Questions Bank
Calculus I
First Class

Part 1

PRELIMINARY MATERIAL

CHAPTER 1

INEQUALITIES AND ABSOLUTE VALUES

1.1. Background

Topics: inequalities, absolute values.

1.1.1. Definition. If x and a are two real numbers the DISTANCE between x and a is $|x-a|$. For most purposes in calculus it is better to think of an inequality like $|x-5|<2$ geometrically rather then algebraically. That is, think "The number x is within 2 units of 5," rather than "The absolute value of x minus 5 is strictly less than 2." The first formulation makes it clear that x is in the open interval $(3,7)$.

1.1.2. Definition. Let a be a real number. A NEIGHBORHOOD of a is an open interval (c,d) in $\mathbb{R}$ which contains a. An open interval $(a-\delta, a+\delta)$ which is centered at a is a SYMMETRIC NEIGHBORHOOD (or a δ-NEIGHBORHOOD) of a.

1.1.3. Definition. A DELETED (or PUNCTURED) NEIGHBORHOOD of a point $a \in \mathbb{R}$ is an open interval around a from which a has been deleted. Thus, for example, the deleted δ-neighborhood about 3 would be $(3-\delta, 3+\delta) \setminus \{3\}$ or, using different notation, $(3-\delta, 3) \cup (3, 3+\delta)$.

1.1.4. Definition. A point a is an ACCUMULATION POINT of a set $B \subseteq \mathbb{R}$ if every deleted neighborhood of a contains at least one point of B.

1.1.5. Notation (For Set Operations)**.** Let A and B be subsets of a set S. Then

(1) $x \in A \cup B$ if $x \in A$ or $x \in B$ (*union*);
(2) $x \in A \cap B$ if $x \in A$ and $x \in B$ (*intersection*);
(3) $x \in A \setminus B$ if $x \in A$ and $x \notin B$ (*set difference*); and
(4) $x \in A^c$ if $x \in S \setminus A$ (*complement*).

If the set S is not specified, it is usually understood to be the set $\mathbb{R}$ of real numbers or, starting in Part 6, the set $\mathbb{R}^n$, Euclidean n-dimensional space.

1.2. Exercises

(1) The inequality $|x-2|<6$ can be expressed in the form $a<x<b$ where $a =$ _____ and $b =$ _____ .

(2) The inequality $-15 \le x \le 7$ can be expressed in the form $|x-a| \le b$ where $a =$ _____ and $b =$ _____ .

(3) Solve the equation $|4x+23| = |4x-9|$. Answer: $x =$ ________ .

(4) Find all numbers x which satisfy $|x^2+2| = |x^2-11|$.

Answer: $x =$ ________ and $x =$ ________ .

(5) Solve the inequality $\dfrac{3x}{x^2+2} \ge \dfrac{1}{x-1}$. Express your answer in interval notation.

Answer: [_____ , _____) $\cup$ [2 , _____) .

(6) Solve the equation $|x-2|^2 + 3|x-2| - 4 = 0$.

Answer: $x =$ _____ and $x =$ _____ .

(7) The inequality $-4 \le x \le 10$ can be expressed in the form $|x-a| \le b$ where $a =$ _____ and $b =$ _____ .

(8) Sketch the graph of the equation $x-2 = |y-3|$.

(9) The inequality $|x+4| < 7$ can be expressed in the form $a<x<b$ where $a =$ _____ and $b =$ _____ .

(10) Solve the inequality $|3x+7| < 5$. Express your answer in interval notation.

Answer: (_____ , _____).

(11) Find all numbers x which satisfy $|x^2-9| = |x^2-5|$.

Answer: $x =$ _____ and $x =$ _____ .

(12) Solve the inequality $\left|\dfrac{2x^2-3}{14}\right| \le \dfrac{1}{2}$. Express your answer in interval notation.

Answer: [_____ , _____].

(13) Solve the inequality $|x-3| \ge 6$. Express your answer in interval notation.

Answer: (_____ , _____] $\cup$ [_____ , _____) .

(14) Solve the inequality $\dfrac{x}{x+2} \ge \dfrac{x+3}{x-4}$. Express your answer in interval notation.

Answer: (_____ , _____) $\cup$ [_____ , _____).

(15) In interval notation the solution set for the inequality $\dfrac{x+1}{x-2} \le \dfrac{x+2}{x+3}$ is $(-\infty,$ _____) $\cup$ [_____ , 2).

(16) Solve the inequality $\dfrac{4x^2-x+19}{x^3+x^2+4x+4} \ge 1$. Express your answer in interval notation.

Answer: (_____ , _____].

(17) Solve the equation $2|x+3|^2 - 15|x+3| + 7 = 0$.

Answer: $x =$ _____ , $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(18) Solve the inequality $x \ge 1 + \dfrac{2}{x}$. Express your answer in interval notation.

Answer: [_____ , 0) $\cup$ [_____ , _____).

1.3. Problems

(1) Let a, $b \in \mathbb{R}$. Show that $\big|\,|a| - |b|\,\big| \le |a - b|$.

(2) Let a, $b \in \mathbb{R}$. Show that $|ab| \le \frac{1}{2}(a^2 + b^2)$.

1.4. Answers to Odd-Numbered Exercises

(1) -4, 8

(3) $-\dfrac{7}{4}$

(5) $[-\frac{1}{2}, 1) \cup [2, \infty)$

(7) 3, 7

(9) -11, 3

(11) $-\sqrt{7}$, $\sqrt{7}$

(13) $(-\infty, -3] \cup [9, \infty)$

(15) $(-\infty, -3) \cup [-\frac{7}{4}, 2)$

(17) -10, $-\dfrac{7}{2}$, $-\dfrac{5}{2}$, 4

CHAPTER 2

LINES IN THE PLANE

2.1. Background

Topics: equations of lines in the plane, slope, x- and y-intercepts, parallel and perpendicular lines.

2.1.1. Definition. Let (x_1, y_1) and (x_2, y_2) be points in the plane such that $x_1 \neq x_2$. The SLOPE of the (nonvertical straight) line L which passes through these points is

$$m_L := \frac{y_2 - y_1}{x_2 - x_1}.$$

The *equation* for L is

$$y - y_0 = m_L(x - x_0)$$

where (x_0, y_0) is any point lying on L. (If the line L is vertical (that is, parallel to the y-axis) it is common to say that it has *infinite slope* and write $m_L = \infty$. The equation for a vertical line is $x = x_0$ where (x_0, y_0) is any point lying on L.)

Two nonvertical lines L and L' are PARALLEL if their respective slopes m_L and $m_{L'}$ are equal. (Any two vertical lines are parallel.) They are PERPENDICULAR if their respective slopes are negative reciprocals; that is, if $m_{L'} = \dfrac{1}{m_L}$. (Vertical lines are always perpendicular to horizontal lines.)

2.2. Exercises

(1) The equation of the line passing through the points $(-7,-3)$ and $(8,2)$ is $ay = x+b$ where $a =$ _____ and $b =$ _____

(2) The equation of the perpendicular bisector of the line segment joining the points $(2,-5)$ and $(4,3)$ is $ax + by + 1 = 0$ where $a =$ _____ and $b =$ _____ .

(3) Let L be the line passing through the point $(4,9)$ with slope $\frac{3}{4}$. The x-intercept of L is _____ and its y-intercept is _____ .

(4) The equation of the line which passes through the point $(4,2)$ and is perpendicular to the line $x + 2y = 1$ is $ax + by + 1 = 0$ where $a =$ ________ and $b =$ ________ .

(5) The equation of the line which is parallel to the line $x + \frac{3}{2}y = \frac{5}{2}$ and passes through the point $(-1,-3)$ is $2x + ay + b = 0$ where $a =$ ________ and $b =$ ________ .

2.3. Problems

(1) The town of Plainfield is 4 miles east and 6 miles north of Burlington. Allentown is 8 miles west and 1 mile north of Plainfield. A straight road passes through Plainfield and Burlington. A second straight road passes through Allentown and intersects the first road at a point somewhere south and west of Burlington. The angle at which the roads intersect is $\pi/4$ radians. Explain how to find the location of the point of intersection and carry out the computation you describe.

(2) Prove that the line segment joining the midpoints of two sides of a triangle is half the length of the third side and is parallel to it. *Hint.* Try not to make things any more complicated than they need to be. A thoughtful choice of a coordinate system may be helpful. One possibility: orient the triangle so that one side runs along the x-axis and one vertex is at the origin.

2.4. Answers to Odd-Numbered Exercises

(1) 3, -2

(3) -8, 6

(1) 3, 11

CHAPTER 3

FUNCTIONS

3.1. Background

Topics: functions, domain, codomain, range, bounded above, bounded below, composition of functions.

3.1.1. Definition. If S and T are sets we say that f is a FUNCTION from S to T if for every x in S there corresponds one and only one element $f(x)$ in T. The set S is called the DOMAIN of f and is denoted by $\operatorname{dom} f$. The set T is called the CODOMAIN of f. The RANGE of f is the set of all $f(x)$ such that x belongs to S. It is denoted by $\operatorname{ran} f$. The words *function, map, mapping*, and *transformation* are synonymous.

A function $f\colon A \to B$ is said to be *real valued* if $B \subseteq \mathbb{R}$ and is called a *function of a real variable* if $A \subseteq \mathbb{R}$.

The notation $f\colon S \to T\colon x \mapsto f(x)$ indicates that f is a function whose domain is S, whose codomain is T, and whose value at x is $f(x)$. Thus, for example, $f\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x^2$ defines the real valued function whose value at each real number x is given by $f(x) = x^2$. We use $\operatorname{dom} f$ to denote the domain of f and $\operatorname{ran} f$ to denote its range.

3.1.2. Definition. A function $f\colon S \to \mathbb{R}$ is BOUNDED ABOVE by a number M is $f(x) \le M$ for every $x \in S$, It is BOUNDED BELOW by a number K if $K \le f(x)$ for every $x \in S$. And it is BOUNDED if it is bounded both above and below; that is, if there exists $N > 0$ such that $|f(x)| \le N$ for every $x \in S$.

3.1.3. Definition. Let f and g be real valued functions of a real variable. Define the COMPOSITE of g and f, denoted by $g \circ f$, by

$$(g \circ f)(x) := g(f(x))$$

for all $x \in \operatorname{dom} f$ such that $f(x) \in \operatorname{dom} g$. The operation $\circ$ is called *composition*.

For problem 2, the following fact may be useful.

3.1.4. Theorem. *Every nonempty open interval in* $\mathbb{R}$ *contains both rational and irrational numbers.*

3.2. Exercises

(1) Let $f(x) = \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{x}}}$. Then:

(a) $f(\frac{1}{2}) =$ ________ .

(b) The domain of f is the set of all real numbers except _____ , ______ , and _____ .

(2) Let $f(x) = \dfrac{7 - \sqrt{x^2 - 9}}{\sqrt{25 - x^2}}$. Then $\operatorname{dom} f = (\ \_\_\_\_\ ,\ \_\_\_\_\ ] \cup [\ \_\_\_\_\ ,\ \_\_\_\_\ )$.

(3) Find the domain and range of the function $f(x) = 2\sqrt{4 - x^2} - 3$.

Answer: $\operatorname{dom} f = [\ \_\_\_\_\ ,\ \_\_\_\_\ ]$ and $\operatorname{ran} f = [\ \_\_\_\_\ ,\ \_\_\_\_\ ]$.

(4) Let $f(x) = x^3 - 4x^2 - 11x - 190$. The set of all numbers x such that $|f(x) - 40| < 260$ is $(\ \_\_\_\_\ ,\ \_\_\_\_\ ) \cup (\ \_\_\_\_\ ,\ \_\_\_\_\ )$.

(5) Let $f(x) = x + 5$, $g(x) = \sqrt{x}$, and $h(x) = x^2$. Then $(g \circ (h - (g \circ f)))(4) =$ ________ .

(6) Let $f(x) = \dfrac{1}{1 - \dfrac{2}{1 + \dfrac{1}{1 - x}}}$.

(a) Find $f(1/2)$. Answer. _____ .

(b) Find the domain of f. Answer. The domain of f is the set of all real numbers *except* _____ , _____ , and _____ .

(7) Let $f(x) = \dfrac{\sqrt{x^2 - 4}}{5 - \sqrt{36 - x^2}}$. Then, in interval notation, that part of the domain of f which is to the right of the origin is $[2, a) \cup (a, b]$ where $a =$ ______ and $b =$ _____ .

(8) Let $f(x) = (-x^2 - 7x - 10)^{-1/2}$.

(a) Then $f(-3) =$ ________ .

(b) The domain of f is (_____ , _____) .

(9) Let $f(x) = x^3 - 4$ for all real numbers x. Then for all $x \neq 0$ define a new function g by $g(x) = (2x)^{-1}(f(1 + x) - f(1 - x))$. Then $g(x)$ can be written in the form $ax^2 + bx + c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(10) The cost of making a widget is 75 cents. If they are sold for \$1.95 each, 3000 widgets can be sold. For every cent the price is lowered, 60 more widgets can be sold.

(a) If x is the price of a widget in cents, then the net profit is $p(x) = ax^2 + bx + c$ where $a =$ ______ , $b =$ ____________ , and $c =$ ______________________ .

(b) The "best" price (that is, the price that maximizes profit) is $x = \$$ ___ . ___ ___ .

(c) At this best price the profit is \$______________ .

(11) Let $f(x) = 3\sqrt{25 - x^2} + 2$. Then $\operatorname{dom} f = [\ \_\_\_\_\ ,\ \_\_\_\_\ ]$ and $\operatorname{ran} f = [\ \_\_\_\_\ ,\ \_\_\_\_\ ]$.

(12) Find a formula exhibiting the area A of an equilateral triangle as a function of the length s of one of its sides.

Answer: $A(s) =$ _________ .

(13) Let $f(x) = 4x^3 - 18x^2 - 4x + 33$. Find the largest set S on which the function f is bounded above by 15 and below by -15.

Answer: $S = [\ \_\_\_\_\ ,\ \_\_\_\_\ ] \cup [\ \_\_\_\_\ ,\ \_\_\_\_\ ] \cup [\ \_\_\_\_\ ,\ \_\_\_\_\ ]$.

(14) Let $f(x) = \sqrt{x}$, $g(x) = \dfrac{4}{5-x}$, and $h(x) = x^2$. Find $(h \circ ((h \circ g \circ f) - f))(4)$.

Answer: ____________ .

(15) Let $f(x) = x + 7$, $g(x) = \sqrt{x+2}$, and $h(x) = x^2$. Find $(h \circ ((f \circ g) - (g \circ f)))(7)$.

Answer: ____________ .

(16) Let $f(x) = \sqrt{5-x}$, $g(x) = \sqrt{x+11}$, $h(x) = 2(x-1)^{-1}$, and $j(x) = 4x - 1$.

Then $(f \circ (g + (h \circ g)(h \circ j)))(5) =$ ____________ .

(17) Let $f(x) = x^2$, $g(x) = \sqrt{9+x}$, and $h(x) = \dfrac{1}{x-2}$. Then $(h \circ (f \circ g - g \circ f))(4) =$ ______ .

(18) Let $f(x) = x^2$, $g(x) = \sqrt{9+x}$, and $h(x) = (x-1)^{1/3}$.

Then $(h \circ ((f \circ g)(g \circ f)))(4) =$ _____ .

(19) Let $f(x) = \dfrac{5}{x}$, $g(x) = \sqrt{x}$, and $h(x) = x + 1$. Then $(g(f \circ g) + (g \circ f \circ h))(4) =$ _____ .

(20) Let $g(x) = 5 - x^2$, $h(x) = \sqrt{x+13}$, and $j(x) = \dfrac{1}{x}$. Then $(j \circ h \circ g)(3) =$ ______ .

(21) Let $h(x) = \dfrac{1}{\sqrt{x+6}}$, $j(x) = \dfrac{1}{x}$, and $g(x) = 5 - x^2$. Then $(g \circ j \circ h)(3) =$ ______ .

(22) Let $f(x) = x^2 + \dfrac{2}{x}$, $g(x) = \dfrac{2}{2x+3}$, and $h(x) = \sqrt{2x}$. Then $(h \circ g \circ f)(4) =$ ______ .

(23) Let $f(x) = 3(x+1)^3$, $g(x) = \dfrac{x^5 + x^4}{x+1}$, and $h(x) = \sqrt{x}$.

Then $(h \circ (g + (h \circ f)))(2) =$ ______ .

(24) Let $f(x) = x^2$, $g(x) = \sqrt{x+11}$, $h(x) = 2(x-1)^{-1}$, and $j(x) = 4x - 1$.

Then $(f \circ ((h \circ g) + (h \circ j)))(5) =$ _______ .

(25) Let $f(x) = x^3 - 5x^2 + x - 7$. Find a function g such that $(f \circ g)(x) = 27x^3 + 90x^2 + 78x - 2$.

Answer: $g(x) =$ ______________ .

(26) Let $f(x) = \cos x$ and $g(x) = x^2$ for all x. Write each of the following functions in terms of f and g. *Example.* If $h(x) = \cos^2 x^2$, then $h = g \circ f \circ g$.

(a) If $h(x) = \cos x^2$, then $h =$ _________ .

(b) If $h(x) = \cos x^4$, then $h=$ ______________ .

(c) If $h(x) = \cos^4 x^2$, then $h =$ ____________________ .

(d) If $h(x) = \cos(\cos^2 x)$, then $h =$ ______________ .

(e) If $h(x) = \cos^2(x^4 + x^2)$, then $h =$ ______________________________ .

(27) Let $f(x) = x^3$, $g(x) = x - 2$, and $h(x) = \sin x$ for all x. Write each of the following functions in terms of f, g, and h. *Example.* If $k(x) = \sin^3(x-2)^3$, then $k = f \circ h \circ f \circ g$.

(a) If $k(x) = \sin^3 x$, then $k =$ ___________ .

(b) If $k(x) = \sin x^3$, then $k =$ ___________ .

(c) If $k(x) = \sin(x^3 - 2)$, then $k =$ ______________ .

(d) If $k(x) = \sin(\sin x - 2)$, then $k =$ ______________ .

(e) If $k(x) = \sin^3(\sin^3(x-2)$, then $k =$ __________________________ .

(f) If $k(x) = \sin^9(x-2)$, then $k =$ ____________________ .

(g) If $k(x) = \sin(x^3 - 8)$, then $k =$ ______________________________ .

(h) If $k(x) = \sin(x^3 - 6x^2 + 12x - 8)$, then $k =$ ______________ .

(28) Let $g(x) = 3x - 2$. Find a function f such that $(f \circ g)(x) = 18x^2 - 36x + 19$.
Answer: $f(x) =$ ____________________ .

(29) Let $h(x) = \arctan x$ for $x \geq 0$, $g(x) = \cos x$, and $f(x) = (1 - x^2)^{-1}$. Find a number p such that $(f \circ g \circ h)(x) = 1 + x^p$. Answer: $p =$ ________ .

(30) Let $f(x) = 3x^2 + 5x + 1$. Find a function g such that $(f \circ g)(x) = 3x^4 + 6x^3 - 4x^2 - 7x + 3$.
Answer: $g(x) =$ ____________________ .

(31) Let $g(x) = 2x - 1$. Find a function f such that $(f \circ g)(x) = 8x^3 - 28x^2 + 28x - 14$.
Answer: $f(x) =$ __________________________ .

(32) Find two solutions to the equation

$$8\cos^3(\pi(x^2 + \tfrac{8}{3}x + 2)) + 16\cos^2(\pi(x^2 + \tfrac{8}{3}x + 2)) + 16\cos(\pi(x^2 + \tfrac{8}{3}x + 2)) = 13.$$

Answer: ________ and ________ .

(33) Let $f(x) = (x + 4)^{-1/2}$, $g(x) = x^2 + 1$, $h(x) = (x - 3)^{1/2}$, and $j(x) = x^{-1}$.
Then $(j \circ ((g \circ h) - (g \circ f)))(5) =$ ________ .

(34) Let $g(x) = 3x - 2$. Find a function f such that $(f \circ g)(x) = 18x^2 - 36x + 19$.
Answer: $f(x) =$ ____________________ .

(35) Let $f(x) = x^3 - 5x^2 + x - 7$. Find a function g such that $(f \circ g)(x) = 27x^3 + 90x^2 + 78x - 2$.
Answer: $g(x) =$ ________________ .

(36) Let $f(x) = x^2 + 1$. Find a function g such that $(f \circ g)(x) = 2 + \dfrac{2}{x} + \dfrac{1}{x^2}$.
Answer: $g(x) =$ _________________ .

(37) Let $f(x) = x^2 + 3x + 4$. Find two functions g such that $(f \circ g)(x) = 4x^2 - 6x + 4$.
Answer: $g(x) =$ _________ and $g(x) =$ _________ .

(38) Let $h(x) = x^{-1}$ and $g(x) = \sqrt{x} + 1$. Find a function f such that $(f \circ g \circ h)(x) = x^{-3/2} + 4x^{-1} + 2x^{-1/2} - 6$.
Answer: $f(x) =$ _____________________ .

(39) Let $g(x) = x^2 + x - 1$. Find a function f such that $(f \circ g)(x) = x^4 + 2x^3 - 3x^2 - 4x + 6$.
Answer: $f(x) =$ ___________________ .

(40) Let $S(x) = x^2$ and $P(x) = 2^x$.
Then $(S \circ S \circ S \circ S \circ P \circ P)(-1) =$ ___________ .

3.3. Problems

(1) Do there exist functions f and g defined on $\mathbb{R}$ such that

$$f(x) + g(y) = xy$$

for all real numbers x and y? Explain.

(2) Your friend Susan has become interested in functions $f\colon \mathbb{R} \to \mathbb{R}$ which preserve both the operation of addition and the operation of multiplication; that is, functions f which satisfy

$$f(x+y) = f(x) + f(y) \tag{3.1}$$

and

$$f(xy) = f(x)f(y) \tag{3.2}$$

for all $x, y \in \mathbb{R}$. Naturally she started her investigation by looking at some examples. The trouble is that she was able to find only two very simple examples: $f(x) = 0$ for all x and $f(x) = x$ for all x. After expending considerable effort she was unable to find additional examples. She now conjectures that there are no other functions satisfying (3.1) and (3.2). Write Susan a letter explaining why she is correct.

Hint. You may choose to pursue the following line of argument. Assume that f is a function (not identically zero) which satisfies (3.1) and (3.2) above.

(a) Show that $f(0) = 0$. [In (3.1) let $y = 0$.]
(b) Show that if $a \neq 0$ and $a = ab$, then $b = 1$.
(c) Show that $f(1) = 1$. [How do we know that there exists a number c such that $f(c) \neq 0$? Let $x = c$ and $y = 1$ in (3.2).]
(d) Show that $f(n) = n$ for every natural number n.
(e) Show that $f(-n) = -n$ for every natural number n. [Let $x = n$ and $y = -n$ in (3.1). Use (d).]
(f) Show that $f(1/n) = 1/n$ for every natural number n. [Let $x = n$ and $y = 1/n$ in (3.2).]
(g) Show that $f(r) = r$ for every rational number r. [If $r \geq 0$ write $r = m/n$ where m and n are natural numbers; then use (3.2), (d), and (e). Next consider the case $r < 0$.]
(h) Show that if $x \geq 0$, then $f(x) \geq 0$. [Write x as $\sqrt{x}\sqrt{x}$ and use (3.2).]
(i) Show that if $x \leq y$, then $f(x) \leq f(y)$. [Show that $f(-x) = -f(x)$ holds for all real numbers x. Use (h).]
(j) Now prove that f must be the identity function on $\mathbb{R}$. [Argue by contradiction: Assume $f(x) \neq x$ for some number x. Then there are two possibilities: either $f(x) > x$ or $f(x) < x$. Show that both of these lead to a contradiction. Apply theorem 3.1.4 to the two cases $f(x) > x$ and $f(x) < x$ to obtain the contradiction $f(x) < f(x)$.]

(3) Let $f(x) = 1 - x$ and $g(x) = 1/x$. Taking composites of these two functions in all possible ways ($f \circ f$, $g \circ f$, $f \circ g \circ f \circ f \circ f$, $g \circ g \circ f \circ g \circ f \circ f$, etc.), how many distinct functions can be produced? Write each of the resulting functions in terms of f and g. How do you know there are no more? Show that each function on your list has an inverse which is also on your list. What is the common domain for these functions? That is, what is the largest set of real numbers for which all these functions are defined?

(4) Prove or disprove: composition of functions is commutative; that is $g \circ f = f \circ g$ when both sides are defined.

(5) Let f, g, $h\colon \mathbb{R} \to \mathbb{R}$. Prove or disprove: $f \circ (g + h) = f \circ g + f \circ h$.

(6) Let f, g, $h\colon \mathbb{R} \to \mathbb{R}$. Prove or disprove: $(f + g) \circ h = (f \circ h) + (g \circ h)$.

(7) Let $a \in \mathbb{R}$ be a constant and let $f(x) = a - x$ for all $x \in \mathbb{R}$. Show that $f \circ f = I$ (where I is the identity function on $\mathbb{R}$: $I(x) = x$ for all x).

3.4. Answers to Odd-Numbered Exercises

(1) (a) $\dfrac{3}{4}$

(b) $-1,\ -\dfrac{1}{2},\ 0$

(3) $[-2,2],\ [-3,1]$

(5) $\sqrt{13}$

(7) $\sqrt{11},\ 6$

(9) 1, 0, 3

(11) $[-5,5],\ [2,17]$

(13) $[-\frac{3}{2},-1] \cup [1,2] \cup [4,\frac{9}{2}]$

(15) 36

(17) $\dfrac{1}{6}$

(19) 6

(21) -4

(23) 5

(25) $3x+5$

(27) (a) $f \circ h$
(b) $h \circ f$
(c) $h \circ g \circ f$
(d) $h \circ g \circ h$
(e) $f \circ h \circ f \circ h \circ g$
(f) $f \circ f \circ h \circ g$
(g) $h \circ g \circ g \circ g \circ g \circ f$
(h) $h \circ f \circ g$

(29) -2

(31) $x^3 - 4x^2 + 3x - 6$

(33) $\dfrac{9}{17}$

(35) $3x+5$

(37) $-2x,\ 2x-3$

(39) $x^2 - 2x + 3$

Part 2

LIMITS AND CONTINUITY

CHAPTER 4

LIMITS

4.1. Background

Topics: limit of $f(x)$ as x approaches a, limit of $f(x)$ as x approaches infinity, left- and right-hand limits.

4.1.1. Definition. Suppose that f is a real valued function of a real variable, a is an accumulation point of the domain of f, and $\ell \in \mathbb{R}$. We say that ℓ is the *limit of* $f(x)$ *as* x *approaches* a if for every neighborhood V of ℓ there exists a corresponding deleted neighborhood U of a which satisfies the following condition:

for every point x in the domain of f which lies in U the point $f(x)$ lies in V.

Once we have convinced ourselves that in this definition it doesn't matter if we work only with symmetric neighborhoods of points, we can rephrase the definition in a more conventional algebraic fashion: ℓ is the *limit of* $f(x)$ *as* x *approaches* a provided that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ and $x \in \operatorname{dom} f$, then $|f(x) - \ell| < \epsilon$.

4.1.2. Notation. To indicate that a number ℓ is the limit of $f(x)$ as x approaches a, we may write either

$$\lim_{x \to a} f(x) = l \qquad \text{or} \qquad f(x) \to \ell \text{ as } x \to a.$$

(See problem 2.)

4.2. Exercises

(1) $\lim_{x\to 3} \dfrac{x^3-13x^2+51x-63}{x^3-4x^2-3x+18} = \dfrac{a}{5}$ where $a =$ ______ .

(2) $\lim_{x\to 0} \dfrac{\sqrt{x^2+9x+9}-3}{x} = \dfrac{a}{2}$ where $a =$ _____ .

(3) $\lim_{x\to 1} \dfrac{x^3-x^2+2x-2}{x^3+3x^2-4x} = \dfrac{3}{a}$ where $a =$ _____ .

(4) $\lim_{t\to 0} \dfrac{t}{\sqrt{4-t}-2} =$ ______ .

(5) $\lim_{x\to 0} \dfrac{\sqrt{x+9}-3}{x} = \dfrac{1}{a}$ where $a =$ _____ .

(6) $\lim_{x\to 2} \dfrac{x^3-3x^2+x+2}{x^3-x-6} = \dfrac{1}{a}$ where $a =$ ______ .

(7) $\lim_{x\to 2} \dfrac{x^3-x^2-8x+12}{x^3-10x^2+28x-24} = -\dfrac{a}{4}$ where $a =$ _____ .

(8) $\lim_{x\to 0} \dfrac{\sqrt{x^2-x+4}-2}{x^2+3x} = -\dfrac{1}{a}$ where $a =$ _______ .

(9) $\lim_{x\to 1} \dfrac{x^3+x^2-5x+3}{x^3-4x^2+5x-2} =$ ______ .

(10) $\lim_{x\to 3} \dfrac{x^3-4x^2-3x+18}{x^3-8x^2+21x-18} =$ _____ .

(11) $\lim_{x\to -1} \dfrac{x^3-x^2-5x-3}{x^3+6x^2+9x+4} = -\dfrac{4}{a}$ where $a =$ _____ .

(12) $\lim_{x\to 0} \dfrac{2x\sin x}{1-\cos x} =$ _____ .

(13) $\lim_{x\to 0} \dfrac{1-\cos x}{3x\sin x} = \dfrac{1}{a}$ where $a =$ _____ .

(14) $\lim_{x\to 0} \dfrac{\tan 3x-\sin 3x}{x^3} = \dfrac{a}{2}$ where $a =$ ______ .

(15) $\lim_{h\to 0} \dfrac{\sin 2h}{5h^2+7h} =$ ________ .

(16) $\lim_{h\to 0} \dfrac{\cot 7h}{\cot 5h} =$ ________ .

(17) $\lim_{x\to 0} \dfrac{\sec x-\cos x}{3x^2} = \dfrac{1}{a}$ where $a =$ _____ .

(18) $\lim_{x\to \infty} \dfrac{(9x^8-6x^5+4)^{1/2}}{(64x^{12}+14x^7-7)^{1/3}} = \dfrac{a}{4}$ where $a =$ _____ .

(19) $\lim_{x\to \infty} \sqrt{x}(\sqrt{x+3}-\sqrt{x-2}) = \dfrac{a}{2}$ where $a =$ _____ .

(20) $\lim_{x\to \infty} \dfrac{7-x+2x^2-3x^3-5x^4}{4+3x-x^2+x^3+2x^4} = \dfrac{a}{2}$ where $a =$ ______ .

(21) $\lim_{x\to \infty} \dfrac{(2x^4-137)^5}{(x^2+429)^{10}} =$ _______ .

(22) $\displaystyle\lim_{x\to\infty} \frac{(5x^{10}+32)^3}{(1-2x^6)^5} = -\frac{a}{32}$ where $a =$ __________ .

(23) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \frac{1}{a}$ where $a =$ _____ .

(24) $\displaystyle\lim_{x\to\infty} x(256x^4+81x^2+49)^{-1/4} = \frac{1}{a}$ where $a =$ _____ .

(25) $\displaystyle\lim_{x\to\infty} x\left(\sqrt{3x^2+22} - \sqrt{3x^2+4}\right) = a\sqrt{a}$ where $a =$ _____ .

(26) $\displaystyle\lim_{x\to\infty} x^{\frac{2}{3}}\left((x+1)^{\frac{1}{3}} - x^{\frac{1}{3}}\right) = \frac{1}{a}$ where $a =$ _____ .

(27) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}}\right) =$ _____ .

(28) Let $f(x) = \begin{cases} 2x-1, & \text{if } x<2; \\ x^2+1, & \text{if } x>2. \end{cases}$ Then $\displaystyle\lim_{x\to 2^-} f(x) =$ _____ and $\displaystyle\lim_{x\to 2^+} f(x) =$ _____ .

(29) Let $f(x) = \dfrac{|x-1|}{x-1}$. Then $\displaystyle\lim_{x\to 1^-} f(x) =$ ______ and $\displaystyle\lim_{x\to 1^+} f(x) =$ _____ .

(30) Let $f(x) = \begin{cases} 5x-3, & \text{if } x<1; \\ x^2, & \text{if } x\ge 1. \end{cases}$ Then $\displaystyle\lim_{x\to 1^-} f(x) =$ _____ and $\displaystyle\lim_{x\to 1^+} f(x) =$ _____ .

(31) Let $f(x) = \begin{cases} 3x+2, & \text{if } x<-2; \\ x^2+3x-1, & \text{if } x\ge -2. \end{cases}$ Then $\displaystyle\lim_{x\to -2^-} f(x) =$ ______ and $\displaystyle\lim_{x\to -2^+} f(x) =$ ______ .

(32) Suppose $y = f(x)$ is the equation of a curve which always lies between the parabola $x^2 = y-1$ and the hyperbola $yx+y-1=0$. Then $\displaystyle\lim_{x\to 0} f(x) =$ _____ .

4.3. Problems

(1) Find $\lim_{x\to 0^+}\left(e^{-1/x}\sin(1/x)-(x+2)^3\right)$ (if it exists) and give a careful argument showing that your answer is correct.

(2) The notation $\lim_{x\to a} f(x)=\ell$ that we use for limits is somewhat optimistic. It assumes the uniqueness of limits. Prove that limits, if they exist, are indeed unique. That is, suppose that f is a real valued function of a real variable, a is an accumulation point of the domain of f, and ℓ, $m\in\mathbb{R}$. Prove that if $f(x)\to\ell$ as $x\to a$ and $f(x)\to m$ as $x\to a$, then $l=m$. (Explain carefully why it was important that we require a to be an accumulation point of the domain of f.)

(3) Let $f(x)=\dfrac{\sin\pi x}{x+1}$ for all $x\neq -1$. The following information is known about a function g defined for all real numbers $x\neq 1$:

(i) $g=\dfrac{p}{q}$ where $p(x)=ax^2+bx+c$ and $q(x)=dx+e$ for some constants a,b,c,d,e;
(ii) the only x-intercept of the curve $y=g(x)$ occurs at the origin;
(iii) $g(x)\geq 0$ on the interval $[0,1)$ and is negative elsewhere on its domain;
(iv) g has a vertical asymptote at $x=1$; and
(v) $g(1/2)=3$.

Either find $\lim_{x\to 1} g(x)f(x)$ or else show that this limit does not exist.

Hints. Write an explicit formula for g by determining the constants $a\dots e$. Use (ii) to find c; use (ii) and (iii) to find a; use (iv) to find a relationship between d and e; then use (v) to obtain an explicit form for g. Finally look at $f(x)g(x)$; replace $\sin\pi x$ by $\sin(\pi(x-1)+\pi)$ and use the formula for the sine of the sum of two numbers.

(4) Evaluate $\lim_{x\to 0}\dfrac{\sqrt{|x|}\cos\left(\pi^{1/x^2}\right)}{2+\sqrt{x^2+3}}$ (if it exists). Give a careful proof that your conclusion is correct.

4.4. Answers to Odd-Numbered Exercises

(1) -4

(3) 5

(5) 6

(7) 5

(9) -4

(11) 3

(13) 6

(15) $\dfrac{2}{7}$

(17) 3

(19) 5

(21) 32

(23) 2

(25) 3

(27) 1

(29) -1, 1

(31) -4, -3

CHAPTER 5

CONTINUITY

5.1. Background

Topics: continuous functions, *intermediate value theorem. extreme value theorem.*

There are many ways of stating the *intermediate value theorem.* The simplest says that *continuous functions take intervals to intervals.*

5.1.1. Definition. A subset J of the real line $\mathbb{R}$ is an INTERVAL if $z \in J$ whenever a, $b \in J$ and $a < z < b$.

5.1.2. Theorem (Intermediate Value Theorem)**.** *Let J be an interval in $\mathbb{R}$ and $f\colon J \to \mathbb{R}$ be continuous. Then the range of f is an interval.*

5.1.3. Definition. A real-valued function f on a set A is said to have a MAXIMUM at a point a in A if $f(a) \geq f(x)$ for every x in A; the number $f(a)$ is the MAXIMUM VALUE of f. The function has a MINIMUM at a if $f(a) \leq f(x)$ for every x in A; and in this case $f(a)$ is the MINIMUM VALUE of f. A number is an EXTREME VALUE of f if it is either a maximum or a minimum value. It is clear that a function may fail to have maximum or minimum values. For example, on the open interval $(0, 1)$ the function $f : x \mapsto x$ assumes neither a maximum nor a minimum.

The concepts we have just defined are frequently called GLOBAL (or ABSOLUTE) MAXIMUM and GLOBAL (or ABSOLUTE) MINIMUM.

5.1.4. Definition. Let $f\colon A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$. The function f has a LOCAL (or RELATIVE) MAXIMUM at a point $a \in A$ if there exists a neighborhood J of a such that $f(a) \geq f(x)$ whenever $x \in J$ and $x \in \operatorname{dom} f$. It has a LOCAL (or RELATIVE) MINIMUM at a point $a \in A$ if there exists a neighborhood J of a such that $f(a) \leq f(x)$ whenever $x \in J$ and $x \in \operatorname{dom} f$.

5.1.5. Theorem (Extreme Value Theorem)**.** *Every continuous real valued function on a closed and bounded interval in $\mathbb{R}$ achieves its (global) maximum and minimum value at some points in the interval.*

5.1.6. Definition. A number p is a FIXED POINT of a function $f\colon \mathbb{R} \to \mathbb{R}$ if $f(p) = p$.

5.1.7. Example. If $f(x) = x^2 - 6$ for all $x \in \mathbb{R}$, then 3 is a fixed point of f.

5.2. Exercises

(1) Let $f(x) = \dfrac{x^3 - 2x^2 - 2x - 3}{x^3 - 4x^2 + 4x - 3}$ for $x \neq 3$. How should f be defined at $x = 3$ so that it becomes a continuous function on all of $\mathbb{R}$?

Answer: $f(3) = \dfrac{a}{7}$ where $a =$ ________ .

(2) Let $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 3 \\ x - 4 & \text{if } x > 3 \end{cases}$.

(a) Is it possible to define f at $x = 0$ in such a way that f becomes continuous at $x = 0$? Answer: ______ . If so, then we should set $f(0) =$ ______ .

(b) Is it possible to define f at $x = 1$ in such a way that f becomes continuous at $x = 1$? Answer: ______ . If so, then we should set $f(1) =$ ______ .

(c) Is it possible to define f at $x = 3$ in such a way that f becomes continuous at $x = 3$? Answer: ______ . If so, then we should set $f(3) =$ ______ .

(3) Let $f(x) = \begin{cases} x + 4 & \text{if } x < -2 \\ -x & \text{if } -2 < x < 1 \\ x^2 - 2x + 1 & \text{if } 1 < x < 3 \\ 10 - 2x & \text{if } x > 3 \end{cases}$.

(a) Is it possible to define f at $x = -2$ in such a way that f becomes continuous at $x = -2$? Answer: ______ . If so, then we should set $f(-2) =$ ______ .

(b) Is it possible to define f at $x = 1$ in such a way that f becomes continuous at $x = 1$? Answer: ______ . If so, then we should set $f(1) =$ ______ .

(c) Is it possible to define f at $x = 3$ in such a way that f becomes continuous at $x = 3$? Answer: ______ . If so, then we should set $f(3) =$ ______ .

(4) The equation $x^5 + x^3 + 2x = 2x^4 + 3x^2 + 4$ has a solution in the open interval $(n, n+1)$ where n is the positive integer ______ .

(5) The equation $x^4 - 6x^2 - 53 = 22x - 2x^3$ has a solution in the open interval $(n, n+1)$ where n is the positive integer ______ .

(6) The equation $x^4 + x + 1 = 3x^3 + x^2$ has solutions in the open intervals $(m, m+1)$ and $(n, n+1)$ where m and n are the distinct positive integers ______ and ______ .

(7) The equation $x^5 + 8x = 2x^4 + 6x^2$ has solutions in the open intervals $(m, m+1)$ and $(n, n+1)$ where m and n are the distinct positive integers ______ and ______ .

5.3. Problems

(1) Prove that the equation

$$x^{180} + \frac{84}{1 + x^2 + \cos^2 x} = 119$$

has at least two solutions.

(2) (a) Find all the fixed points of the function f defined in example 5.1.7.

Theorem: Every continuous function $f\colon [0,1] \to [0,1]$ has a fixed point.

(b) Prove the preceding theorem. *Hint.* Let $g(x) = x - f(x)$ for $0 \le x \le 1$. Apply the *intermediate value theorem* 5.1.2 to g.

(c) Let $g(x) = 0.1\,x^3 + 0.2$ for all $x \in \mathbb{R}$, and h be the restriction of g to $[0,1]$. Show that h satisfies the hypotheses of the theorem.

(d) For the function h defined in (c) find an approximate value for at least one fixed point with an error of less than 10^{-6}. Give a careful justification of your answer.

(e) Let g be as in (c). Are there other fixed points (that is, points not in the unit square where the curve $y = g(x)$ crosses the line $y = x$)? If so, find an approximation to each such point with an error of less than 10^{-6}. Again provide careful justification.

(3) Define f on $[0,4]$ by $f(x) = x + 1$ for $0 \le x < 2$ and $f(x) = 1$ for $2 \le x \le 4$. Use the *extreme value theorem* 5.1.5 to show that f is not continuous.

(4) Give an example of a function defined on $[0,1]$ which has no maximum and no minimum on the interval. Explain why the existence of such a function does not contradict the *extreme value theorem* 5.1.5.

(5) Give an example of a continuous function defined on the interval $(1,2]$ which does not achieve a maximum value on the interval. Explain why the existence of such a function does not contradict the *extreme value theorem* 5.1.5.

(6) Give an example of a continuous function on the closed interval $[3,\infty)$ which does not achieve a minimum value on the interval. Explain why the existence of such a function does not contradict the *extreme value theorem* 5.1.5.

(7) Define f on $[-2,0]$ by $f(x) = \dfrac{-1}{(x+1)^2}$ for $-2 \le x < -1$ and $-1 < x \le 0$, and $f(-1) = -3$. Use the *extreme value theorem* 5.1.5 to show that f is not continuous.

(8) Let $f(x) = \dfrac{1}{x}$ for $0 < x \le 1$ and $f(0) = 0$. Use the *extreme value theorem* 5.1.5 to show that f is not continuous on $[0,1]$.

5.4. Answers to Odd-Numbered Exercises

(1) 13

(3) (a) yes, 2
  (b) no, —
  (c) yes, 4

(5) 3

(7) 1, 2

Part 3

DIFFERENTIATION OF FUNCTIONS OF A SINGLE VARIABLE

CHAPTER 6

DEFINITION OF THE DERIVATIVE

6.1. Background

Topics: definition of the derivative of a real valued function of a real variable at a point

6.1.1. Notation. Let f be a real valued function of a real variable which is differentiable at a point a in its domain. When thinking of a function in terms of its graph, we often write $y = f(x)$, call x the *independent variable*, and call y the *dependent variable*. There are many notations for the derivative of f at a. Among the most common are

$$Df(a), \quad f'(a), \quad \left.\frac{df}{dx}\right|_a, \quad y'(a), \quad \dot{y}(a), \quad \text{and} \quad \left.\frac{dy}{dx}\right|_a.$$

6.2. Exercises

(1) Suppose you know that the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$ for every $x > 0$. Then

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(2) Suppose you know that the derivatives of $x^{\frac{1}{3}}$ is $\frac{1}{3}x^{-\frac{2}{3}}$ for every $x \neq 0$. Then

$$\lim_{x \to 8} \frac{\left(\frac{x}{8}\right)^{\frac{1}{3}} - 1}{x - 8} = \frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(3) Suppose you know that the derivative of e^x is e^x for every x. Then

$$\lim_{x \to 2} \frac{e^x - e^2}{x - 2} = \underline{\qquad}.$$

(4) Suppose you know that the derivative of $\ln x$ is $\dfrac{1}{x}$ for every $x > 0$. Then

$$\lim_{x \to e} \frac{\ln x^3 - 3}{x - e} = \underline{\qquad}.$$

(5) Suppose you know that the derivative of $\tan x$ is $\sec^2 x$ for every x. Then

$$\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{4x - \pi} = \underline{\qquad}.$$

(6) Suppose you know that the derivative of $\arctan x$ is $\dfrac{1}{1 + x^2}$ for every x. Then

$$\lim_{x \to \sqrt{3}} \frac{3 \arctan x - \pi}{x - \sqrt{3}} = \underline{\qquad}.$$

(7) Suppose you know that the derivative of $\cos x$ is $-\sin x$ for every x. Then

$$\lim_{x \to \frac{\pi}{3}} \frac{2 \cos x - 1}{3x - \pi} = -\frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(8) Suppose you know that the derivative of $\cos x$ is $-\sin x$ for every x. Then

$$\lim_{t \to 0} \frac{\cos(\frac{\pi}{6} + t) - \frac{\sqrt{3}}{2}}{t} = -\frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(9) Suppose you know that the derivative of $\sin x$ is $\cos x$ for every x. Then

$$\lim_{x \to -\pi/4} \frac{\sqrt{2} \sin x + 1}{4x + \pi} = \frac{1}{a} \text{ where } a = \underline{\qquad}.$$

(10) Suppose you know that the derivative of $\sin x$ is $\cos x$ for every x. Then

$$\lim_{x \to \frac{7\pi}{12}} \frac{2\sqrt{2} \sin x - \sqrt{3} - 1}{12x - 7\pi} = \frac{1 - \sqrt{a}}{b} \text{ where } a = \underline{\qquad} \text{ and } b = \underline{\qquad}.$$

(11) Let $f(x) = \begin{cases} x^2, & \text{for } x \leq 1 \\ 1, & \text{for } 1 < x \leq 3 \\ 5 - 2x, & \text{for } x > 3 \end{cases}$. Then $f'(0) = \underline{\qquad}$, $f'(2) = \underline{\qquad}$, and $f'(6) = \underline{\qquad}$.

(12) Suppose that the tangent line to the graph of a function f at $x = 1$ passes through the point $(4, 9)$ and that $f(1) = 3$. Then $f'(1) =$ ______ .

(13) Suppose that g is a differentiable function and that $f(x) = g(x) + 5$ for all x. If $g'(1) = 3$, then $f'(1) =$ _____ .

(14) Suppose that g is a differentiable function and that $f(x) = g(x + 5)$ for all x. If $g'(1) = 3$, then $f'(a) = 3$ where $a =$ _____ .

(15) Suppose that f is a differentiable function, that $f'(x) = -2$ for all x, and that $f(-3) = 11$. Find an algebraic expression for $f(x)$. Answer: $f(x) =$ ______________ .

(16) Suppose that f is a differentiable function, that $f'(x) = 3$ for all x, and that $f(3) = 3$. Find an algebraic expression for $f(x)$. Answer: $f(x) =$ _______________ .

6.3. Problems

(1) Let $f(x) = \dfrac{1}{x^2 - 1}$ and $a = -3$. Show how to use the definition of *derivative* to find $Df(a)$.

(2) Let $f(x) = \dfrac{1}{\sqrt{x+7}}$. Show how to use the definition of *derivative* to find $f'(2)$.

(3) Let $f(x) = \dfrac{1}{\sqrt{x+3}}$. Show how to use the definition of *derivative* to find $f'(1)$.

(4) Let $f(x) = \sqrt{x^2 - 5}$. Show how to use the definition of *derivative* to find$f'(3)$.

(5) Let $f(x) = \sqrt{8 - x}$. Show how to use the definition of *derivative* to find $f'(-1)$.

(6) Let $f(x) = \sqrt{x - 2}$. Show how to use the definition of *derivative* to find $f'(6)$.

(7) Let $f(x) = \dfrac{x}{x^2 + 2}$. Show how to use the definition of *derivative* to find $Df(2)$.

(8) Let $f(x) = (2x^2 - 3)^{-1}$. Show how to use the definition of *derivative* to find $Df(-2)$.

(9) Let $f(x) = x + 2x^2 \sin\dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. What is the derivative of f at 0 (if it exists)? Is the function f' continuous at 0?

6.4. Answers to Odd-Numbered Exercises

(1) 6

(3) e^2

(5) $\dfrac{1}{2}$

(7) $\sqrt{3}$

(9) 4

(11) 0, 0, -2

(13) 3

(15) $-2x + 5$

CHAPTER 7

TECHNIQUES OF DIFFERENTIATION

7.1. Background

Topics: *rule for differentiating products*, *rule for differentiating quotients*, *chain rule*, tangent lines, implicit differentiation.

7.1.1. Notation. We use $f^{(n)}(a)$ to denote the n$^{\text{th}}$ derivative of f at a.

7.1.2. Definition. A point a in the domain of a function f is a STATIONARY POINT of f is $f\,'(a) = 0$. It is a CRITICAL POINT of f if it is either a stationary point of f or if it is a point where the derivative of f does not exist.

Some authors use the terms *stationary point* and *critical point* interchangeably—especially in higher dimensions.

7.2. Exercises

(1) If $f(x) = 5x^{-1/2} + 6x^{3/2}$, then $f'(x) = ax^p + bx^q$ where $a =$ ________ , $p =$ ________ , $b =$ ______ , and $q =$ ______ .

(2) If $f(x) = 10\sqrt[5]{x^3} + \dfrac{12}{\sqrt[6]{x^5}}$, then $f'(x) = ax^p + bx^q$ where $a =$ _____ , $p =$ ________ , $b =$ ________ , and $q =$ __________ .

(3) If $f(x) = 9x^{4/3} + 25x^{2/5}$, then $f''(x) = ax^p + bx^q$ where $a =$ _____ , $p =$ __________ , $b =$ ______ , and $q =$ ___________ .

(4) If $f(x) = 18\sqrt[6]{x} + \dfrac{8}{\sqrt[4]{x^3}}$, then $f''(x) = ax^p + bx^q$ where $a =$ __________ , $p =$ ___________ , $b =$ ___________ , and $q =$ _____________ .

(5) Find a point a such that the tangent line to the graph of the curve $y = \sqrt{x}$ at $x = a$ has y-intercept 3. Answer: $a =$ ______ .

(6) Let $f(x) = ax^2 + bx + c$ for all x. We know that $f(2) = 26$, $f'(2) = 23$, and $f''(2) = 14$. Then $f(1) =$ ______ .

(7) Find a number k such that the line $y = 6x + 4$ is tangent to the parabola $y = x^2 + k$. Answer: $k =$ ________ .

(8) The equation for the tangent line to the curve $y = x^3$ which passes through the point $(0, 2)$ is $y = mx + b$ where $m =$ _____ and $b =$ _____ .

(9) Let $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2 + \frac{7}{4}$. Find all points x_0 such that the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is horizontal. Answer: $x_0 =$ ______ , _____ , and _____ .

(10) In the land of Oz there is an enormous statue of the Good Witch Glinda. Its base is 20 feet high and, on a surveyor's chart, covers the region determined by the inequalities

$$-1 \le y \le 24 - x^2 .$$

(The chart coordinates are measured in feet.) Dorothy is looking for her little dog Toto. She walks along the curved side of the base of the statue in the direction of increasing x and Toto is, for a change, sitting quietly. He is at the point on the positive x-axis 7 feet from the origin. How far from Toto is Dorothy when she is first able to see him? Answer: $5\sqrt{a}$ ft. where $a =$ ______ .

(11) Let $f(x) = \dfrac{x - \frac{3}{2}}{x^2 + 2}$ and $g(x) = \dfrac{x^2 + 1}{x^2 + 2}$. At what values of x do the curves $y = f(x)$ and $y = g(x)$ have parallel tangent lines? Answer: at $x =$ ______ and $x =$ _____ .

(12) The tangent line to the graph of a function f at the point $x = 2$ has x-intercept $\dfrac{10}{3}$ and y-intercept -10. Then $f(2) =$ ______ and $f'(2) =$ _____ .

(13) The tangent line to the graph of a function f at $x = 2$ passes through the points $(0, -20)$ and $(5, 40)$. Then $f(2) =$ _____ and $f'(2) =$ ______ .

(14) Suppose that the tangent line to the graph of a function f at $x = 2$ passes through the point $(5, 19)$ and that $f(2) = -2$. Then $f'(2) =$ _____ .

(15) Let $f(x) = \begin{cases} x^2, & \text{for } x \le 1 \\ 1, & \text{for } 1 < x \le 3 \\ 5 - 2x, & \text{for } x > 3 \end{cases}$. Then $f'(0) =$ _____ , $f'(2) =$ _____ , and $f'(6) =$ ______ .

(16) Suppose that g is a differentiable function and that $f(x) = g(x + 5)$ for all x. If $g'(1) = 3$, then $f'(a) = 3$ where $a =$ ______ .

(17) Let $f(x) = |2 - |x-1|| - 1$ for every real number x. Then

$f'(-2) =$ ______ , $f'(0) =$ _____ , $f'(2) =$ ______ , and $f'(4) =$ _____ .

(18) Let $f(x) = \tan^3 x$. Then $Df(\pi/3) =$ ________ .

(19) Let $f(x) = \dfrac{1}{x}\csc^2\dfrac{1}{x}$. Then $Df(6/\pi) = \dfrac{\pi^2}{a}\left(\dfrac{\pi}{\sqrt{b}} - 1\right)$ where $a =$ _____ and $b =$ _____ .

(20) Let $f(x) = \sin^2(3x^5 + 7)$. Then $f'(x) = ax^4\sin(3x^5+7)f(x)$ where $a =$ ______ and $f(x) =$ ______________________ .

(21) Let $f(x) = (x^4 + 7x^2 - 5)\sin(x^2+3)$. Then $f'(x) = f(x)\cos(x^2+3) + g(x)\sin(x^2+3)$ where $f(x) =$ _________________________ and $g(x) =$ ________________ .

(22) Let $j(x) = \sin^5(\tan(x^2+6x-5)^{1/2})$. Then $Dj(x) = p(x)\sin^n(g(a(x)))f(g(a(x)))h(a(x))\big(a(x)\big)^r$ where

$f(x) =$ _________ ,
$g(x) =$ _________ ,
$h(x) =$ ___________ ,
$a(x) =$ _____________________ ,
$p(x) =$ ___________ ,
$n =$ _____ , and
$r =$ ______ .

(23) Let $j(x) = \sin^4(\tan(x^3-3x^2+6x-11)^{2/3})$. Then $j\,'(x) = 8p(x)f(g(a(x)))\cos(g(a(x)))h(a(x))\big(a(x)\big)^r$ where

$f(x) =$ ___________ ,
$g(x) =$ _________ ,
$h(x) =$ ___________ ,
$a(x) =$ ____________________________ ,
$p(x) =$ ______________ ,
$r =$ ________ .

(24) Let $j(x) = \sin^{11}(\sin^6(x^3 - 7x + 9)^3)$. Then

$$Dj(x) = 198(3x^2 + b)\sin^p(g(a(x)))h(g(a(x)))\sin^q(a(x))h(a(x))\big(a(x)\big)^r$$

where

$g(x) =$ _________ ,
$h(x) =$ ___________ ,
$a(x) =$ _____________________ ,
$p =$ ______ ,
$q =$ _____ ,
$r =$ ______ , and
$b =$ ______ ,

(25) Let $f(x) = (x^2 + \sin\pi x)^{100}$. Then $f'(1) =$ _________________ .

(26) Let $f(x) = (x^2-15)^9(x^2-17)^{10}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 4 is $y = ax + b$ where $a =$ ______ and $b =$ ______ .

(27) Let $f(x) = (x^2-3)^{10}(x^3+9)^{20}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is -2 is $y = ax + b$ where $a =$ ________ and $b =$ ________ .

(28) Let $f(x) = (x^3-9)^8(x^3-7)^{10}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 2 is $y = ax + b$ where $a =$ ______ and $b =$ ________ .

(29) Let $f(x) = (x^2 - 10)^{10}(x^2 - 8)^{12}$. Then the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 3 is $y = ax+b$ where $a =$ ______ and $b =$ ________ .

(30) Let $h = g \circ f$ and $j = f \circ g$ where f and g are differentiable functions on $\mathbb{R}$. Fill in the missing entries in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$	$j(x)$	$j'(x)$
0		-3			1		1	$-\frac{3}{2}$
1	0			$\frac{3}{2}$	0	$\frac{1}{2}$		

(31) Let $f = g \circ h$ and $j = g \cdot h$ where g and h are differentiable functions on $\mathbb{R}$. Fill in the missing entries in the table below.

x	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$	$f(x)$	$f'(x)$	$j(x)$	$j'(x)$
0			2			-4	-6	3
1	-2					4	-4	2
2		4	4		13	24	4	19

Also, $g(4) =$ ______ and $g'(4) =$ _____ .

(32) Let $h = g \circ f$, $j = g \cdot f$, and $k = g + f$ where f and g are differentiable functions on $\mathbb{R}$. Fill in the missing entries in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$	$j(x)$	$j'(x)$	$k(x)$	$k'(x)$
-1			-2	4				4	-2	
0		0					0	-1		1
1			2			2	0			6

(33) Let $y = \log_3(x^2 + 1)^{1/3}$. Then $\dfrac{dy}{dx} = \dfrac{2x}{a(x^2+1)}$ where $a =$ ___________ .

(34) Let $f(x) = \ln \dfrac{(6 + \sin^2 x)^{10}}{(7 + \sin x)^3}$. Then $Df(\pi/6) = \dfrac{a}{5}$ where $a =$ ___________ .

(35) Let $f(x) = \ln(\ln x)$. What is the domain of f? Answer: (_____ , _____). What is the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is e^2? Answer: $y - a = \dfrac{1}{b}(x - e^2)$ where $a =$ ________ and $b =$ ________ .

(36) Find when $y = (\tan x)^{\sin x}$ for $0 < x < \pi/2$. Then $\dfrac{dy}{dx} = (\tan x)^{\sin x}(f(x) + \cos x \ln \tan x)$ where $f(x) =$ ________ .

(37) Find when $y = (\sin x)^{\tan x}$ for $0 < x < \pi/2$. Then $\dfrac{dy}{dx} = (\sin x)^{\tan x}(a + f(x)\sec^2 x)$ where $a =$ ____ and $f(x) =$ ______________ .

(38) $\dfrac{d}{dx}\sqrt{x}\ln x = x^p(1 + g(x))$ where $p =$ _______ and $g(x) =$ _________ .

(39) If $f(x) = x^3e^x$, then $f'''(x) = (ax^3 + bx^2 + cx + d)e^x$ where $a =$ ____ , $b =$ ____ , $c =$ ______ , and $d =$ ____ .

(40) Let $f(x) = x^2\cos x$. Then $(ax^2 + bx + c)\sin x + (Ax^2 + Bx + C)\cos x$ is an antiderivative of $f(x)$ if $a =$ ____ , $b =$ ____ , $c =$ _____ , $A =$ ____ , $B =$ ____ , and $C =$ ____ .

(41) Let $f(x) = (x^4 - x^3 + x^2 - x + 1)(3x^3 - 2x^2 + x - 1)$. Use the rule for differentiating products to find $f'(1)$. Answer: ____ .

(42) Let $f(x) = \dfrac{x^{3/2} - x}{3x - x^{1/2}}$. Then $f'(4) = \dfrac{9}{a}$ where $a =$ _________ .

(43) Find a point on the curve $y = \dfrac{x^2}{x^3 - 2}$ where the tangent line is parallel to the line $4x + 6y - 5 = 0$. Answer: (____ , ______).

(44) Let $f(x) = 5x\cos x - x^2\sin x$. Then $(ax^2 + bx + c)\sin x + (Ax^2 + Bx + C)\cos x$ is an antiderivative of $f(x)$ if $a =$ ____ , $b =$ ____ , $c =$ ____ , $A =$ ____ , $B =$ ____ , and $C =$ ____ .

(45) Let $f(x) = (2x - 3)\csc x + (2 + 3x - x^2)\cot x\csc x$. Then $\dfrac{ax^2 + bx + c}{\sin x}$ is an antiderivative of $f(x)$ if $a =$ ____ , $b =$ _____ , and $c =$ _____ .

(46) Let $f(x) = \dfrac{x^2 - 10}{x^2 - 8}$. Find the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is 3. Answer: $y = ax + b$ where $a =$ _____ and $b =$ _______ .

(47) Let $f(x) = (x^4 + x^3 + x^2 + x + 1)(x^5 + x^3 + x + 2)$. Find the equation of the tangent line to the curve $y = f(x)$ at the point on the curve whose x-coordinate is -1. Answer: $y = ax + b$ where $a =$ _____ and $b =$ _____ .

(48) Let $y = \dfrac{x^2 - 2x + 1}{x^3 + 1}$. Then $\left.\dfrac{dy}{dx}\right|_{x=2} = \dfrac{2}{a}$ where $a =$ _______ .

(49) Let $f(x) = \dfrac{x - \frac{3}{2}}{x^2 + 2}$ and $g(x) = \dfrac{x^2 + 1}{x^2 + 2}$. At what values of x do the curves $y = f(x)$ and $y = g(x)$ have parallel tangent lines? Answer: at $x =$ _____ and ____ .

(50) Let $f(x) = x\sin x$. Find constants a, b, A, and B so that $(ax + b)\cos x + (Ax + B)\sin x$ is an antiderivative of $f(x)$. Answer: $a =$ _____ , $b =$ ____ , $A =$ ____ , and $B =$ ____ .

(51) Find $\dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{d^2}{dx^2}\left(\dfrac{1}{1+x}\right)\right) = a\,\dfrac{bx + 1}{x^2(1 + x)^b}$ where $a =$ _____ and $b =$ ____ .

(52) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\cdot\dfrac{d^2}{dx^2}\left(\dfrac{1}{x^2}\right)\right) = ax^p$ where $a =$ _______ and $p =$ _____ .

(53) Let $f(x) = \dfrac{x + 3}{4 - x}$. Find $f^{(15)}(x)$. Answer: $\dfrac{7n!}{(4 - x)^p}$ where $n =$ _____ and $p =$ _____ .

(54) Let $f(x) = \dfrac{x}{x + 1}$. Then $f^{(4)}(x) = a(x + 1)^p$ where $a =$ _________ and $p =$ _____ .

(55) Let $f(x) = \dfrac{x+1}{2-x}$. Then $f^{(4)}(x) = a(2-x)^p$ where $a =$ ______ and $p =$ ______ .

(56) Find the equation of the tangent line to the curve $2x^6 + y^4 = 9xy$ at the point $(1, 2)$. Answer: $23y = ax + b$ where $a =$ _____ and $b =$ ______ .

(57) For the curve $x^3 + 2xy + \frac{1}{3}y^3 = \frac{11}{3}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(2, -1)$.

Answer: $y'(2) =$ ________ and $y''(2) = \dfrac{a}{5}$ where $a =$ _____ .

(58) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for the devil's curve $y^4 + 5y^2 = x^4 - 5x^2$ at the point $(3, 2)$.
Answer: $y'(3) =$ ______ and $y''(3) =$ ________ .

(59) Find $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, and $\dfrac{d^3y}{dx^3}$ at the point $(1, 8)$ on the astroid $x^{2/3} + y^{2/3} = 5$.

Answer: $y'(1) =$ ______ : $y''(1) = \dfrac{a}{6}$ where $a =$ _____ ; and $y'''(1) = \dfrac{b}{24}$ where $b =$ __________ .

(60) Find the point of intersection of the tangent lines to the curve $x^2 + y^3 - 3x + 3y - xy = 18$ at the points where the curve crosses the x-axis. Answer: (_______ , _______).

(61) Find the equation of the tangent line to the curve $x \sin y + x^3 = \arctan e^y + x - \frac{\pi}{4}$ at the point $(1, 0)$. Answer: $y = ax + b$ where $a =$ ______ and $b =$ _____ .

(62) The equation of the tangent line to the lemniscate $3(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(2, 1)$ is $y - 1 = m(x - 2)$ where $m =$ __________ .

(63) The points on the ovals of Cassini $(x^2+y^2)^2 - 4(x^2-y^2) + 3 = 0$ where there is a horizontal tangent line are $\left(\pm\dfrac{\sqrt{a}}{b\sqrt{b}}, \pm\dfrac{1}{b\sqrt{b}}\right)$ where $a =$ ______ and $b =$ _____ .

(64) The points on the ovals of Cassini $(x^2 + y^2)^2 - 4(x^2 - y^2) + 3 = 0$ where there is a vertical tangent line are $(\pm\sqrt{a}, b)$ and $(\pm c, b)$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(65) At the point $(1, 2)$ on the curve $4x^2 + 2xy + y^2 = 12$, $\dfrac{dy}{dx} =$ ______and $\dfrac{d^2y}{dx^2} =$ ________ .

(66) Let f and g be differentiable real valued functions on $\mathbb{R}$. We know that the points $(-4, 1)$ and $(3, 4)$ lie on the graph of the curve $y = f(x)$ and the points $(-4, 3)$ and $(3, -2)$ lie on the graph of $y = g(x)$. We know also that $f'(-4) = 3$, $f'(3) = -4$, $g'(-4) = -2$, and $g'(3) = 6$.
  (a) If $h = f \cdot g$, then $h'(-4) =$ _____ .
  (b) If $j = (2f + 3g)^4$, then $j'(3) =$ ________ .
  (c) If $k = f \circ g$, then $k'(-4) =$ _____ .
  (d) If $\ell = \dfrac{f}{g}$, then $\ell'(3) =$ ______ .

(67) Let $f(x) = 5 \sin x + 3 \cos x$. Then $f^{(117)}(\pi) =$ ______ .

(68) Let $f(x) = 4 \cos x - 7 \sin x$. Then $f^{(87)}(0) =$ ______ .

7.3. Problems

(1) Let (x_0, y_0) be a point in $\mathbb{R}^2$. How many tangent lines to the curve $y = x^2$ pass through the point (x_0, y_0)? What are the equations of these lines? *Hint.* Consider the three cases: $y_0 > {x_0}^2$, $y_0 = {x_0}^2$, and $y_0 < {x_0}^2$.

(2) For the purposes of this problem you may assume that the differential equation

$$y'' + y = 0 \tag{$*$}$$

has at least one nontrivial solution on the real line. (That is, there exists at least one twice differentiable function y, not identically zero, such that $y''(x) + y(x) = 0$ for all $x \in \mathbb{R}$.)

(a) Show that if u and v are solutions of $(*)$ and a, $b \in \mathbb{R}$, then $w = au + bv$ and u' are also solutions of $(*)$.
(b) Show that if y is a solution of $(*)$ then $y^2 + (y')^2$ is constant.
(c) Show that if y is a nontrivial solution of $(*)$, then either $y(0) \neq 0$ or $y'(0) \neq 0$. *Hint.* Argue by contradiction. Show that if y is a solution of $(*)$ such that both $y(0) = 0$ and $y'(0) = 0$, then $y(x) = 0$ for all x.
(d) Show that there exists a solution s of $(*)$ such that $s(0) = 0$ and $s'(0) = 1$. *Hint.* Let y be a nontrivial solution of $(*)$. Look for a solution s of the form $a\,y + b\,y'$ (with a, $b \in \mathbb{R}$) satisfying the desired conditions.
(e) Show that if y is a solution of $(*)$ such that $y(0) = a$ and $y'(0) = b$, then $y = b\,s + a\,s'$. *Hint.* Let $u(x) = y(x) - b\,s(x) - a\,s'(x)$ and show that u is a solution of $(*)$ such that $u(0) = u'(0) = 0$. Use (c).
(f) Define $c(x) = s'(x)$ for all x. Show that $(s(x))^2 + (c(x))^2 = 1$ for all x.
(g) Show that s is an odd function and that c is even. *Hint.* To see that s is odd let $u(x) = s(-x)$ for all x. Show that u is a solution of $(*)$. Use (e). Once you know that s is odd, differentiate to see that c is even.
(h) Show that $s(a + b) = s(a)c(b) + c(a)s(b)$ for all real numbers a and b. *Hint.* Let $y(x) = s(x + b)$ for all x. Show that y is a solution of $(*)$. Use (e).
(i) Show that $c(a+b) = c(a)c(b) - s(a)s(b)$ for all real numbers a and b. *Hint.* Differentiate the formula for $s(x + b)$ that you derived in (h).
(j) Define $t(x) = \dfrac{s(x)}{c(x)}$ and $\sigma(x) = \dfrac{1}{c(x)}$ for all x such that $c(x) \neq 0$. Show that $t'(x) = (\sigma(x))^2$ and $\sigma'(x) = t(x)\sigma(x)$ wherever $c(x) \neq 0$.
(k) Show that $1 + (t(x))^2 = (\sigma(x))^2$ wherever $c(x) \neq 0$.
(l) Explain carefully what the (mathematical) point of this problem is.

(3) Suppose that f is a differentiable function such that $f'(x) \geq \dfrac{3}{2}$ for all x and that $f(1) = 2$. Prove that $f(5) \geq 8$.

(4) Suppose that f is a differentiable function such that $f'(x) \geq 3$ for all x and that $f(0) = -4$. Prove that $f(3) \geq 5$.

(5) Suppose that f is a differentiable function such that $f'(x) \leq -2$ for all $x \in [0, 4]$ and that $f(1) = 6$.
(a) Prove that $f(4) \leq 0$.
(b) Prove that $f(0) \geq 8$.

(6) Give a careful proof that $\sin x \leq x$ for all $x \geq 0$.

(7) Give a careful proof that $1 - \cos x \leq x$ for all $x \geq 0$.

(8) Prove that if $x^2 = \dfrac{1 - y^2}{1 + y^2}$, then $\left(\dfrac{dx}{dy}\right)^2 = \dfrac{1 - x^4}{1 - y^4}$ at points where $y \neq \pm 1$.

(9) For the circle $x^2 + y^2 - 1 = 0$ use implicit differentiation to show that $y'' = -\dfrac{1}{y^3}$ and $y''' = -\dfrac{3x}{y^5}$.

(10) Explain how to calculate $\dfrac{d^2y}{dx^2}$ at the point on the folium of Déscartes

$$x^3 + y^3 = 9xy$$

where the tangent line is parallel to the asymptote of the folium.

(11) Explain carefully how to find the curve passing through the point $(2, 3)$ which has the following property: the segment of any tangent line to the curve contained between the (positive) coordinate axes is bisected at the point of tangency. Carry out the computation you have described.

7.4. Answers to Odd-Numbered Exercises

(1) $-\frac{5}{2}$, $-\frac{3}{2}$, 9, $\frac{1}{2}$

(3) 4, $-\frac{2}{3}$, -6, $-\frac{8}{5}$

(5) 36

(7) 13

(9) -3, 0, 2

(11) -1, 2

(13) 4, 12

(15) 0, 0, -2

(17) -1, 1, -1, 1

(19) 9, 3

(21) $2x^5 + 14x^3 - 10x$, $4x^3 + 14x$

(23) $\sin^3 x$, $\tan x$, $\sec^2 x$, $\left(x^3 - 3x^2 + 6x - 11\right)^{\frac{2}{3}}$, $x^2 - 2x + 2$, $-\frac{1}{2}$

(25) $100(2 - \pi)$

(27) 200, 401

(29) 12, -35

(31) -3, 0, -1, 1 (first row)
2, 2, 1, 1 (second row)
1, 3 (third row)
13, 8

(33) $3\ln 3$

(35) 1, ∞, $\ln 2$, $2e^2$

(37) 1, $\ln\sin x$

(39) 1, 9, 18, 6

(41) 8

(43) 2, $\frac{2}{3}$

(45) 1, -3, -2

(47) 11, 10

(49) -1, 2

(51) -2, 4

(53) 15, 16

(55) 72, -5

(57) -2, 4

(59) -2, 5, -25

(61) -4, 4

(63) 15, 2

(65) $-2, -\dfrac{4}{3}$

(67) -5

CHAPTER 8

THE MEAN VALUE THEOREM

8.1. Background

Topics: *Rolle's theorem*, the *mean value theorem*, the *intermediate value theorem.*

8.1.1. Definition. A real valued function f defined on an interval J is INCREASING on J if $f(a) \leq f(b)$ whenever a, $b \in J$ and $a \leq b$. It is STRICTLY INCREASING on J if $f(a) < f(b)$ whenever a, $b \in J$ and $a < b$. The function f is DECREASING on J if $f(a) \geq f(b)$ whenever a, $b \in J$ and $a \leq b$. It is STRICTLY DECREASING on J if $f(a) > f(b)$ whenever a, $b \in J$ and $a < b$.
NOTE: In many texts the word "nondecreasing" is used where "increasing" in these notes; and "increasing" is used for "strictly increasing".

8.2. Exercises

(1) Let $M > 0$ and $f(x) = x^3$ for $0 \leq x \leq M$. Find a value of c which satisfies the conclusion of the *mean value theorem* for the function f over the interval $[0, M]$. Answer: $c = \dfrac{M}{a}$ where $a =$ ______ .

(2) Let $f(x) = x^4 + x + 3$ for $0 \leq x \leq 2$. Find a point c whose existence is guaranteed by the *mean value theorem*. Answer: $c = 2^p$ where $p =$ ______ .

(3) Let $f(x) = \sqrt{x}$ for $4 \leq x \leq 16$. Find a point c whose existence is guaranteed by the *mean value theorem*. Answer: $c =$ _____ .

(4) Let $f(x) = \dfrac{x}{x+1}$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Find a point c whose existence is guaranteed by the *mean value theorem*. Answer: $c = \dfrac{a}{2} - 1$ where $a =$ ______ .

8.3. Problems

(1) Use *Rolle's theorem* to derive the *mean value theorem*.

(2) Use the *mean value theorem* to derive *Rolle's theorem*.

(3) Use the *mean value theorem* to prove that if a function f has a positive derivative at every point in an interval, then it is increasing on that interval.

(4) Let $a \in \mathbb{R}$. Prove that if f and g are differentiable functions with $f'(x) \leq g'(x)$ for every x in some interval containing a and if $f(a) = g(a)$, then $f(x) \leq g(x)$ for every x in the interval such that $x \geq a$.

(5) Suppose that f is a differentiable function such that $f'(x) \leq -2$ for all $x \in [0, 4]$ and that $f(1) = 6$.
  (a) Prove that $f(4) \leq 0$.
  (b) Prove that $f(0) \geq 8$.

(6) Your friend Fred is confused. The function $f \colon x \mapsto x^{\frac{2}{3}}$ takes on the same values at $x = -1$ and at $x = 1$. So, he concludes, according to *Rolle's theorem* there should be a point c in the open interval $(-1, 1)$ where $f'(c) = 0$. But he cannot find such a point. Help your friend out.

(7) Consider the equation $\cos x = 2x$.
  (a) Use the *intermediate value theorem* to show that the equation has at least one solution.
  (b) Use the *mean value theorem* to show that the equation has at most one solution.

(8) Let $m \in \mathbb{R}$. Use *Rolle's theorem* to show that the function f defined by $f(x) = x^3 - 3x + m$ can not have two zeros in the interval $[-1, 1]$.

(9) Use the *mean value theorem* to show that if $0 < x \leq \pi/3$, then $\frac{1}{2}x \leq \sin x \leq x$.

(10) Use the *mean value theorem* to show that on the interval $[0, \pi/4]$ the graph of the curve $y = \tan x$ lies between the lines $y = x$ and $y = 2x$.

(11) Let $x > 0$. Use the *mean value theorem* to show that $\dfrac{x}{x^2 + 1} < \arctan x < x$.

(12) Use the *mean value theorem* to show that

$$x + 1 < e^x < 2x + 1$$

whenever $0 < x \leq \ln 2$.

(13) Show that the equation $e^x + x = 0$ has exactly one solution. Locate this solution between consecutive integers.

(14) Prove that the equation $\sin x = 1 - 2x$ has exactly one solution. Explain how the *intermediate value theorem* can be used to produce an approximation to the solution which is correct to two decimal places.

(15) Give a careful proof that at one time your height (in inches) was exactly equal to your weight (in pounds). Be explicit about any physical assumptions you make.

8.4. Answers to Odd-Numbered Exercises

(1) $\sqrt{3}$

(3) 9

CHAPTER 9

L'HÔPITAL'S RULE

9.1. Background

Topics: l'Hôpital's rule.

9.2. Exercises

(1) $\displaystyle\lim_{x\to 0} \frac{\frac{1}{3}x^3 + 2x - 2\sin x}{4x^3} = \frac{1}{a}$ where $a =$ _____ .

(2) $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) =$ _____ .

(3) $\displaystyle\lim_{t\to 1} \frac{nt^{n+1} - (n+1)t^n + 1}{(t-1)^2} =$ ____________________ when $n \geq 2$.

(4) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} =$ _____ .

(5) $\displaystyle\lim_{x\to 2} \frac{x^4 - 4x^3 + 5x^2 - 4x + 4}{x^4 - 4x^3 + 6x^2 - 8x + 8} = \frac{a}{6}$ where $a =$ _____ .

(6) Let n be a fixed integer. Then the function f given by $f(x) = \dfrac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}$ is not defined at points $x = 2m\pi$ where m is an integer. The function f can be extended to a function continuous on all of $\mathbb{R}$ by defining
$f(2m\pi) =$ _____________ for every integer m.

(7) $\displaystyle\lim_{x\to 1} \frac{x^5 - 1}{6x^5 - 4x^3 + x - 3} = \frac{5}{a}$ where $a =$ _______ .

(8) Suppose that g has derivatives of all orders, that $g(0) = g'(0) = g''(0) = 0$, that $g'''(0) = 27$, and that there is a deleted neighborhood U of 0 such that $g^{(n)}(x) \neq 0$ whenever $x \in U$ and $n \geq 0$. Define $f(x) = x^{-4}g(x)(1 - \cos x)$ for $x \neq 0$ and $f(0) = 0$. Then $f'(0) = \dfrac{a}{4}$ where $a =$ _____ .

(9) Suppose that g has derivatives of all orders, that $g(0) = g'(0) = 0$, that $g''(0) = 10$, that $g'''(0) = 12$, and that there is a deleted neighborhood of 0 in which $g(x)$, $g'(x)$, $xg'(x) - g(x) - 5x^2$, and $g''(x) - 10$ are never zero. Let $f(x) = \dfrac{g(x)}{x}$ for $x \neq 0$ and $f(0) = 0$. Then $f''(0) =$ _____ .

(10) Suppose that g has derivatives of all orders, that $g(0) = g'(0) = g''(0) = g'''(0) = 0$, and that $g^{(4)}(0) = 5$. Define $f(x) = \dfrac{x\,g(x)}{2\cos x + x^2 - 2}$ for $x \neq 0$ and $f(0) = 0$. Then $f'(0) = \dfrac{a}{2}$ where $a =$ _____ .

(11) $\displaystyle\lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x^4} = \frac{1}{a}$ where $a =$ ______ .

(12) $\displaystyle\lim_{x\to 0} \frac{\cos x + \frac{1}{2}x^2 - 1}{5x^4} = \frac{1}{a}$ where $a =$ ________ .

(13) $\displaystyle\lim_{x\to 0} \frac{x^2 + 2\ln(\cos x)}{x^4} = -\frac{1}{a}$ where $a =$ _____ .

(14) $\displaystyle\lim_{x\to\infty} \left(1 - \frac{5}{2x}\right)^{4x} =$ ________ .

(15) $\displaystyle\lim_{x\to\infty} \left(\frac{\ln x}{x}\right)^{1/\ln x} =$ ________ .

(16) $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{x^3} - \frac{1}{x^2}\right) = -\frac{1}{a}$ where $a =$ _____ .

(17) $\displaystyle\lim_{x\to 1}\left[\frac{1}{x-1}-\frac{\ln x}{(x-1)^2}\right]=\frac{1}{a}$ where $a=$ _____ .

(18) $\displaystyle\lim_{x\to 1}\left[\frac{1}{2(x-1)}-\frac{1}{(x-1)^2}+\frac{\ln x}{(x-1)^3}\right]=\frac{1}{a}$ where $a=$ _____ .

(19) $\displaystyle\lim_{x\to 0^+}\left(\sqrt{\frac{1}{x^2}+\frac{1}{x}}-\sqrt{\frac{1}{x^2}-\frac{1}{x}}\right)=$ _____ .

(20) $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\arctan x}=$ ______ .

(21) Let $f(x)=x^2e^{1/x}$ for all $x\neq 0$. Then $\displaystyle\lim_{x\to 0^-}f(x)=$ _____ and $\displaystyle\lim_{x\to 0^+}f(x)=$ _____ .

(22) $\displaystyle\lim_{x\to\infty}(x\ln(5x))^{3/\ln x}=$ ______ .

(23) $\displaystyle\lim_{x\to 0}\left[\frac{1}{x^2}+\frac{2}{x^4}\ln\cos x\right]=$ ________ .

(24) $\lim_{x\to 0^+}(\sin x)^{\tan x}=$ _____ .

(25) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)=$ _____ .

(26) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\sec^2 x-\tan^2 x)=$ _____ .

(27) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\sec^3 x-\tan^3 x)=$ _____ .

(28) $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^{25}}{x}=$ _____ .

(29) $\displaystyle\lim_{x\to\infty}\left(1-\frac{5}{7x}\right)^{2x}=e^{-a/7}$ where $a=$ ______ .

(30) $\displaystyle\lim_{x\to\infty}\left(\frac{3x}{e^{2x}+7x^2}\right)^{1/x}=e^a$ where $a=$ ______ .

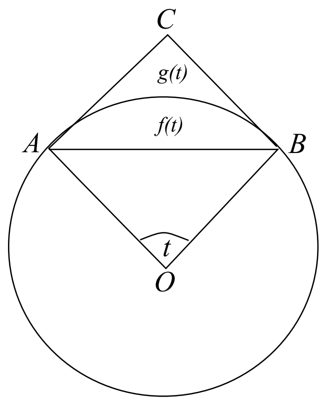
(31) $\displaystyle\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)^{1/\ln x}=e^a$ where $a=$ ______ .

9.3. Problems

(1) Is the following a correct application of l'Hôpital's rule? Explain.

$$\lim_{x\to 1}\frac{2x^3-3x+1}{x^4-1}=\lim_{x\to 1}\frac{6x^2-3}{4x^3}=\lim_{x\to 1}\frac{12x}{12x^2}=\lim_{x\to 1}\frac{1}{x}=1.$$

(2) Let t be the measure of a central angle $\angle AOB$ of a circle. The segments AC and BC are tangent to the circle at points A and B, respectively. The triangular region $\triangle ABC$ is divided into the region outside the circle whose area is $g(t)$ and the region inside the circle with area $f(t)$. Find $\displaystyle\lim_{t\to 0}\frac{f(t)}{g(t)}$.



(3) Let $f(x)=x^{(x-1)^{-1}}$ for $x>0$, $x\neq 1$. How should $f(1)$ be defined so that f is continuous on $(0,\infty)$? Explain your reasoning carefully.

(4) Show that the curve $y=x(\ln x)^2$ does not have a vertical asymptote at $x=0$.

(5) Define $f(x)=(x^2)^x$ for all $x\neq 0$. Define $f(0)$ in such a way as to make f a continuous function on $\mathbb{R}$. Find all critical points of f. Determine the intervals on which f is increasing, decreasing, concave up, concave down. Take special care to describe what happens at $x=0$. Use Newton's method to find to 4 decimal place accuracy any points of inflection which may occur.

(6) Let $f(x)=\dfrac{x\ln x}{x-1}$ for $x>0$ and $x\neq 1$.
  (a) How should f be defined at $x=1$ so that f will be continuous on $(0,\infty)$? Explain how you know your answer is correct.
  (b) Suppose $f(1)$ has the value you found in (a). Then find $f'(1)$ (and explain what you are doing).
  (c) Suppose $f(1)$ has the value you found in (a). Find $f''(1)$ (and explain what you are doing).
  (d) Suppose $f(1)$ has the value you found in (a). Give a careful proof that f'' is continuous at $x=1$.

(7) Your good friend Fred is confused again. He is trying to find $\ell=\displaystyle\lim_{x\to 0^+}(1+x)^{1/x}$. It is clear him that for $x>0$ the quantity in parentheses, $1+x$, is always strictly greater than 1. Further more the power $\dfrac{1}{x}$ is going to infinity as x approaches 0 from the right. So ℓ is the result of taking a number strictly greater than 1 to higher and higher powers and, therefore, $l=\infty$. On the other hand he sees that $1+x$ is approaching 1 as x approaches 0, and 1 taken to any power whatever is 1. So $\ell=1$. Help Fred by pointing out to him the error of his ways.

9.4. Answers to Odd-Numbered Exercises

(1) 6

(3) $\dfrac{1}{2}n(n+1)$

(5) 5

(7) 19

(9) 4

(11) 12

(13) 6

(15) $\dfrac{1}{e}$

(17) 2

(19) 1

(21) 0, ∞

(23) $-\dfrac{1}{6}$

(25) 0

(27) ∞

(29) 10

(31) -1

CHAPTER 10

MONOTONICITY AND CONCAVITY

10.1. Background

Topics: increasing, decreasing, monotone, concave up, concave down.

10.1.1. Definition. A real valued function f defined on an interval J is INCREASING on J if $f(a) \le f(b)$ whenever $a, b \in J$ and $a \le b$. It is STRICTLY INCREASING on J if $f(a) < f(b)$ whenever $a, b \in J$ and $a < b$. The function f is DECREASING on J if $f(a) \ge f(b)$ whenever $a, b \in J$ and $a \le b$. It is STRICTLY DECREASING on J if $f(a) > f(b)$ whenever $a, b \in J$ and $a < b$.

10.1.2. Definition. Let $f \colon A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$. The function f has a LOCAL (or RELATIVE) MAXIMUM at a point $a \in A$ if there exists $r > 0$ such that $f(a) \ge f(x)$ whenever $|x - a| < r$ and $x \in \operatorname{dom} f$. It has a LOCAL (or RELATIVE) MINIMUM at a point $a \in A$ if there exists $r > 0$ such that $f(a) \le f(x)$ whenever $|x - a| < r$ and $x \in \operatorname{dom} f$. The point a is a RELATIVE EXTREMUM of f if it is either a relative maximum or a relatives minimum.

The function $f \colon A \to \mathbb{R}$ is said to attain a MAXIMUM at a if $f(a) \ge f(x)$ for all $x \in \operatorname{dom} f$. This is often called a GLOBAL (or ABSOLUTE) MAXIMUM to help distinguish it from the local version defined above. It is clear that every global maximum is also a local maximum but not *vice versa.* (Of course, similar definitions hold for *global* or *absolute minima* and *global* or *absolute extrema.*)

10.1.3. Definition. A real valued function f defined on an interval J is CONCAVE UP on J if the chord line connecting any two points $(a, f(a))$ and $(b, f(b))$ on the curve (where $a, b \in J$) always lies on or above the curve. It is CONCAVE DOWN if the chord line always lies on or below the curve. A point on the curve where the concavity changes is a POINT OF INFLECTION.

When f is twice differentiable it is concave up on J if and only if $f''(c) \ge 0$ for all $c \in J$ and is concave down on J if and only if $f''(c) \le 0$ for all $c \in J$.

10.2. Exercises

(1) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{x}{x+2} - \dfrac{x+3}{x-4}$. Then the intervals on which f is increasing are (______ , ______) and (________ , _____).

(2) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{-x}{(x^2+1)^2}$.

(a) The interval on which the function f is increasing is (______ , _____).

(b) Estimate to two decimal places the location of a point $x > 0$ where f has a point of inflection. Answer: ___ . ___ ___ .

(3) A function f is defined on the interval $[0, \pi]$. Its derivative is given by $f'(x) = \cos x - \sin 2x$.

(a) The intervals on which f is increasing are $\left(a\,,\dfrac{\pi}{b}\right)$ and $\left(\dfrac{\pi}{c}\,,\dfrac{d\pi}{b}\right)$ where $a =$ _____, $b =$ _____, $c =$ _____, and $d =$ _____.

(b) Estimate to two decimal places the location of points of inflection.
Answer: 1. ___ ___ and 2. ___ ___ .

(4) Suppose that the derivative of a function f is given by $f'(x) = (x-2)^2(x+4)$.

(a) The interval on which f is increasing is (______ , _____) .

(b) f has no local ________________ .

(c) f has a local ________________ at $x =$ ______ .

(5) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{x+1}{\sqrt{x^2+1}}$.

(a) The interval on which f is increasing is (______ , _____) .

(b) f has no local ________________ .

(c) f has a local ________________ at $x =$ ______ .

(6) Suppose that the derivative of a function f is given by $f'(x) = \ln(1 + x^2)$.

(a) The interval on which f is increasing is (______ , _____) .

(b) At how many points does f have a local maximum? Answer: _____ .

(c) At how many points does f have a local minimum? Answer: _____ .

(d) The interval on which f is concave up is (_____ , _____) .

(e) f has a point of inflection at $x =$ _____ .

(7) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{1}{x^2+1}$.

(a) The interval on which f is increasing is (______ , _____) .

(b) At how many points does f have a local maximum? Answer: _____ .

(c) At how many points does f have a local minimum? Answer: _____ .

(d) The interval on which f is concave up is (______ , _____) .

(e) f has a point of inflection at $x =$ _____ .

(8) Suppose that the derivative of a function f is given by $f'(x) = \dfrac{x}{x^2+1}$.

(a) The interval on which f is increasing is (_____ , _____) .

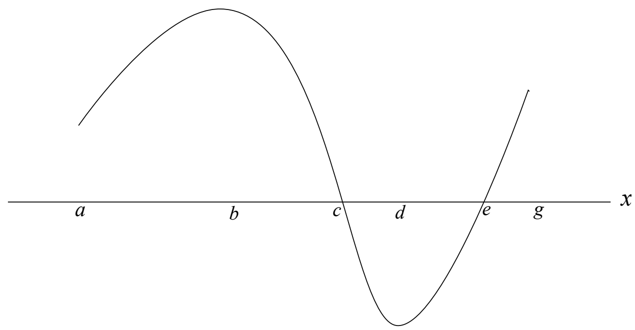
(b) At how many points does f have a local maximum? Answer: _____ .

(c) At how many points does f have a local minimum? Answer: _____ .

(d) The interval on which f is concave up is (______ , _____) .

(e) f has points of inflection at $x =$ ______ and $x =$ _____ .

(9) The domain of a function f is $[a, g]$. Below is a sketch of the graph of the *derivative* of f.



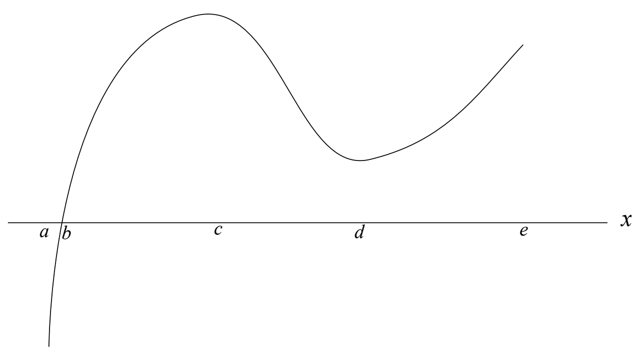
(a) The largest intervals on which f is increasing are (_____ , _____) and (_____ , _____) .

(b) f has local minima at $x =$ _____ and $x =$ _____ .

(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .

(d) f has points of inflection at: $x =$ _____ and $x =$ _____ .

(10) The domain of a function f is $[a, e]$. Below is a sketch of the graph of the *derivative* of f.



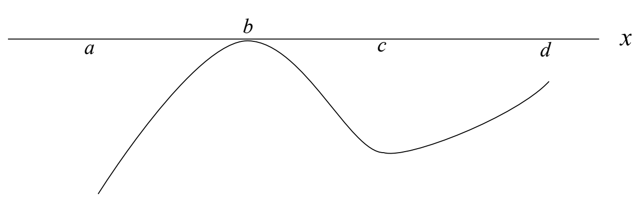
(a) The largest interval on which f is increasing is (_____ , _____) .

(b) f has local maxima at $x =$ _____ and $x =$ _____ .

(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .

(d) f has points of inflection at: $x =$ _____ and $x =$ _____ .

(11) The domain of a function f is $[a, d]$. Below is a sketch of the graph of the *derivative* of f.



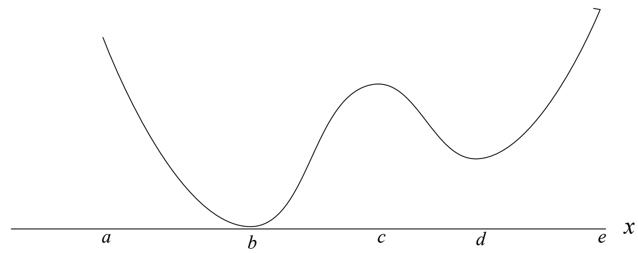
(a) The largest interval on which f is decreasing is (_____ , _____) .

(b) f has a local maximum at $x =$ _____ .

(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .

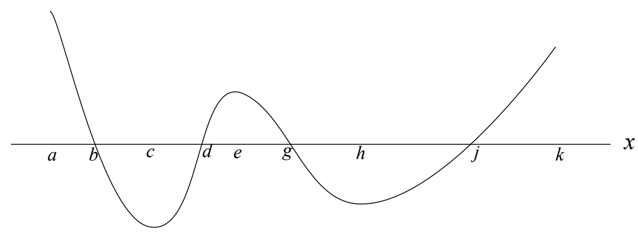
(d) f has points of inflection at: $x =$ _____ and $x =$ _____ .

(12) The domain of a function f is $[a, e]$. Below is a sketch of the graph of the *derivative* of f.



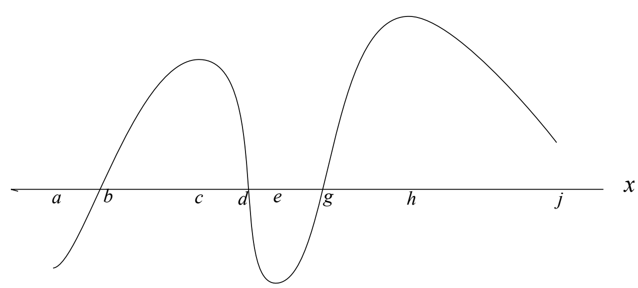
(a) The largest interval on which f is increasing is (_____ , _____) .
(b) f has local minimum at $x =$ _____ .
(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .
(d) f has points of inflection at: $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(13) The domain of a function f is $[a, k]$. Below is a sketch of the graph of the *derivative* of f.



(a) The largest intervals on which f is decreasing are (_____ , _____) and (_____ , _____) .
(b) f has local minima at $x =$ _____ , $x =$ _____ , and $x =$ _____ .
(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .
(d) f has points of inflection at: $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(14) The domain of a function f is $[a, j]$. Below is a sketch of the graph of the *derivative* of f.



(a) The largest intervals on which f is increasing are (_____ , _____) and (_____ , _____) .
(b) f has local maxima at $x =$ _____ , $x =$ _____ , and $x =$ _____ .
(c) The largest intervals on which f is concave up are (_____ , _____) and (_____ , _____) .
(d) f has points of inflection at: $x =$ _____ , $x =$ _____ , and $x =$ _____ .

(15) Consider the function $f\colon x \mapsto x^2 e^{-x}$.
(a) The interval on which f is increasing is (_____ , _____) .
(b) f has a local minimum at $x =$ _____ .
(c) The interval on which f is concave down is $(a - \sqrt{a}, a + \sqrt{a})$ where $a =$ _____ .

(16) The intervals on which the function $f(x) = x^2 + \dfrac{16}{x^2}$ is increasing are (_____ , _____) and (_____ , _____) .

(17) Consider the function $f\colon x \mapsto \dfrac{1}{x}e^x$.

(a) The intervals on which f is decreasing are (______ , _____) and (_____ , _____) .

(b) How many local maxima does f have? Answer: _____ .

(c) The interval on which f is concave up is (_____ , _____) .

(18) Consider the function $f\colon x \mapsto x\exp\left(-\frac{1}{2}x^2\right)$.

(a) The intervals on which f is decreasing are (______ , ______) and (_____ , _____) .

(b) The intervals on which f is concave up are $(-a, 0)$ and (a, ∞) where $a =$ ______ .

(c) f has how many points of inflection? Answer: _____ .

(19) Consider the function $f\colon x \mapsto \ln(4 - x^2)$.

(a) The domain of f is the interval (_____ , _____) .

(b) The interval on which f is increasing is (_____ , _____) .

(c) f is concave __________ .

(20) Consider the function $f\colon x \mapsto x\ln x$.

(a) The domain of f is the interval (_____ , _____) .

(b) $\displaystyle\lim_{x\to 0^+} f(x) =$ _____ .

(c) The interval on which f is positive is (_____ , _____) .

(d) The interval on which f is increasing is (________ , _____) .

(e) The function f attains its minimum value of $-\dfrac{1}{a}$ at $x = \dfrac{1}{b}$ where $a =$ ________ and $b =$ ______ .

(f) f is concave __________ .

(21) Consider the function $f\colon x \mapsto x^2\ln x$.

(a) The domain of f is the interval (_____ , _____) .

(b) $\displaystyle\lim_{x\to 0^+} f(x) =$ _____ .

(c) The interval on which f is positive is (_____ , _____) .

(d) The interval on which f is increasing is (________ , _____) .

(e) The function f attains its minimum value of $-\dfrac{1}{a}$ at $x = \dfrac{1}{b}$ where $a =$ ________ and $b =$ ______ .

(f) f has a point of inflection at $x = e^p$ where $p =$ ________ .

(22) Consider the function $f\colon x \mapsto x(\ln x)^2$.

(a) The domain of f is the interval (_____ , _____) .

(b) $\displaystyle\lim_{x\to 0^+} f(x) =$ _____ .

(c) The intervals on which f is increasing are $\left(0, \dfrac{1}{a}\right)$, where $a =$ ________ , and (_____ , _____) .

(d) The function f attains its minimum value of _____ at $x =$ _____ .

(e) f has a point of inflection at $x = e^p$ where $p =$ ______ .

(23) Consider the function $f\colon x \mapsto \dfrac{1}{x}\ln x$.

(a) The domain of f is the interval (_____ , _____) .

(b) $\lim\limits_{x\to 0^+} f(x) =$ ______ .

(c) The interval on which f is increasing is (_____ , _____) .

(d) The function f attains its maximum value of ________ at $x =$ _____ .

(e) f has a point of inflection at $x = e^p$ where $p =$ ________ .

(24) Let $f(x) = e^x \sin x$ for $0 \le x \le \pi$. Then f has its global maximum at $x =$ __________ ; it has its global minimum at $x =$ _____; and it has a point of inflection at $x =$ ________ .

(25) Find real numbers a and b such that $x = 1$ is a critical point of the function f where $f(x) = ax + \dfrac{b}{x^2}$ for all $x \ne 0$ and $f(1) = 3$. Answer: $a =$ _____ and $b =$ _____ . Then the point $(1, 3)$ a local ____________.

(26) Let $f(x) = \dfrac{x^2 - 5}{x^2 + 3}$ for all $x \ge -1$. The function f has a global minimum at $x =$ _____ and a local maximum at $x =$ _____ .

(27) Consider the function $f\colon x \mapsto \dfrac{x^2 - 2x}{(x + 1)^2}$.

(a) The intervals on which f is increasing are (______ , ______) and (______ , _____) .

(b) f has a global minimum at $x =$ _____ .

(c) The intervals on which f is concave up are (______ , ______) and (______ , ________) .

(d) f has a point of inflection at: $x =$ ________ .

(28) Consider the function $f\colon x \mapsto \dfrac{2x^2}{x^2 + 2}$.

(a) The interval on which f is increasing is (_____ , _____) .

(b) f has global minimum at $x =$ _____ .

(c) The interval on which f is concave up is $(-\sqrt{a}, \sqrt{a})$ where $a =$ ________ .

(29) Consider the function $f\colon x \mapsto \dfrac{6}{x^2} - \dfrac{6}{x}$.

(a) The intervals on which f is increasing are (______ , _____) and (_____ , _____) .

(b) f has a global minimum at $x =$ _____ .

(c) The intervals on which f is concave up are (______ , _____) and (_____ , _____) .

(30) Consider the function $f\colon x \mapsto \dfrac{|x - 1|}{|x| - 1}$.

(a) The intervals on which f is strictly increasing are (______ , ______) and (______ , _____) .

(b) f is constant on the intervals [_____ , _____) and (_____ , _____) .

(c) f has a vertical asymptote at $x =$ ______ and a horizontal asymptote at $y =$ _____ .

(d) The only point in the domain of f at which f is not differentiable is $x =$ _____ .

(e) f has how many points of inflection? Answer: _____ .

(31) Let $f(x) = \frac{1}{4}x^3 - 3x + 7$ for $-4 \le x \le 3$. Then f has (local) maxima at $x =$ _____ and $x =$ _____ . The global maximum of f occurs at $x =$ _____ . The maximum value of f is _____ .

(32) Let $f(x) = \sqrt{x} + \dfrac{4}{x}$ for $\frac{1}{4} \le x \le 100$. The maximum value attained by $f(x)$ is $\dfrac{a}{2}$ where $a =$ ______ .

10.3. Problems

(1) Suppose that a function f is increasing on the interval $(-\infty, -5)$ and is also increasing on the interval $(-5, \infty)$. Is it necessarily the case that f must be increasing on the set $(-\infty, -5) \cup (-5, \infty)$? Explain.

(2) An EQUATION OF STATE of a substance is an equation expressing a relationship between the pressure P, the volume V, and the temperature T of the substance. A VAN DER WAALS GAS is a gas for which there exist positive constants a and b (depending on the particular gas) such that the following equation of state holds:

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT. \tag{$*$}$$

(Here R is a universal constant, *not* depending on the particular gas.) For each fixed value of T the equation of state ($*$) can be used to express P as a function of V, say $P = f(V)$. A CRITICAL TEMPERATURE, which we denote by T_c, is a value of T for which the corresponding function f possesses a critical point which is also a point of inflection. The V and P coordinates of this critical point are denoted by V_c and P_c and are called the CRITICAL VOLUME and the CRITICAL PRESSURE.

Show that every van der Waals gas has a critical temperature. Compute the critical values RT_c, V_c, and P_c (in terms of the gas constants a and b). Explain how you know that the point (V_c, P_c) is a point of inflection.

(3) A water storage tank consists of two parts: the bottom portion is a cylinder with radius 10 feet and height 50 feet; the top portion is a sphere of radius 25 feet. (A small bottom portion of the sphere is missing where it connects to the cylinder.) The tank is being filled from the bottom of the cylindrical portion with water flowing in at a constant rate of 100 cubic feet per minute. Let $h(t)$ be the height of the water in the tank at time t. Sketch a graph of the function h from the time the filling starts to the time the tank is full. Explain carefully the reasoning behind all properties of your graph—paying particular attention to its concavity properties.

(4) For what values of $k > 0$ does the function f defined by

$$f(x) = \frac{\ln x}{k} - \frac{kx}{x+1}$$

have local extrema? For each such k locate and classify the extrema. Explain the reasons for your conclusions carefully.

10.4. Answers to Odd-Numbered Exercises

(1) $-\infty$, -2, $-\dfrac{2}{3}$, 4

(3) (a) 0, 6, 2, 5
(b) 0, 0, 1, 4

(5) (a) -1, ∞
(b) maximum
(c) minimum. -1

(7) (a) $-\infty$, ∞
(b) 0
(c) 0
(d) $-\infty$, 0
(e) 0

(9) (a) a, c, e, g
(b) a, e
(c) a, b, d, g
(d) b, d

(11) (a) a, d
(b) a
(c) a, b, c, d
(d) b, c

(13) (a) b, d, g, j
(b) a, d, j
(c) c, e, h, k
(d) c, e, h

(15) (a) 0, 2
(b) 0
(c) 2

(17) (a) $-\infty$, 0, 0, 1
(b) 0
(c) 0, ∞

(19) (a) -2, 2
(b) -2, 0
(c) down

(21) (a) 0, ∞
(b) 0
(c) 1, ∞
(d) $\dfrac{1}{\sqrt{e}}$, ∞
(e) $2e$, $\sqrt{e}$
(f) $-\dfrac{3}{2}$

(23) (a) 0, ∞
(b) $-\infty$
(c) 0, e
(d) $\dfrac{1}{e}$, e
(e) $\dfrac{3}{2}$

(25) 2, 1, minimum

(27) (a) $-\infty$, -1, $\frac{1}{2}$, ∞
(b) $\frac{1}{2}$
(c) $-\infty$, -1, -1, $\frac{5}{4}$
(d) $\frac{5}{4}$

(29) (a) $-\infty$, 0, 2, ∞
(b) 2
(c) $-\infty$, 0, 0, 3

(31) -2, 3, -2, 11

CHAPTER 11

INVERSE FUNCTIONS

11.1. Background

Topics: inverse functions and their derivatives, logarithmic functions, the natural logarithm, exponential functions, trigonometric and inverse trigonometric functions, implicit differentiation.

The following two facts may be helpful in solving problem 1

11.1.1. Proposition. *Every real number is the limit of a sequence of rational numbers. That is, if a is a real number, then there are rational numbers x_1, x_2, x_3, ... such that* $\lim_{n\to\infty} x_n = a$.

11.1.2. Proposition. *If g is a continuous function and x_1, x_2, x_3, ... are real numbers such that* $\lim_{n\to\infty} x_n = a$, *then* $\lim_{n\to\infty} g(x_n) = g(a)$.

11.2. Exercises

(1) Let $f(x) = x^5 + 3x^3 + x - 10$. Then $Df^{-1}(48) = \dfrac{1}{a}$ where $a =$ ________ .

(2) Let $f(x) = \dfrac{3}{(x-1)^4}$ for $x \geq 1$. Then $f^{-1}(243) = -a\,3^p$ where $a =$ _____ and $p =$ _____ .

(3) Let $f(x) = \ln(x-2) + e^{x^2}$ for $x > 2$. Then $Df^{-1}(e^9) = (1 + ae^b)^{-1}$ where $a =$ _____ and $b =$ _____ .

(4) Let $f(x) = \ln \dfrac{1+x}{1-x}$ for $-1 < x < 1$. Then $Df^{-1}(\ln 5) = \dfrac{5}{a}$ where $a =$ ________ .

(5) Let $f(x) = \dfrac{2+x}{5-x}$. Then $f^{-1}(x) = \dfrac{ax+b}{x+1}$ where $a =$ _____ and $b =$ ______ .

(6) Let $f(x) = \exp\left(\dfrac{1}{1-x}\right)$. Then $f^{-1}(x) = 1 - \big(g(x)\big)^p$ where $g(x) =$ ________ and $p =$ _____ .

(7) Let $f(x) = \arctan\big(8x^3 + 2\big)$. Then $f^{-1}(x) = \dfrac{1}{a}(\tan x + b)^p$ where $a =$ _____ , $b =$ _____ , and $p =$ ________ .

(8) Let $f(x) = \sin^3 2x$ for $\frac{-\pi}{4} \leq x \leq \frac{\pi}{4}$. Then $Df^{-1}(\frac{1}{8}) = \dfrac{a}{b\sqrt{b}}$ where $a =$ _____ and where $b =$ _____ .

(9) Let $f(x) = \frac{4}{3}x^4 - 8x^3 + 18x^2 - 18x + \frac{27}{4}$ for $x < \frac{3}{2}$. Then $Df^{-1}(\frac{27}{4}) = -\dfrac{1}{a}$ where $a =$ ______ .

(10) Let $f(x) = x^3 + \ln(x-1)$ for $x > 1$. Then $Df^{-1}(8) = \dfrac{1}{a}$ where $a =$ ______ .

(11) Let $f(x) = \ln \dfrac{x^2+1}{x^2-1}$ for $x > 1$. Then $Df^{-1}(\ln 5 - \ln 3) = -\dfrac{a}{8}$ where $a =$ ______ .

(12) What is the area of the largest rectangle that has one corner at the origin, one corner on the negative y-axis, one corner on the positive x-axis, and one corner on the curve $y = \ln x$?

Answer: the area is ___________ .

(13) What is the area of the largest rectangle that has one corner at the origin, one corner on the negative x-axis, one corner on the positive y-axis, and one corner on the curve $y = e^x$?

Answer: the area is ___________ .

(14) Solve the equation: $1 + \log_{10}(x-4) = \log_{10}(x+5)$. Answer: $x =$ _____ .

(15) Let $f(x) = \log_3(\log_2 x)$. Then $Df(e) = \dfrac{1}{ae}$ where $a =$ ________ .

(16) Suppose p, $q > 0$ and $\log_9(p) = \log_{12}(q) = \log_{16}(p+q)$. Find $\dfrac{q}{p}$. Express your answer in a form that involves neither exponentials nor logarithms.

Answer: $\dfrac{q}{p} = \dfrac{1+a}{2}$ where $a =$ _____ .

(17) A triangle is bounded by the x-axis, the y-axis, and the tangent line to the curve $y = 2^x$ at $x = 0$. The area of this triangle is $\dfrac{1}{a \ln a}$ where $a =$ _____ .

(18) $\displaystyle\lim_{x\to 0} \frac{3^{4+x} - 3^4}{x} = a \ln b$ where $a =$ ______ and $b =$ _____ .

(19) $\displaystyle\lim_{t\to 0}\frac{\log_5(t+0.04)+2}{t}=\frac{a}{\ln b}$ where $a =$ ________ and $b =$ _____ .

(20) If $y=\arcsin\left(\dfrac{x^2}{3}\right)$, then $\dfrac{dy}{dx}=\dfrac{ax}{\sqrt{b-x^4}}$ where $a =$ _____ and $b =$ _____ .

(21) A tapestry 30 feet high is hung so that its lower edge is 24 feet above the eye of an observer. How far from the tapestry should the observer stand in order to maximize the visual angle subtended by the tapestry? Answer: ________ ft.

(22) Let $f(x)=\arctan\left(\dfrac{x^2}{1+x}\right)$. Then $Df(1)=\dfrac{a}{5}$ where $a =$ _____ .

(23) Let $f(x)=\arctan\left(\dfrac{1}{x}\right)$. Then $f\left(\dfrac{1}{\sqrt{3}}\right)=\dfrac{\pi}{a}$ and $f'\left(\dfrac{1}{\sqrt{3}}\right)=-\dfrac{a}{b}$ where $a =$ _____ and $b =$ _____ .

(24) Let $f(x)=x^{\arcsin x}$. Then $f(1)=$ _____ and $f'(1)=$ __________ .

(25) A solution to the equation $\arcsin x-\arccos x=0$ is $x=\dfrac{1}{a}$ where $a =$ ________ .

(26) Let $f(x)=\arctan 2x-\arctan x$ for $x\geq 0$.
  (a) The function f is increasing on the interval (_____ , ________).
  (b) The function f has a local maximum at $x =$ ________ .
  (c) The function f has a local minimum at $x =$ ________ .

(27) Let $f(x)=\ln(\arctan\sqrt{x^2-1})$. Then $f'(2)=\dfrac{\sqrt{a}}{b\pi}$ where $a =$ _____ and $b =$ _____ .

(28) The expression $e^{-\frac{3}{4}\ln 81}$ is a complicated way of writing the integer _____ .

(29) The expression $\dfrac{\ln 81}{(\ln 27)^2}\ln 3\sqrt{3}$ is a complicated way of writing the fraction $\dfrac{a}{3}$ where $a =$ _____ .

(30) The solution to the differential equation $y'=(2x-1)y$ which satisfies the initial condition $y(0)=3$ is $y=ae^{f(x)}$ where $a =$ _____ and $f(x) =$ _____________ .

(31) The solution to the differential equation $y'=4x^3y$ which satisfies the initial condition $y(0)=7$ is $y=ae^{f(x)}$ where $a =$ _____ and $f(x) =$ ________ .

(32) Let $f(x)=e^{x^2+\ln x}$ for $x>0$. Then $Df^{-1}(e)=\dfrac{1}{ae}$ where $a =$ _____ .

(33) The equation of the tangent line at the point $(1,0)$ to the curve whose equation is

$$x\sin y+x^3=\arctan(e^y)+x-\frac{\pi}{4}$$

is $y=-ax+a$ where $a =$ _____.

11.3. Problems

(1) Show that the natural logarithm is the only continuous function f defined on the interval $(0, \infty)$ which satisfies

$$f(xy) = f(x) + f(y) \qquad \text{for all } x, y > 0$$

and

$$f(e) = 1.$$

Hint. Assume that you are given a function $f\colon (0, \infty) \to \mathbb{R}$ about which you know only three things:

(i) f is continuous;
(ii) $f(xy) = f(x) + f(y)$ for all $x, y > 0$; and
(iii) $f(e) = 1$.

What you must prove is that

$$f(x) = \ln x \qquad \text{for every } x > 0. \tag{11.1}$$

The crucial result that you will need to prove is that

$$f(u^r) = r\, f(u) \tag{11.2}$$

holds for every real number $u > 0$ and every rational number r. Once you have this, then you can use propositions 11.1.1 and 11.1.2 to conclude that

$$f(e^y) = y \qquad \text{for every real number } y.$$

Then substituting $\ln x$ for y will give you the desired result (11.1).

Prove (11.2) first for the case $r = n$ where n is a natural number. Then prove it for the case $r = 1/n$ where n is a natural number. Use these results to show that (11.2) holds for every positive rational number. Next deal with the case $r = 0$. Finally verify (11.2) for the case where r is a negative rational number. (To do this prove that $f(1/v) = -f(v)$ for all $v > 0$ by substituting v for x and $1/v$ for y in (ii).)

(2) Prove that

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

whenever $xy \neq 1$. *Hint.* Let y be an arbitrary, but fixed, real number. Define $f(x) = \arctan x + \arctan y$ and $g(x) = \arctan \dfrac{x+y}{1-xy}$. Compare the derivatives of f and g.

(3) Prove that $\arctan x$ and $\arctan \dfrac{1+x}{1-x}$ differ by constants on the intervals $(-\infty, 1)$ and $(1, \infty)$. Find the appropriate constants. Show how to use this information to find $\displaystyle\lim_{x\to 1^-} \arctan \frac{1+x}{1-x}$ and $\displaystyle\lim_{x\to 1^+} \arctan \frac{1+x}{1-x}$.

(4) Give a careful proof that

$$\frac{x}{x^2+1} \leq \arctan x \leq x$$

for all $x \geq 0$.

(5) Define $f(x) = (x^2)^x$ for all $x \neq 0$. Define $f(0)$ in such a way as to make f a continuous function on $\mathbb{R}$. Sketch the function f. Locate all critical points and identify the intervals on which f is increasing, is decreasing, is concave up, and is concave down. Take special care to describe what happens at $x = 0$. Use Newton's method to find to 4 decimal place accuracy any points of inflection which may occur.

(6) Let $f(x) = 2x + \cos x + \sin^2 x$ for $-10 \leq x \leq 10$. Show that f has an inverse.

(7) Let $f(x) = \dfrac{4x+3}{x+2}$.
  (a) Show that f is one-to-one.
  (b) Find $f^{-1}(-2)$.
  (c) Find dom f^{-1}.

(8) Let $f(x) = e^{3x} + \ln x$ for $x > 0$. Prove that f has an inverse and calculate $Df^{-1}(e^3)$.

(9) Let $f(x) = \ln(1+x) - \ln(1-x)$ for $-1 < x < 1$. Prove that f has an inverse and find $f^{-1}(x)$.

(10) Show that there is exactly one number x such that $e^{-x} = x^3 - 9$. Locate the number between consecutive integers.

(11) Show that there is exactly one number x such that $e^{2x} = 10 - x^3$. Locate the number between consecutive integers.

(12) Show that there is exactly one number x such that $\ln x + x = 0$.

(13) Use the *mean value theorem* to show that $x + 1 < e^x < 2x + 1$ whenever $0 < x \leq \ln 2$.

(14) (a) Find $\displaystyle\lim_{x\to 1} \frac{1}{x-1}$.
  (b) Find $\displaystyle\lim_{x\to 1} \frac{\ln x}{(x-1)^2}$.
  (c) Find $\displaystyle\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{\ln x}{(x-1)^2}\right)$.

(15) Suppose that $f\colon (0,\infty) \to \mathbb{R}$ is a continuous function on $(0,\infty)$ such that $f(x) = \dfrac{x\ln x}{x-1}$ for every $x > 0$ except $x = 1$. Prove that $f''(x)$ (exists and) is continuous at $x = 1$.

(16) Let $0 < a < b$. Use the *mean value theorem* to show that

$$1 - \frac{a}{b} < \ln\frac{b}{a} < \frac{b}{a} - 1.$$

11.4. Answers to Odd-Numbered Exercises

(1) 117

(3) 6, 9

(5) 5, -2

(7) 2, -2, $\frac{1}{3}$

(9) 18

(11) 15

(13) $\frac{1}{e}$

(15) $\ln 3$

(17) 2

(19) 25, 5

(21) 36

(23) 3, 4

(25) $\sqrt{2}$

(27) 3, 2

(29) 2

(31) 7, x^4

(33) 4

CHAPTER 12

APPLICATIONS OF THE DERIVATIVE

12.1. Background

Topics: antiderivatives, related rates, optimization, Newton's method.

This chapter makes no pretense of presenting interesting "real-world" applications of the differential calculus. Its purpose is simply to make some elementary connections between the mathematical concept of *derivative* and various instances of *rates of change* of physical quantities.

Newton's Law of Cooling: the rate of cooling of a hot body is proportional to the difference between its temperature and that of the surrounding medium.

12.2. Exercises

(1) One leg of a right triangle decreases at 1 in./min. and the other leg increases at 2 in./min. At what rate is the area changing when the first leg is 8 inches and the second leg is 6 inches? Answer: _____ in^2/min.

(2) The volume of a sphere is increasing at the rate of 3 cubic feet per minute. At what rate is the radius increasing when the radius is 8 feet? Answer: $\dfrac{a}{b\pi}$ ft/min where $a =$ _____ and $b =$ ___________ .

(3) A beacon on a lighthouse 1 mile from shore revolves at the rate of 10π radians per minute. Assuming that the shoreline is straight, calculate the speed at which the spotlight is sweeping across the shoreline as it lights up the sand 2 miles from the lighthouse. Answer: _________ miles/min.

(4) Two boats are moving with constant speed toward a marker, boat A sailing from the south at 8 mph and boat B approaching from the east. When equidistant from the marker the boats are $4\sqrt{2}$ miles apart and the distance between them is decreasing by $7\sqrt{2}$ mph. How fast is boat B going? Answer: _____ mph.

(5) A (right circular) cylinder is expanding in such a way that its height is increasing three times as rapidly as the radius of its base. At the moment when its height is 5 inches and the radius of its base is 3 inches its height is increasing at a rate of 12 inches per minute. At that moment its volume is increasing at a rate of ___________ cubic inches per minute.

(6) A cube is expanding in such a way that its edge is increasing at a rate of 4 inches per second. When its edge is 5 inches long, what is the rate of change of its volume? Answer: ________ in^3/sec.

(7) A kite 100 feet above the ground is being blown away from the person holding its string in a direction parallel to the ground and at a rate of 10 feet per second. At what rate must the string be let out when the length of string already let out is 200 feet? Answer: ________ ft/sec.

(8) A plane flying 4000 feet above the ground at a speed of 16,000 feet per minute is followed by a searchlight. It is flying in a straight line and passes directly over the light. When the angle between the beam and the ground is $\pi/3$ radians, what is the angular velocity of the beam? Answer: _____ radians/min.

(9) A lighthouse is 3 miles from (a straight) shore. The light makes 4 revolutions per minute. How fast does the light move along the shoreline when it makes an angle of $\pi/4$ radians with the shoreline? Answer: ____________ mi/min.

(10) Water leaking onto a floor creates a circular pool with an area that increases at the rate of 3 square inches per minute. How fast is the radius of the pool increasing when the radius is 10 inches? Answer: $\dfrac{a}{b\pi}$ in/min where $a =$ _____ and $b =$ ________ .

(11) A cube is expanding in such a way that the length of its diagonal is increasing at a rate of 5 inches per second. When its edge is 4 inches long, the rate at which its volume is increasing is ___________ in^3/sec.

(12) You are standing on a road, which intersects a railroad track at right angles, one quarter of a mile from the intersection. You observe that the distance between you and the approaching train is decreasing at a constant rate of 25 miles per hour. How far from the intersection is the train when its speed is 40 miles per hour? Answer: $\dfrac{5}{4\sqrt{a}}$ mi where $a =$ _________.

(13) A light shines on top of a lamppost 30 feet above the ground. A woman 5 feet tall walks away from the light. Find the rate at which her shadow is increasing if she is walking at 3 ft./sec. Answer: ______________ ft/sec.

(14) A balloon is going up, starting at a point on the ground. An observer 300 feet away looks at the balloon. The angle θ which a line to the balloon makes with the horizontal is observed to increase at $\frac{1}{10}$ rad./sec. How rapidly is the balloon rising when $\theta = \pi/6$? Answer: _________ ft/sec.

(15) A man is walking along a sidewalk at 6 ft./sec. A searchlight on the ground 24 feet from the walk is kept trained on him. At what rate is the searchlight revolving when the man is 18 feet from the point on the walk nearest the light? Answer: ____________ rad/sec.

(16) A 20 foot long ramp has one end on the ground and the other end at a loading dock 5 feet off the ground. A person is pushing a box up the ramp at the rate of 3 feet per second. How fast is the box rising? Answer: _________ ft/sec.

(17) What is the area of the largest rectangle (with sides parallel to the coordinate axes) which lies above the x-axis and below the parabola $y = 48 - x^2$? Answer: Area is _________ .

(18) A piece of cardboard is to be made into an open box by cutting out the corners and folding up the sides. Given a piece of cardboard 12 in. $\times$ 12 in. what size should the corner notches be so that the resulting box has maximum volume? Answer: they should be squares _____ inches on each side.

(19) Express 20 as the sum of two positive numbers x and y such that $x^3 + y^2$ is as small as possible. Answer: $x =$ ________ and $y =$ ________.

(20) The combined resistance R of two resistors R_1 and R_2 is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ (where R_1, $R_2 > 0$). Suppose $R_1 + R_2$ is a constant. How does one obtain maximum combined resistance? Answer: ________________ .

(21) Find the point on the curve $y^2 = \frac{5}{2}(x+1)$ which is nearest the origin. Answer: (___ , ___).

(22) Find the point on the curve $y = x^2$ which is closest to the point $(3, 0)$. Answer: (___ , ___).

(23) Find the lengths of the sides of the rectangle of largest area which can be inscribed in a semicircle of radius 8. (The lower base of the rectangle lies along the diameter of the semicircle.) Answer: the sides should have lengths ________ and ________ .

(24) Consider triangles in the first quadrant bounded by the x-axis, the y-axis, and a tangent line to the curve $y = e^{-x}$. The largest possible area for such a triangle is _________ .

(25) An open cylindrical tank of volume 192π cubic feet is to be constructed. If the material for the sides costs \$3 per square foot, and the material for the bottom costs \$9 per square foot, find the radius and height of the tank which will be most economical.

Answer: radius = _____ ft; height = ______ ft.

(26) Find the dimensions of the cylinder with the greatest volume which can be inscribed in a sphere of radius 1.

Answer: radius = ________ ; height = ________ .

(27) A farmer has 100 pigs each weighing 300 pounds. It costs \$.50 a day to keep one pig. The pigs gain weight at 10 pounds a day. They sell today for \$.75 a pound, but the price is falling by \$.01 a day. How many days should the farmer wait to sell his pigs in order to maximize his profit? Answer: ________ days.

(28) Consider a parallelogram inscribed in a triangle ABC in such a way that one vertex coincides with A while the others fall one on each side of the triangle. The maximum

possible area for such a parallelogram is what fraction of the area of the original triangle? *Hint.* Orient the triangle so that its vertices are $A = (0,0)$, $B = (a,0)$, and $C = (b,c)$. Let $(t,0)$ be the vertex of the parallelogram lying on AB and (x,y) be the vertex lying on BC. Use t as the independent variable. Find x and y in terms of t (and the constants a, b, and c). Answer: $\dfrac{r}{s}$ where $r =$ ____ and $s =$ ____ .

(29) Two men carry a $14\sqrt{7}$ ft. ladder down a $10\sqrt{5}$ ft. wide corridor. They turn into a second corridor, perpendicular to the first one, while keeping the ladder horizontal. Find the minimum possible width of the second corridor. Answer: ________ feet.

(30) At each point $a > 0$ the tangent line to the parabola $y = 1 - x^2$ and the positive coordinate axes form a triangle. The minimum possible area of such a triangle is $\dfrac{a}{b\sqrt{b}}$ where $a =$ ____ and $b =$ ____ .

(31) A window is in the shape of a rectangle surmounted by a semicircle. If the perimeter is to be 18 feet, find the dimensions which maximize the area.

Answer: the radius of the semicircle should be $\dfrac{a}{4+\pi}$ ft and the height of the rectangle should be $\dfrac{b}{4+\pi}$ ft where $a =$ ________ and $b =$ ________.

(32) What is the distance from the point $(8,4)$ to the tangent line to the curve $f(x) = 3x^2 - 4x + 6$ at $x = 1$? Answer: ________ .

(33) What are the dimensions of a rectangular box—with no top—of greatest volume that can be constructed from 120 sq. in. of material if the base of the box is to be twice as long as it is wide? Answer: width of base $= 2\sqrt{a}$ and height of box $= \dfrac{4}{b}\sqrt{a}$ where $a =$ ____ in. and $b =$ ____ in.

(34) Consider all rectangles which have two sides on the positive coordinate axes and which lie under the curve $y = 2\cos x$. The one with the largest perimeter has width ______ and height ______ .

(35) Consider all rectangles which have one side on the positive x-axis and which lie under the curve $y = 4\sin x$ with $0 \le x \le \pi$. The one with the largest perimeter has width ______ and height ______ .

(36) Suppose that $f(-1) = -6$ and that $f'(x) = 6x^2 - 2x + 7$ for all real numbers x. Then $f(1) =$ ________ .

(37) Suppose that $f''(x) = 18x - 14$, that $f'(-1) = 8$, and that $f(-1) = 9$. Then $f(1) =$ ________ .

(38) Suppose $f''(x) = 12x - 10$, $f(2) = -6$, and $f(-1) = -18$. Then $f(1) =$ ________ .

(39) Suppose that $f'''(x) = 6x + 6$, $f(0) = -7$, $f(1) = \frac{1}{4}$, and $f(2) = 19$. Then $f(x) = ax^4 + x^3 + bx^2 + cx + d$ where $a =$ ____ , $b =$ ____ , $c =$ ____ , and $d =$ ____ .

(40) A pan of warm water (109°F) was put in a refrigerator. Fifteen minutes later, the water's temperature was 97° F; fifteen minutes after that, it was 87°F. Using *Newton's law of cooling* we can conclude that the temperature of the refrigerator was ________ ^{o} F.

(41) An object is heated to 838° and then allowed to cool in air that is 70°. Suppose that it takes 2 hours to cool the object to 313°. Then it takes ________ minutes to cool the object to 646°. *Hint.* Use *Newton's law of cooling.*

(42) A quantity y varies with time. The rate of increase of y is proportional to $\cos^2 y$. The initial value of y is $\pi/6$, while its value at $t = 1$ is $\pi/3$.

(a) For what value of t does $y = \pi/4$? Answer: $t =$ ____________ .

(b) What is the long-run value of y? Answer: $\lim_{t\longrightarrow\infty} y(t) =$ ______ .

(43) A point is moving along the x-axis in such a way that its acceleration at each time t is $\frac{3}{4}\pi^2 \sin \frac{\pi}{2}t$. Initially the point is located 4 units to the left of the origin. One second later it is at the origin. Where is it at time $t = 5$?

Answer: ________ units to the ___________ of the origin.

(44) A cylindrical water tank standing on end has diameter 9 ft and height 16 ft. The tank is emptied through a valve at the bottom of the tank. The rate at which the water level decreases when the valve is open is proportional to the square root of the depth of the water in the tank. Initially the tank is full of water. Three minutes after the valve is opened the tank is only 1/4 full. How long does it take from the time the valve is opened to empty the tank? Answer: _____ minutes.

(45) A function f satisfies the following conditions:
(i) $f''(x) = 6x - 12$ for all x, and
(ii) the graph of the curve $y = f(x)$ passes through the point $(2, 5)$ and has a horizontal tangent at that point.
Then $f(x) = x^3 + ax^2 + bx + c$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(46) A physical quantity y, which takes on only positive values, varies with time t. It is known that the rate of change of y is proportional to $y^3(t + 1)^{-1/2}$, that initially $y = 1/3$, and that after 8 minutes $y = 1/5$.
(a) What is the value of y after 35 minutes? Answer: $y(35) =$ ________ .
(b) Approximately how many hours must one wait for y to become less than 1/15? Answer: ________ hours.

(47) The solution to the differential equation $\dfrac{dy}{dx} = 3x^{1/3}$ subject to the condition $y = 25$ when $x = 8$ is $y(x) = \frac{a}{4}x^{p/3} + b$ where $a =$ _____ , $p =$ _____ , and $b =$ _____ .

(48) The solution to the differential equation $\dfrac{d^2y}{dx^2} = \dfrac{6}{x^4}$ which satisfies the conditions $\dfrac{dy}{dx} = 3$ and $y = 2$ when $x = 1$ is $y(x) = ax^p + bx + c$ where $a =$ _____ , $p =$ _____ , $b =$ _____ , and $c =$ _____ .

(49) The solution to the differential equation $y'(x) = \sin x e^{\cos x}$ which satisfies the condition $y = 2$ when $x = \dfrac{\pi}{2}$ is $y(x) =$ ___________________ .

(50) The decay equation for (radioactive) radon gas is $y = y_0 e^{-0.18\,t}$ with t in days. About how long will it take the radon in a sealed sample of air to fall to 80% of its original value? (Give an approximate answer to two decimal places.)

Answer: __.2__ days.

(51) If the half-life of carbon 14 is approximately 5730 years, how old is a wooden axe handle that is found to contain only $\dfrac{1}{2\sqrt{2}}$ times the atmospheric proportion of carbon 14? Answer: ______________ years.

(52) The half-life of a radioactive substance is 10 years. If we start with 20 grams of this substance, then the amount remaining after 5 years is $a\sqrt{b}$ where $a =$ _____ and $b =$ _____ .

(53) If we assume exponential growth, what was the population of a city in 1930 if its population in 1940 was 750,000 and in 1970 was 1,296,000? Answer: _____________ .

(54) In 1920 the population of a city was 135,000 and in 1950 it was 320,000. Assuming exponential growth, the population in 1940 was approximately ________________ .

(55) An electric condenser discharges through a resistance, losing voltage at a rate proportional to the voltage remaining. If the initial voltage of 100 volts decreases to 50 volts in 3 seconds, then the function representing the voltage on the condenser at any time t is $100a^{f(t)}$ where $a =$ ______ and $f(t) =$ ________ .

(56) Find a function ϕ such that $y(x) = \sin \dfrac{1}{x}$ is a solution to the differential equation

$$\big(\phi(x)\, y'(x)\big)' + \frac{y(x)}{x^2} = 0\,.$$

Answer: $\phi(x) =$ _________ will work.

(57) Use *Newton's method* to find the first three estimates to $\sqrt{5}$ starting at $x = 2$.

Answer: $x_1 = \dfrac{a}{4}$ where $a =$ _____.

$x_2 = \dfrac{a}{72}$ where $a =$ __________.

$x_3 = \dfrac{a}{23184}$ where $a =$ ________________.

(58) Use *Newton's method* to find the first four estimates to $\sqrt{3}$ starting at $x = 1$.

Answer: $x_1 =$ _____.

$x_2 = \dfrac{a}{4}$ where $a =$ __________.

$x_3 = \dfrac{a}{56}$ where $a =$ ________________.

$x_4 = \dfrac{a}{10864}$ where $a =$ ________________.

(59) Find the first 6 approximations given by *Newton's method* to the root of the polynomial $x^3 - x - 1$ starting with $x_0 = 1$. Carry out your answers to 9 decimal places.

Answer: $x_0 = 1.000\ 000\ 000$; $x_1 = 1.$_ _0 000 000;

$x_2 = 1.347\ 8$_6 0_7; $x_3 = 1.325\ 2$_0 3_9;

$x_4 = 1.324\ 71$_ 1_4; $x_5 = 1.324\ 717\ 9$_7.

(60) Use *Newton's method* to find six successive approximations to each root of the polynomial $x^4 - 2x^3 - x^2 - 2x + 2$. Carry out your work to nine decimal places. In each case use the starting value of the form n or $n.5$ (where n is an integer) which is closest to the root your are trying to approximate.

Answer: For the first root: $x_0 = 0.$ _00 000 000; $x_1 = 0.640$ _25 _00;

$x_2 = 0.630\ 1$_1 5_1; $x_3 = 0.630\ 115\ 3$_8;

$x_4 = 0.630\ 115\ 39$_; $x_5 = 0.630\ 115\ 39$_.

For the second root: $x_0 =$ _ . _00 000 000; $x_1 = 2.57$_ 86_ 111;

$x_2 = 2.573$ _1_ 023; $x_3 = 2.573\ 271\ 9$_5;

$x_4 = 2.573\ 271\ 96$_; $x_5 = 2.573\ 271\ 96$_.

(61) A ball is thrown upward from the edge of the roof of a building 176 feet high with an initial velocity of 56 ft/sec. (Assume that the acceleration due to gravity is 32 ft/sec^2.)

(a) How high does the ball go? Answer: __________ ft.

(b) When does it reach the ground? Answer: after ________ sec.

(62) A ball is thrown upward from the edge of the roof of a building with a velocity of 40 ft./sec. The ball hits the ground at 120 ft./sec. (Assume that the acceleration due to gravity is 32 ft/sec^2.)

(a) How long does it take the ball to reach the ground? Answer: _____ sec.

(b) How tall is the building? Answer: ________ ft.

(c) What is the maximum height reached by the ball? Answer: ________ ft.

(63) A ball is thrown upward from the edge of the roof of a building at 72 ft./sec. It hits the ground 10 seconds later. (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.)

(a) How tall is the building? Answer: 8_ _ ft.

(b) What is the maximum height reached by the ball? Answer: _ 6 _ ft.

(64) A ball is thrown upward from the edge of the roof of a building 160 feet tall at a velocity of 48 ft./sec. At what velocity does the ball hit the ground? (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.) Answer: _________ ft/sec.

(65) A falling stone is observed to be at a height of 171 feet. Two seconds later it is observed to be at a height of 75 feet. From what height was it dropped? (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.) Answer: _________ ft.

(66) A falling stone is observed to be at a height of 154 feet. Two seconds later it is observed to be at a height of 14 feet. If the stone was initially thrown upwards with a speed of 10 ft./sec., from what height was it thrown? (Use 32 ft./sec.^{2} as the magnitude of the acceleration due to gravity.) Answer: _________ ft.

(67) Two seconds after being thrown upward an object is rising at 176 ft./sec. How far does it travel before returning to the position from which it was thrown? Answer: ___________ ft.

(68) A predator-prey system is modeled by the equations

$$\begin{aligned} \frac{dx}{dt} &= 4x - 5y\sqrt{x} \\ \frac{dy}{dt} &= 7y\sqrt{x} \end{aligned}$$

where the variable y represents the predator population while the variable x represents the prey population. Explain briefly how we know that the predator must have an alternate source of food.

Answer: ____ .

12.3. Problems

(1) A piston P moves within a cylinder. A connecting rod of length 7 inches connects the piston with a point Q on a crankshaft, which is constrained to move in a circle with center C and radius 2 inches. Assuming that the angular velocity of Q is 5π radians per second, find the speed of the piston at the moment when the line segment CQ makes an angle of $\pi/4$ radians with the horizontal.

(2) Part of the northern boundary of a body of water is a straight shoreline running east and west. A lighthouse with a beacon rotating at a constant angular velocity is situated 600 yards offshore. An observer in a boat 200 yards east of the lighthouse watches the light from the beacon move along the shore. At the moment t_1 when the observer is looking directly northeast the angular velocity of his line of sight is 2.5 radians per second.
   (a) How many revolutions per minute does the beacon make?
   (b) How fast (in miles per hour) is the light moving along the shore at time t_1?
   (c) Although the beacon rotates with constant angular velocity, the observer's line of sight does not. Locate the points on the shoreline where the angular velocity of the line of sight is greatest and where it is least. What is the limiting angular velocity of the line of sight as the light disappears down the shoreline?

(3) A wire 24 inches long is cut in two parts. One part is bent into the shape of a circle and the other into the shape of a square. How should it be cut if the sum of the areas of the circle and the square is to be (a) minimum, (b) maximum?

12.3.1. Theorem. *Let f be a function such that $f(x) \geq 0$ for every x in its domain. Then f has a local maximum at a point a if and only if the function f^2 has a local maximum there. Similarly, f has a local minimum at a if and only if f^2 does.*

(4) (a) Prove the preceding theorem.
   (b) Suppose that $0 < k < l$. Let $f(x) = |k \cos x - l \sin x|$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Without using the theorem above find all local maxima and minima of f.
   (c) Let f be as in (b). Use the theorem above to find all local maxima and minima of f.
   (d) Show (if you have not already done so) that the answers you got in parts (b) and (c) are in agreement.

(5) When a sector is removed from a thin circular disk of metal, the portion of the disk which remains can be formed into a cone. Explain how the sector should be chosen so that the resulting cone has the greatest capacity.

(6) Your good friend George, who is working for the Acme Widget Corporation, has a problem. He knows that you are studying calculus and writes a letter asking for your help. His problem concerns solutions to a system of two differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x(t) \\ \dfrac{dy}{dt} = x(t) + y(t) \end{cases} \tag{1}$$

subject to the initial conditions

$$x(0) = a \qquad \text{and} \qquad y(0) = b, \tag{2}$$

where a and b are arbitrary constants. He has already found one set of solutions:

$$\begin{cases} x(t) = ae^t \\ y(t) = (b + at)e^t \end{cases}$$

What Fred is unable to discover is whether or not there are other solutions. Write a letter to Fred helping him out.

Hint. Suppose

$$\begin{cases} x(t) = u(t) \\ y(t) = v(t) \end{cases}$$

is a solution to the system (1) which satisfies the initial conditions (2). Consider the functions $p(t) = e^{-t}u(t)$ and $q(t) = e^{-t}v(t)$.

(7) Use *Newton's method* to approximate the solutions to the equation

$$\sin x = x^2 - x + 0.5$$

to eight decimal places. Use starting approximations of 0.3 and 1.3.

Explain carefully how we know that there are exactly two solutions. Explain how one might reasonably have chosen the numbers 0.3 and 1.3 as initial approximations. Discuss fully the problem of deciding when to stop.

(8) Explain carefully and fully how to use *Newton's method* to find the first point of intersection of the curves $y = \sin x$ and $y = e^{-x}$. Give your answer correct to 8 decimal places.

(9) Suppose we are given $a > 0$. Explain why it is that if x_1 is arbitrary and for each $n \in \mathbb{N}$ we let $x_{n+1} = \frac{1}{2}(x_n + a{x_n}^{-1})$, then (x_n) converges to the square root of a. *Hint.* Use *Newton's method*. Use this sequence to compute the square root of 10^7 to ten decimal places.

(10) Explain carefully and fully how to use *Newton's method* to find, correct to eight decimal places, an approximate value for the reciprocal of 2.74369.

(11) A chord subtends an arc of a circle. The length of the chord is 4 inches; the length of the arc is 5 inches. Find the central angle θ of the circle subtended by the chord (and the arc). The *law of cosines* yields an equation involving the angle θ. Explain carefully and fully how to use *Newton's method* to solve the equation (in radians) to four decimal places.

(12) Explain carefully and fully how to use *Newton's method* to find, correct to six decimal places, the slope of the tangent line to the curve $y = -\sin x$ $(\pi/2 \le x \le 3\pi/2)$ which passes through the origin.

12.4. Answers to Odd-Numbered Exercises

(1) 5

(3) 40π

(5) 228π

(7) $5\sqrt{3}$

(9) 48π

(11) $80\sqrt{3}$

(13) $\dfrac{3}{5}$

(15) $\dfrac{4}{25}$

(17) 256

(19) $\dfrac{10}{3}$, $\dfrac{50}{3}$

(21) -1, 0

(23) $4\sqrt{2}$, $8\sqrt{2}$

(25) 4, 12

(27) 20

(29) $4\sqrt{2}$

(31) 18, 18

(33) 5, 3

(35) $\dfrac{\pi}{3}$, $2\sqrt{3}$

(37) -15

(39) $\dfrac{1}{4}$, 1, 5, -7

(41) 30

(43) 28, right

(45) -6, 12, -3

(47) 9, 4, -11

(49) $3 - \exp(\cos x)$

(51) 8595

(53) $625,000$

(55) $\dfrac{1}{2}$, $\dfrac{t}{3}$

(57) 9, 161, 51841

(59) 5, 0, 2, 8, 0, 9, 8, 7, 5

(61) (a) 225
(b) $\dfrac{11}{2}$

(63) (a) 8, 0

(b) 9, 1

(65) 175

(67) 1800