e^{x^2} + e^{-x^2}}$
13. $y = \frac{2}{e^{x^3} - e^{-x^3}}$
14. $y = \frac{2}{e^{x^4} + e^{-x^4}}$
15. $y = \arcsin\left(\frac{x}{2}\right)$
16. $y = \arccos\left(\frac{x}{3}\right)$
17. $y = \arctan\left(\frac{x}{5}\right)$
18. $y = \operatorname{arccot}\left(\frac{x}{7}\right)$
19. $y = \operatorname{arcsec}\left(\frac{x}{2}\right)$
20. $y = \operatorname{arccsc}\left(\frac{x}{3}\right)$
21. $y = 3\sinh(2x) + 4\cosh 3x$
22. $y = e^x(3\sin 2x + 4\cos 2x)$
23. $y = e^{-x}(4\sin 3x - 3\cos 3x)$
24. $y = 4\sinh 2x + 3\cosh 2x$
25. $y = 3\tanh(2x) - 7\coth(2x)$
26. $y = 3\operatorname{sech}(5x) + 4\operatorname{csch}(3x)$
27. $y = 10^{x^2}$
28. $y = 2^{(x^3+1)}$
29. $y = 5^{(x^4+x^2)}$
30. $y = 3^{\sin x}$

31. $y = 4^{\cos(x^2)}$

32. $y = 10^{\tan(x^3)}$

33. $y = 2^{\cot x}$

34. $y = 10^{\sec(2x)}$

35. $y = 4^{\csc(x^2)}$

36. $y = e^{-x}(2\sin(x^2) + 3\cos(x^3))$

37. $y = \text{arcsinh}\left(\frac{x}{2}\right)$

38. $y = \text{arccosh}\left(\frac{x}{3}\right)$

39. $y = \arctan\left(\frac{x}{4}\right)$

40. $y = x\ \text{arcsinh}\left(\frac{x}{3}\right)$

In exercises 41–50, use the following procedure to compute the derivative of the given functions:

$$\begin{aligned}\frac{d}{dx}\left[(f(x)^{g(x)}\right] &= \frac{d}{dx}\left[e^{g(x)\ln(f(x))}\right] \\ &= e^{g(x)\ln(f(x))} \cdot \left[g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right] \\ &= (f(x))^{g(x)} \cdot \left[g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right].\end{aligned}$$

41. $y = (x^2+4)^{3x}$

42. $y = (2+\sin x)^{\cos x}$

43. $y = (3+\cos x)^{\sin 2x}$

44. $y = (x^2+4)^{x^2+1}$

45. $y = (1+x)^{1/x}$

46. $y = (1+x^2)^{\cos 3x}$

47. $y = (2\sin x + 3\cos x)^{x^3}$

48. $y = (1+\ln x)^{1/x^2}$

49. $y = (1+\sinh x)^{\cosh x}$

50. $y = (\sinh^2 x + \cosh^2 x)^{x^2+3}$

3.4 Implicit Differentiation

So far we have dealt with explicit functions such as x^2, $\sin x$, $\cos x$, $\ln x$, e^x, $\sinh x$ and $\cosh x$ etc. In applications, two variables can be related by an equation such as

(i) $x^2+y^2=16$ (ii) $x^3+y^3=4xy$ (iii) $x\sin y+\cos 3y=\sin 2y$.

In such cases, it is not always practical or desirable to solve for one variable explicitly in terms of the other to compute derivatives. Instead, we may implicitly assume that y is some function of x and differentiate each term of the equation with respect to x. Then we solve for y', noting any conditions under which the derivative may or may not exist. This process is called implicit differentiation. We illustrate it by examples.

Example 3.4.1 Find $\dfrac{dy}{dx}$ if $x^2+y^2=16$.

Assuming that y is to be considered as a function of x, we differentiate each term of the equation with respect to x.

graph

$$\begin{aligned}
\frac{d}{dx}\,(x^2)+\frac{d}{dx}\,(y^2) &= \frac{d}{dx}\,(16)\\
2x+2y\left(\frac{dy}{dx}\right) &= 0 \qquad\qquad \text{(Why?)}\\
2y\,\frac{dy}{dx} &= -2x\\
\frac{dy}{dx} &= -\frac{x}{y}\ , \ \text{provided } y\neq 0.
\end{aligned}$$

We observe that there are two points, namely $(4,0)$ and $(-4,0)$ that satisfy the equation. At each of these points, the tangent line is vertical and hence, has no slope.

If we solve for y in terms of x, we get two solutions, each representing a function of x:

$$y=(16-x^2)^{1/2} \quad \text{or} \quad y=-(16-x^2)^{1/2}.$$

On differentiating each function with respect to x, we get, respectively,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\,(16-x^2)^{-1/2}(-2x)\ ; \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{2}\,(16-x^2)^{-1/2}(-2x) \\
\frac{dy}{dx} &= -\frac{x}{(16-x^2)^{1/2}}\ ; \quad \text{or} \quad \frac{x}{-(16-x^2)^{1/2}} \\
\frac{dy}{dx} &= -\frac{x}{y}\ , y \neq 0; \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}\ , y \neq 0.
\end{aligned}$$

In each case, the final form is the same as obtained by implicit differentiation.

Example 3.4.2 Compute $\frac{dy}{dx}$ for the equation $x^3 + y^3 = 4xy$.

As in Example 2.4.1, we differentiate each term with respect to x, assuming that y is a function of x.

$$\begin{aligned}
\frac{dy}{dx}\,(x^3) + \frac{d}{dx}(y^3) &= \frac{d}{dx}\,(4xy) \\
3x^2 + 3y^2\left(\frac{dy}{dx}\right) &= 4\left[\frac{dx}{dx}\,y + x\,\frac{dy}{dx}\right] && \text{(Why?)} \\
(3y^2)\,\frac{dy}{dx} - 4x\frac{dy}{dx} &= 4y - 3x^2 && \text{(Why?)} \\
(3y^2 - 4x)\,\frac{dy}{dx} &= 4y - 3x^2 && \text{(Why?)} \\
\frac{dy}{dx} &= \frac{4y-3x^2}{3y^2-4x}\ , \text{if } 3y^2 - 4x \neq 0. && \text{(Why?)}
\end{aligned}$$

This differentiation formula is valid for all points (x, y) on the given curve, where $3y^2 - 4x \neq 0$.

Example 3.4.3 Compute $\frac{dy}{dx}$ for the equation $x \sin y + \cos 3y = \sin 2y$. In this example, it certainly is not desirable to solve for y explicitly in terms of

x. We consider y to be a function of x, differentiate each term of the equation with respect to x and then algebraically solve for y in terms of x and y.

$$\frac{d}{dx}\,(x\sin y) + \frac{d}{dx}\,(\cos 3y) = \frac{d}{dx}\,(\sin 2y)$$
$$\left[\left(\frac{dx}{dx}\right)\,(\sin y) + x\,\frac{d}{dx}\,(\sin y)\right] + (-3\sin 3y)\,\frac{dy}{dx} = (\cos 2y)\left(2\frac{dy}{dx}\right)$$
$$\sin y + x(\cos y)\,\frac{dy}{dx} - 3\sin(3y)\,\frac{dy}{dx} = (2\cos 2y)\,\frac{dy}{dx}.$$

Upon collecting all terms containing $\frac{dy}{dx}$ on the left-side, we get

$$[x\cos y - 3\sin 3y - 2\cos 2y]\,\frac{dy}{dx} = -\sin y$$
$$\frac{dy}{dx} = -\frac{\sin y}{x\cos y - 3\sin 3y - 2\cos 2y}$$

whenever

$$x\cos y - 3\sin 3y - 2\cos 2y \neq 0.$$

Example 3.4.4 Find $\frac{dy}{dx}$ for $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1.$

On differentiating each term with respect to x, we get

graph

$$\frac{d}{dx}\,\left(\frac{(x-2)^2}{9}\right) + \frac{d}{dx}\,\left(\frac{(y-3)^2}{16}\right) = \frac{d}{dx}\,(1)$$
$$\frac{2}{9}\,(x-2) + \frac{2}{16}\,(y-3)\,\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2(x-2)/9}{2(y-3)/16}\,,\ \text{if } y \neq 3$$
$$= -\frac{16(x-2)}{9(y-3)}\,,\ \text{if } y = 3.$$

The tangent lines are vertical at $(-1,3)$ and $(5,3)$. The graph of this equation is an ellipse.

Example 3.4.5 Find $\dfrac{dy}{dx}$ for the *astroid* $x^{2/3}+y^{2/3}=16$.

graph

$$\begin{aligned} \frac{d}{dx}\,(x^{2/3})+\frac{d}{dx}\,(y^{2/3}) &= 0\\ \frac{2}{3}\,x^{-1/3}+\frac{2}{3}\,y^{-1/3}\,\frac{dy}{dx} &= 0, \text{ if } x\neq 0 \text{ and } y\neq 0\\ \frac{dy}{dx} = -\frac{y^{-1/3}}{x^{-1/3}} &= -\left(\frac{x}{y}\right)^{1/3}, \text{ if } x\neq 0 \text{ and } y\neq 0. \end{aligned}$$

Example 3.4.6 Find $\dfrac{dy}{dx}$ for the lemniscate with equation $(x^2+y^2)^2 = 4(x^2-y^2)$.

graph

$$\begin{aligned}
\frac{d}{dx}\left((x^2+y^2)^2\right) &= 4\,\frac{d}{dx}\,(x^2-y^2)\\
2(x^2+y^2)\left(2x+2y\,\frac{dy}{dx}\right) &= 4\left[2x-2y\,\frac{dy}{dx}\right]\\
\left[4y(x^2+y^2)+8y\right]\frac{dy}{dx} &= 8x-4x(x^2+y^2) \qquad \text{(Why?)}\\
\frac{dy}{dx} &= \frac{8x-4x(x^2+y^2)}{4y(x^2+y^2)+8y}\ ,\ \text{if } 4y(x^2+y^2)+8y\neq 0, y\neq 0.
\end{aligned}$$

Example 3.4.7 Find the equations of the tangent and normal lines at (x_0, y_0) to the graph of an ellipse of the form

$$\frac{(x-k)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

First, we find $\frac{dy}{dx}$ by implicit differentiation as follows:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{(x-h)^2}{a^2}\right)+\frac{d}{dx}\left(\frac{(y-k)^2}{b^2}\right) &= \frac{d}{dx}\ (1)\\
\frac{2}{a^2}\,(x-h)+\frac{2}{b^2}\,(y-k)\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{2}{a^2}\,(x-h)\cdot\frac{b^2}{2(y-k)}\ ,\ \text{if } y\neq k\\
&= \frac{-b^2}{a^2}\left(\frac{x-h}{y-k}\right)\ , y\neq k.
\end{aligned}$$

It is clear that at $(a+h, k)$ and $(-a+h, k)$, the tangent lines are vertical and have the equations

$$x = a+h \quad \text{and} \quad x = -a+h.$$

Let (x_0, y_0) be a point on the ellipse such that $y_0 \neq k$. Then the equation of the line tangent to the ellipse at (x_0, y_0) is

$$y-y_0 = \frac{-b^2}{a^2}\left(\frac{x_0-h}{y_0-k}\right)(x-x_0).$$

We may express this in the form

$$\frac{(y-y_0)(y_0-k)}{b^2}+\frac{(x-x_0)(x_0-h)}{a^2}=0.$$

By rearranging some terms, we can simplify the equation in the following traditional form:

$$\frac{(y-k)+(k-y_0)}{b^2}\cdot(y_0-k)+\frac{(x-h)+(h-x_0)}{a^2}\,(x_0-h)=0$$

$$\frac{(y-k)(y_0-k)}{b^2}+\frac{(x-h)(x_0-h)}{a^2}=\frac{(x_0-h)^2}{a^2}+\frac{(y_0-k)^2}{b^2}=1.$$

$$\boxed{\frac{(y-k)(y_0-k)}{b^2}+\frac{(x-h)(x_0-h)}{a^2}=1}\,.$$

Exercises 3.4 In each of the following, find $\frac{dy}{dx}$ by implicit differentiation.

1. $y^2+3xy+2x^2=16$
2. $x^{3/4}+y^{3/4}=10^{3/4}$
3. $x^5+4x^3y^2+3y^4=8$
4. $\sin(x-y)=x^2y\cos x$
5. $\frac{x^2}{4}-\frac{y^2}{9}=1$
6. $\frac{x^2}{16}+\frac{y^2}{9}=1$

Find the equation of the line tangent to the graph of the given equation at the given point.

7. $\frac{x^2}{9}+\frac{y^2}{4}=1$ at $\left(2,\frac{2\sqrt{5}}{3}\right)$
8. $\frac{x^2}{9}-\frac{y^2}{4}=1$ at $\left(\frac{3}{2}\sqrt{5},1\right)$
9. $x^2y^2=(y+1)^2(9-y^2)$ at $\left(\frac{3}{2}\sqrt{5},2\right)$
10. $y^2=x^3(4-x)$ at $(2,4)$

Two curves are said to be *orthogonal* at each point (x_0, y_0) of their intersection if their tangent lines are perpendicular. Show that the following families of curves are orthogonal.

11. $x^2 + y^2 = r^2,\ y + mx = 0$

12. $(x-h)^2 + y^2 = h^2,\ x^2 + (y-k)^2 = k^2$

Compute y' and y'' in exercises 13–20.

13. $4x^2 + 9y^2 = 36$

14. $4x^2 - 9y^2 = 36$

15. $x^{2/3} + y^{2/3} = 16$

16. $x^3 + y^3 = a^3$

17. $x^2 + 4xy + y^2 = 6$

18. $\sin(xy) = x^2 + y^2$

19. $x^4 + 2x^2y^2 + 4y^4 = 26$

20. $(x^2 + y^2)^2 = x^2 - y^2$

3.5 Higher Order Derivatives

If the vertical height y of an object is a function f of time t, then $y'(t)$ is called its velocity, denoted $v(t)$. The derivative $v'(t)$ is called the acceleration of the object and is denoted $a(t)$. That is,

$$y(t) = f(t),\ y'(t) = v(t),\ v'(t) = a(t).$$

We say that $a(t)$ is the second derivative of y, with respect to t, and write

$$y''(t) = a(t) \quad \text{or} \quad \frac{d^2y}{dt^2} = a(t).$$

Derivatives of order two or more are called higher derivatives and are represented by the following notation:

$$y'(x) = \frac{dy}{dx}\ ,\ y''(x) = \frac{d^2y}{dx^2}\ ,\ y'''(x) = \frac{d^3y}{dx^3}\ , \ldots\ ,\ y^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The definition is given as follows by induction:

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) \quad \text{and} \quad \frac{d^nf}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}f}{dx^{n-1}}\right), n = 2, 3, 4, \cdots .$$

A convenient notation is

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

which is read as "the nth derivative of f with respect to x."

Example 3.5.1 Compute the second derivative y'' for each of the following functions:

(i) $y = \sin(3x)$ (ii) $y = \cos(4x^2)$ (iii) $y = \tan(3x)$

(iv) $y = \cot(5x)$ (v) $y = \sec(2x)$ (vi) $y = \csc(x^2)$

Part (i) $y' = 3\cos(3x),\ y'' = -9\sin(3x)$

Part (ii) $y' = -8x\sin(4x^2),\ y'' = -8[\sin(4x^2) + x\cdot(8x)\cdot\cos(4x^2)]$

Part(iii) $y' = 3\sec^2(3x),\ y'' = 3[2\sec(3x)\cdot\sec(3x)\tan(3x)\cdot 3]$

$$y'' = 18\sec^2(3x)\tan(3x)$$

Part(iv) $y' = -5\csc^2(5x),\ y'' = -10\csc(5x)[(-\csc 5x\cot 5x)\cdot 5]$

$$y'' = 50\csc^2(5x)\cot(5x)$$

Part(v) $y' = 2\sec(2x)\tan(2x)$

$$y'' = 2[(2\sec(2x)\tan(2x))\cdot\tan(2x) + \sec(2x)\cdot(2\sec^2(2x))]$$

$$y'' = 4\sec(2x)\tan^2(2x) + 4\sec^3(2x)$$

Part(vi) $y' = -2x\csc(x^2)\cot(x^2)$

$$y'' = -2[1\cdot\csc(x^2)\cot(x^2) + x(-2x\csc(x^2)\cot(x^2))\cdot\cot(x^2)$$

$$+ \, x\csc(x^2)\cdot(-2x\csc^2(x^2))]$$

$$= -2\csc(x^2)\cot(x^2) + 4x^2\csc(x^2)\cot^2(x^2) + 4x^2\csc^3(x^2)$$

Example 3.5.2 Compute the second order derivative of each of the following functions:

(i) $y = \sinh(3x)$ (ii) $y = \cosh(x^2)$ (iii) $y = \tanh(2x)$

(iv) $y = \coth(4x)$ (v) $y = \text{sech}(5x)$ (vi) $y = \text{csch}(10x)$

Part (i) $y' = 3\cosh(3x),\ y'' = 9\sinh(3x)$

Part (ii) $y' = 2x\sinh(x^2),\ y'' = 2\sinh(x^2) + 2x(2x\cosh x^2)$ or

$y'' = 2\sinh(x^2) + 4x^2\cosh(x^2)$

Part (iii) $y' = 2\ \text{sech}^2(2x),\ y'' = 2\cdot(2\ \text{sech}(2x)\cdot(-\text{sech}(2x)\tanh(2x)\cdot 2)),$

$y'' = -8\ \text{sech}^2(2x)\tanh(2x)$

Part (iv) $y' = -4\ \text{csch}^2(4x),\ y'' = -4(2(\text{csch}(4x))\cdot(-\text{csch}(4x)\ \coth(4x)\cdot 4))$

$y'' = 32\ \text{csch}^2(4x)\ \coth(4x)$

Part (v) $y' = -5\ \text{sech}\ (5x)\tanh(5x)$

$y' = -5[-5\ \text{sech}(5x)\tanh(5x)\cdot\tanh(5x) +\ \text{sech}(5x)\cdot\ \text{sech}^2(5x)\cdot 5]$

$y' = 25\ \text{sech}(5x)\tanh^2(5x) - 25\ \text{sech}^3(5x).$

Part (vi) $y' = -10\ \text{csch}(10x)\ \coth(10x)$

$y'' = -10[-10\ \text{csch}(10x)\ \coth(10x)\cdot\coth(10x)$

$$+ \text{csch}(10x)(-10 \text{ csch}^2(10x))]$$

$$y'' = 100 \text{ csch}(10x) \coth^2(10x) + 100 \text{ csch}^3(10x)$$

Example 3.5.3 Compute the second order derivatives for the following functions:

(i) $y = \ln(x^2)$ (ii) $y = e^{x^2}$ (iii) $\log_{10}(x^2 + 1)$

(iv) $y = 10^{x^2}$ (v) $y = \arcsin x$ (vi) $y = \arctan x$

Part (i) $y' = \dfrac{2x}{x2} = \dfrac{2}{x} = 2x^{-1}$

$$y'' = -2x^{-2} = \frac{-2}{x^2}.$$

Part (ii) $y' = 2xe^{x^2}$, $y'' = 2e^{x^2} + 4x^2e^{x^2} = (2 + 4x^2)e^{x^2}$.

Part (iii) $y' = \dfrac{1}{\ln 10} \cdot \dfrac{2x}{x^2 + 1}$, $y'' = \dfrac{2}{\ln 10} \left[\dfrac{(x^2 + 1) \cdot 1 - x \cdot 2x}{(x^2 + 1)^2}\right]$,

$$y'' = \frac{2}{\ln 10} \cdot \left[\frac{1 - x^2}{(x^2 + 1)^2}\right]$$

Part (iv) $y' = 10^{x^2} \cdot (\ln 10) \cdot 2x$

$$y'' = 2 \ln 10[10^{x^2} + x \cdot 10^{x^2} \ln 10 \cdot 2x]$$

$$y'' = 10^{x^2}[2 \ln 10 + (2 \ln 10)^2 x^2]$$

Part (v) $y' = \dfrac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2}$

$$y'' = \frac{-1}{2}\,(1-x^2)^{-3/2}(-2x)$$

$$y'' = \frac{x}{(1-x^2)^{3/2}}.$$

Part (vi) $y' = \dfrac{1}{1+x^2} = (1+x^2)^{-1}$

$$y'' = -1(1+x^2)^{-2}\cdot 2x = \frac{-2x}{(1+x^2)^2}$$

Example 3.5.4 Compute the second derivatives of the following functions:

(i) $y = \operatorname{arcsinh} x$ (ii) $y = \operatorname{arccosh} x$ (iii) $y = \operatorname{arctanh} x$

From Section 1.4, we recall that

arcsinh $x = \ln(x + \sqrt{1+x^2})$

arccosh $x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$

arctanh $x = \dfrac{1}{2}\,\ln\left(\dfrac{1+x}{1-x}\right) = \dfrac{1}{2}\,[\ln(1+x) - \ln(1-x)],\ |x| < 1.$

Then

Part (i)

$$\begin{aligned} y' &= \frac{1}{\sqrt{1+x^2}} \\ \frac{d^2}{dx^2}\,(\operatorname{arcsinh} x) &= \frac{d}{dx}\,(1+x^2)^{-1/2} \\ &= \frac{-1}{2}\,(2x)(1+x^2)^{-3/2} \\ &= -\frac{x}{(1+x^2)^{3/2}}. \end{aligned}$$

Part (ii)

$$\begin{aligned}
y' &= \frac{1}{\sqrt{x^2-1}}\ ,\ x>1 \\
\frac{d^2}{dx^2}\,(\operatorname{arccosh} x) &= \frac{d}{dx}\,(x^2-1)^{-1/2} \\
&= \frac{-1}{2}\,(2x)(x^2-1)^{-3/2} \\
&= -\frac{x}{(x^2-1)^{3/2}}\ ,\ x>1
\end{aligned}$$

Part (iii)

$$\begin{aligned}
y' &= \frac{1}{1-x^2}\ ,\ |x|<1. \\
\frac{d^2}{dx}\,(\operatorname{arctanh} x) &= \frac{d}{dx}\,(1-x^2)^{-1} \\
&= (-1)(1-x^2)^{-2}(-2x) \\
&= \frac{x}{(1-x^2)^2}\ ,\ |x|<1.
\end{aligned}$$

Example 3.5.5 Find y'' for the equation $x^2+y^2=4$.

First, we find y' by implicit differentiation.

$$2x+2yy'=0 \to y', \frac{x}{y}.$$

Now, we differentiate again with respect to x.

$$\begin{aligned}
y'' &= \frac{y\cdot 1 - xy'}{y^2} \\
&= -\frac{y-x(-x/y)}{y^2} && \text{(replace } y' \text{ by } -x/y) \\
&= -\frac{y^2+x^2}{y^3} && \text{(Why?)} \\
&= -\frac{4}{y^3} && \text{(since } x^2+y^2=4)
\end{aligned}$$

Example 3.5.6 Compute y'' for $x^3 + y^3 = 4xy$.

From Example 25 in the last section we found that

$$y' = \frac{4y - 3x^2}{3y^2 - 4x} \quad \text{if} \quad 3y^2 - 4x \neq 0.$$

To find y'', we differentiate y' with respect to x to get

$$y'' = \frac{(3y^2 - 4x)(4y' - 3x^2) - (4y - 3x^2)(6yy' - 4)}{3y^2 - 4x}\ ,\ 3y^2 - 4x \neq 0.$$

In order to simplify any further, we must first replace y' by its computed value. We leave this as an exercise.

Example 3.5.7 Compute $f^{(n)}(c)$ for the given f and c and all natural numbers n:

(i) $f(x) = \sin x,\ c = 0$ (ii) $f(x) = \cos x,\ x = 0$ (iii) $f(x) = \ln(x),\ c = 1$

(iv) $f(x) = e^x,\ c = 0$ (v) $f(x) = \sinh x,\ x = 0$ (vi) $f(x) = \cosh x,\ x = 0$

To compute the general nth derivative formula we must discover a pattern and then generalize the pattern.

Part (i) $f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = \cos x, f^4(x) = \sin x$. Then the next four derivatives are repeated and so on. We get

$$f^{(4n)}(n) = \sin x,\ f^{(4n+1)}(x) = \cos x,\ f^{(4n+2)}(x) = -\sin x,\ f^{(4n+3)}(x) = -\cos x.$$

By evaluating these at $c = 0$, we get

$$f^{(4n)}(0) = 0,\ f^{(4n+2)}(0);\ f^{(4n+1)}(0) = 1 \ \text{ and } \ f^{(4n+3)}(0) = -1,$$

for $n = 0, 1, 2, \cdots$

Part (ii) This part is similar to Part (i) and is left as an exercise.

Part (iii) $f(x) = \ln x,\ f'(x) = x^{-1},\ f''(x) = (-1)x^{-2},\ f^{(3)}(x) = (-1)(-2)x^{-3}, \ldots.,$ $f^{(n)}(x) = (-1)(-2)\ldots(-(n-1))x^{-n} = (-1)^{n-1}(n-1)!x^{-n},\ f^{(n)}(1) = (-1)^{n-1}(n-1)!,\ n = 1, 2, \ldots$

Part (iv) $f(x) = e^x,\ f'(x) = e^x,\ f''(x) = e^x, \ldots,\ f^{(n)}(x) = e^x,\ f^{(n)}(0) = 1,\ n = 0, 1, 2, \ldots$

Part (v) $f(x) = \sinh x, f'(x) = \cosh x, f''(x) = \sinh x, \ldots f^{(2n)}(x) = \sinh x$, $f^{(2n+1)}(x) = \cosh x, f^{(2n)}(0) = 0,\ f^{(2n+1)}(0) = 1,\ n = 0, 1, 2, \ldots$

Part (vi) $f(x) = \cosh x,\ f'(x) = \sinh x,\ f''(x) = \cosh x, \ldots, f^{(2n)}(x) = \cosh x,\ f^{(2n+1)}(x) = \sinh x,\ f^{(2n)}(0) = 1, f^{(2n+1)}(0) = 0, n = 0, 1, 2, \ldots$

Exercises 3.5 Find the first two derivatives of each of the following functions f.

1. $f(t) = 4t^3 - 3t^2 + 10$
2. $f(x) = 4\sin(3x) + 3\cos(4x)$
3. $f(x) = (x^2 + 1)^3$
4. $f(x) = x^2 \sin(3x)$
5. $f(x) = e^{3x} \sin 4x$
6. $f(x) = e^{2x} \cos 4x$
7. $f(x) = \dfrac{x^2}{2x + 1}$
8. $f(x) = (x^2 + 1)^{10}$
9. $f(x) = \ln(x^2 + 1)$
10. $f(x) = \log_{10}(x^4 + 1)$
11. $f(x) = 3\sinh(4x) + 5\cosh(4x)$
12. $f(x) = \tanh(3x)$
13. $f(x) = x \tan x$
14. $f(x) = x^2 e^x$
15. $f(x) = \arctan(3x)$
16. $f(x) = \operatorname{arcsinh}(2x)$
17. $f(x) = \cos(nx)$
18. $f(x) = (x^2 + 1)^{100}$

Show that the given $y(x)$ satisfies the given equation:

19. $y = A\sin(4x) + B\cos(4x)$ satisfies $y'' + 16y = 0$
20. $y = A\sinh(4x) + B\cosh(4x)$ satisfies $y'' - 16y = 0$

21. $y = e^{-x}(a\sin(2x) + b\cos(2x))$ satisfies $y'' - 2y' + 2y = 0$

22. $y = e^{x}(a\sin(3x) + b\cos(3x))$ satisfies $y'' - 2y' + 10y = 0$

Compute the general nth derivative for each of the following:

23. $f(x) = e^{2x}$

24. $f(x) = \sin 3x$

25. $f(x) = \cos 4x$

26. $f(x) = \ln(x+1)$

27. $f(x) = \sinh(2x)$

28. $f(x) = \cosh(3x)$

29. $f(x) = (x+1)^{100}$

30. $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

Find y' and y'' for the following equations:

31. $x^4 + y^4 = 20$

32. $x^2 + xy + y^2 = 16$