Calculus II

Mathematics Department

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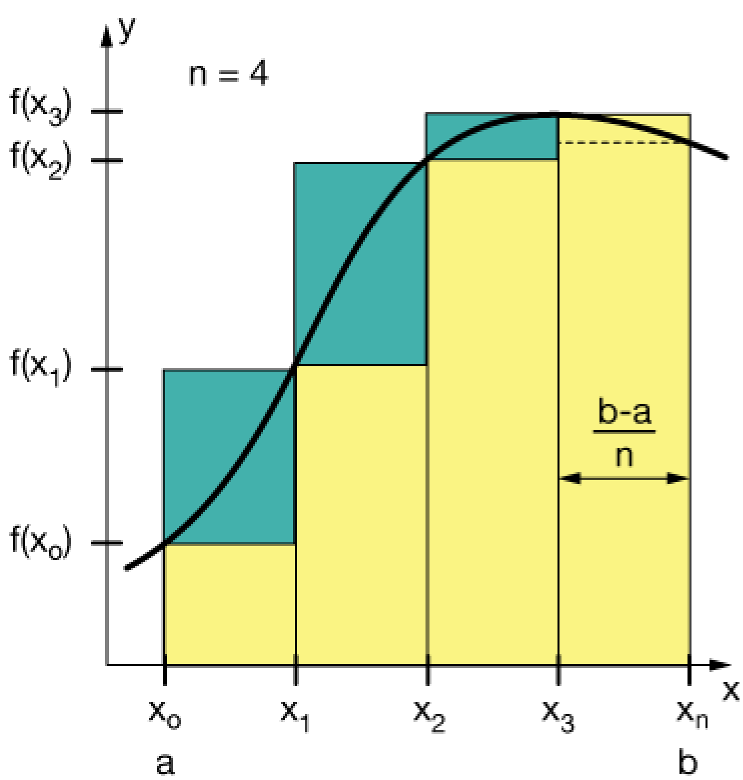

Chapter 5

The Definite Integral

5.1 Area Approximation

In Chapter 4, we have seen the role played by the indefinite integral in finding antiderivatives and in solving first order and second order differential equations. The definite integral is very closely related to the indefinite integral. We begin the discussion with finding areas under the graphs of positive functions.

Example 5.1.1 Find the area bounded by the graph of the function $y = 4,\ y = 0,\ x = 0,\ x = 3.$

graph

From geometry, we know that the area is the height 4 times the width 3 of the rectangle.

Area = 12.

Example 5.1.2 Find the area bounded by the graphs of $y = 4x,\ y = 0,\ x = 0,\ x = 3.$

graph

From geometry, the area of the triangle is $\frac{1}{2}$ times the base, 3, times the height, 12.

Area = 18.

Example 5.1.3 Find the area bounded by the graphs of $y = 2x,\ y = 0,\ x = 1,\ x = 4$.

graph

The required area is covered by a trapezoid. The area of a trapezoid is $\frac{1}{2}$ times the sum of the parallel sides times the distance between the parallel sides.

Area $= \frac{1}{2}\ (2 + 8)(3) = 15$.

Example 5.1.4 Find the area bounded by the curves $y = \sqrt{4 - x^2},\ y = 0,\ x = -2,\ x = 2$.

graph

By inspection, we recognize that this is the area bounded by the upper half of the circle with center at $(0, 0)$ and radius 2. Its equation is

$$x^2 + y^2 = 4 \quad \text{or} \quad y = \sqrt{4 - x^2},\ -2 \le x \le 2.$$

Again from geometry, we know that the area of a circle with radius 2 is $\pi r^2 = 4\pi$. The upper half of the circle will have one half of the total area. Therefore, the required area is 2π.

Example 5.1.5 Approximate the area bounded by $y = x^2,\ y = 0,\ x = 0,$ and $x = 3$. Given that the exact area is 9, compute the error of your approximation.

Method 1. We divide the interval $[0, 3]$ into six equal subdivisions at the points $0, \frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2}$ and 2. Such a subdivision is called a *partition* of $[0, 3]$. We draw vertical segments joining these points of division to the curve. On each subinterval $[x_1, x_2]$, the minimum value of the function x^2 is at x_1^2. The maximum value x_2^2 of the function is at the right hand end point x_2. Therefore,

graph

The lower approximation, denoted L, is given by

$$\begin{aligned} L &= 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} + \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} \\ &= \frac{1}{2} \cdot \left[0 + 1 + \frac{9}{4} + 4 + \frac{25}{4}\right] \\ &= \frac{27}{4} \approx 8 \cdot 75. \end{aligned}$$

This approximation is called the *left-hand* approximation of the area. The error of approximation is -0.25.

The Upper approximation, denoted U, is given by

$$\begin{aligned} U &= \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} + \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} + (3)^2 \cdot \frac{1}{2} \\ &= \frac{1}{2}\left[\frac{1}{4} + 1 + \frac{9}{4} + 4 + \frac{25}{4} + 9\right] \\ &= \frac{1}{2}\left[\frac{91}{4}\right] \\ &= \frac{91}{8} \approx 11 \cdot 38. \end{aligned}$$

The error of approximation is +2.28.
This approximation is called the *right-hand* approximation.

Method 2. (Trapezoidal Rule) In this method, for each subinterval $[x_1, x_2]$, we join the point (x_1, x_1^2) with the point (x_2, x_2^2) by a straight line and find the area under this line to be a trapezoid with area $\frac{1}{2}\,(x_2 - x_1)(x_1^2 + x_2^2)$. We add up these areas as the Trapezoidal Rule approximation, T, that is given by

$$\begin{aligned}
T &= \frac{1}{2}\left(\frac{1}{2} - 0\right)\left(0^2 + \left(\frac{1}{2}\right)^2\right) + \frac{1}{2}\left(1 - \frac{1}{2}\right)\left(1^2 + \left(\frac{1}{2}\right)^2\right) \\
&\quad + \frac{1}{2}\left(\frac{3}{2} - 1\right)\left(\left(\frac{3}{2}\right)^2 + 1^2\right) + \frac{1}{2}\left(2 - \frac{3}{2}\right)\left(2^2 + \left(\frac{3}{2}\right)^2\right) \\
&\quad + \frac{1}{2}\left(\frac{5}{2} - 2\right)\left(\left(\frac{5}{2}\right)^2 + 2^2\right) + \frac{1}{2}\left(3 - \frac{5}{2}\right)\left(3^2 + \left(\frac{5}{2}\right)^2\right) \\
&= \frac{1}{4}\left[0^2 + 2\cdot\left(\frac{1}{2}\right)^2 + 2(1^2) + 2\cdot\left(\frac{3}{2}\right)^2 + 2(2)^2 + 2\cdot\left(\frac{5}{2}\right)^2 + 3^2\right] \\
&= \frac{1}{4}\left[1 + 2 + \frac{9}{2} + 8 + \frac{25}{2} + 9\right] \\
&= \frac{37}{4} = 9\cdot 25.
\end{aligned}$$

The error of this Trapezoidal approximation is +0.25.

Method 3. (Simpson's Rule) In this case we take two intervals, say $[x_1, x_2] \cup [x_2, x_3]$, and approximate the area over this interval by

$$\frac{1}{6}\,[f(x_1) + 4f(x_2) + f(x_3)]\cdot(x_3 - x_1)$$

and then add them up. In our case, let $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $x_4 = 2$, $x_5 = \frac{5}{2}$ and $x_6 = 3$. Then the Simpson's rule approximation, S, is given by

$$
\begin{aligned}
S &= \frac{1}{6}\left[0^2 + 4\cdot\left(\frac{1}{2}\right)^2 + (1)^2\right]\cdot(1) + \frac{1}{6}\left[(1)^2 + 4\cdot\left(\frac{3}{2}\right)^2 + 2^2\right](1) \\
&\quad + \frac{1}{6}\left[2^2 + 4\cdot\left(\frac{5}{2}\right)^2 + 3^2\right]\cdot(1) \\
&= \frac{1}{6}\left[0^2 + 4\left(\frac{1}{2}\right)^2 + 2\cdot 1^2 + 4\cdot\left(\frac{3}{2}\right)^2 + 2\cdot 2^2 + 4\cdot\left(\frac{5}{2}\right)^2 + 3^2\right] \\
&= \frac{54}{6} = 9 = \text{ Exact Value!}
\end{aligned}
$$

For positive functions, $y = f(x)$, defined over a closed and bounded interval $[a, b]$, we define the following methods for approximating the area A, bounded by the curves $y = f(x),\ y = 0,\ x = a$ and $x = b$. We begin with a common equally-spaced partition,

$$P = \{a = x_0 < x_1 < x_2 < x_3 < \ldots < x_n = b\},$$

such that $x_i = a + \dfrac{b-a}{n}\, i$, for $i = 0, 1, 2, \ldots, n$.

Definition 5.1.1 (Left-hand Rule) The left-hand rule approximation for A, denoted L, is defined by

$$L = \frac{b-a}{n}\cdot[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})].$$

Definition 5.1.2 (Right-hand Rule) The right-hand rule approximation for A, denoted R, is defined by

$$R = \frac{b-a}{n}\cdot[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)].$$

Definition 5.1.3 (Mid-point Rule) The mid-point rule approximation for A, denoted M, is defined by

$$M = \frac{b-a}{n}\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right].$$

Definition 5.1.4 (Trapezoidal Rule) The trapezoidal rule approximation for A, denoted T, is defined by

$$\begin{aligned} T &= \frac{b-a}{n}\left[\frac{1}{2}\left(f(x_0)+f(x_1)\right)+\frac{1}{2}\left(f(x_1)+f(x_2)\right)+\cdots+\frac{1}{2}\left(f(x_{n-1})+f(x_n)\right)\right] \\ &= \frac{b-a}{n}\left[\frac{1}{2}\,f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right]. \end{aligned}$$

Definition 5.1.5 (Simpson's Rule) The Simpson's rule approximation for A, denoted S, is defined by

$$\begin{aligned} S = \frac{b-a}{n}&\left[\frac{1}{6}\left\{f(x_0)+4\,f\left(\frac{x_0+x_1}{2}\right)+f(x_1)\right\}\right. \\ &+\frac{1}{6}\left\{f(x_1)+4\,f\left(\frac{x_1+x_2}{2}\right)+f(x_2)\right\} \\ &\left.+\cdots+\frac{1}{6}\left\{f(x_{n-1}+4\,f\left(\frac{x_{n-1}+x_n}{2}\right)+f(x_n)\right\}\right] \\ = \left(\frac{b-a}{n}\right)\cdot\frac{1}{6}\cdot&\left[f(x_0)+4\,f\left(\frac{x_0+x_1}{2}\right)+2\,f(x_1)+4\,f\left(\frac{x_1+x_2}{2}\right)\right. \\ &\left.+\cdots 2\,f(x_{n-1})+4\,f\left(\frac{x_{n-1}+x_n}{2}\right)+f(x_n)\right]. \end{aligned}$$

Examples

Exercises 5.1

1. The sum of n terms $a_1, a_2, \cdots, a_n$ is written in compact form in the so called sigma notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The variable k is called the index, the number 1 is called the lower limit and the number n is called the upper limit. The symbol $\sum_{k=1}^{n} a_k$ is read "the sum of a_k from $k = 1$ to $k = n$."

Verify the following sums for $n = 5$:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

(d) $\displaystyle\sum_{k=1}^{n} 2^r = 2^{n+1} = 1$

2. Prove the following statements by using mathematical induction:

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

(d) $\displaystyle\sum_{k=1}^{n} 2^r = 2^{n+1} - 1$

3. Prove the following statements:

(a) $\displaystyle\sum_{k=1}^{n} (c\ a_k) = c\ \sum_{k=1}^{n} a_k$

(b) $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(d) $\displaystyle\sum_{k=1}^{n} (a\ a_k + b\ b_k) = a\sum_{k=1}^{n} a_k + b\sum_{k=1}^{n} b_k$

4. Evaluate the following sums:

(a) $\sum_{i=0}^{6}(2i)$

(b) $\sum_{j=1}^{5}\left(\frac{1}{j}\right)$

(c) $\sum_{k=0}^{4}(1+(-1)^k)^2$

(d) $\sum_{m=2}^{5}(3m-2)$

5. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $x_k = a + \left(\frac{b-a}{n}\right)k,\ k = 0, 1, 2, \cdots, n$. Let $f(x) = x^2$. Let A denote the area bounded by $y = f(x),\ y = 0,\ x = 0$ and $x = 2$. Show that

(a) Left-hand Rule approximation of A is $\frac{2}{n}\sum_{k=1}^{n-1} x_{k-1}^2$.

(b) Right-hand Rule approximation of A is $\frac{2}{n}\sum_{k=1}^{n-1} x_k^2$.

(c) Mid-point Rule approximation of A is $\frac{2}{n}\sum_{k=1}^{n}\left(\frac{x_{k-1}+x_k}{2}\right)^2$.

(d) Trapezoidal Rule approximation of A is $\frac{2}{n}\left\{2+\sum_{k=1}^{n-1} x_k^2\right\}$.

(e) Simpson's Rule approximation of A

$$\frac{1}{3n}\left\{4+4\sum_{k=1}^{n}\left(\frac{x_{k-1}+x_k}{2}\right)^2+2\sum_{k=1}^{n-1}x_k^2\right\}.$$

In problems 6–20, use the function f, numbers a, b and n, and compute the approximations LH, RH, MP, T, S for the area bounded by $y = f(x),\ y = 0,\ x = a,\ x = b$ using the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\},\ \text{where } x_k = a + k\left(\frac{b-a}{n}\right),\ \text{and}$$

(a) $LH = \dfrac{b-a}{n}\displaystyle\sum_{k=1}^{n} f(x_{k-1})$

(b) $RH = \dfrac{b-a}{n}\displaystyle\sum_{k=1}^{n} f(x_k)$

(c) $MP = \dfrac{b-a}{n}\displaystyle\sum_{k=1}^{n} f\left(\frac{x_{n-1}+x_k}{2}\right)$

(d) $T = \dfrac{b-a}{n}\left\{\displaystyle\sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}(f(x_0)+f(x_n))\right\}$

(e) $S = \dfrac{b-a}{6n}\left\{(f(x_0)+f(x_n)) + 2\displaystyle\sum_{k=1}^{n-1} f(x_n) + 4\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\right\}$

$\quad = \dfrac{1}{6}\{LH + 4MP + RH\}$

6. $f(x) = 2x,\ a = 0,\ b = 2,\ n = 6$

7. $f(x) = \dfrac{1}{x},\ a = 1,\ b = 3,\ n = 6$

8. $f(x) = x^2,\ a = 0,\ b = 3,\ n = 6$

9. $f(x) = x^3,\ a = 0,\ b = 2,\ n = 4$

10. $f(x) = \dfrac{1}{1+x}, a = 0,\ b = 3,\ n = 6$

11. $f(x) = \dfrac{1}{1+x^2},\ a = 0,\ b = 1,\ n = 4$

12. $f(x) = \dfrac{1}{\sqrt{4-x^2}},\ a = 0,\ b = 1,\ n = 4$

13. $f(x) = \dfrac{1}{4-x^2},\ a = 0,\ b = 1,\ n = 4$

14. $f(x) = \dfrac{1}{4+x^2},\ a = 0,\ b = 2,\ n = 4$

15. $f(x) = \dfrac{1}{\sqrt{4+x^2}},\ a = 0,\ b = 2,\ n = 4$

16. $f(x) = \sqrt{4+x^2},\ a = 0,\ b = 2,\ n = 4$

17. $f(x) = \sqrt{4-x^2},\ a = 0,\ b = 2,\ n = 4$

18. $f(x) = \sin x,\ a = 0,\ b = \pi,\ n = 4$

19. $f(x) = \cos x,\ a = -\dfrac{\pi}{2},\ b = \dfrac{\pi}{2}, n = 4$

20. $f(x) = \sin^2 x,\ a = 0,\ b = \pi,\ n = 4$

5.2 The Definite Integral

Let f be a function that is continuous on a bounded and closed interval $[a, b]$. Let $p = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of $[a, b]$, not necessarily equally spaced. Let

$$\begin{aligned} m_i &= \min\{f(x) : x_{i-1} \le x \le x_i\},\ i = 1, 2, \ldots, n; \\ M_i &= \max\{f(x) : x_{i-1} \le x \le x_i\},\ i = 1, 2, \ldots, n; \\ \Delta x_i &= x_i - x_{i-1},\ i = 1, 2, \ldots, n; \\ \Delta &= \max\{\Delta x_i : i = 1, 2, \ldots, n\}; \end{aligned}$$

$L(p) = m_1\Delta x_1 + m_2\Delta x_2 + \ldots + m_n\Delta x_n$
$U(p) = M_1\Delta x_i + M_2\Delta x_2 + \ldots + M_n\Delta x_n.$

We call $L(p)$ the lower Riemann sum. We call $U(p)$ the upper Riemann sum. Clearly $L(p) \le U(p)$, for every partition. Let

$$\begin{aligned} L_f &= \text{lub}\{L(p) : p \text{ is a partition of } [a, b]\} \\ U_f &= \text{glb}\{U(p) : p \text{ is a partition of } [a, b]\}. \end{aligned}$$

Definition 5.2.1 If f is continuous on $[a, b]$ and $L_f = U_f = I$, then we say that:

(i) f is integrable on $[a, b]$;

(ii) the definite integral of $f(x)$ from $x = a$ to $x = b$ is I;

(iii) I is expressed, in symbols, by the equation

$$I = \int_a^b f(x)dx;$$

(iv) the symbol "$\int$" is called the "*integral sign*"; the number "a" is called the "*lower limit*"; the number "b" is called the "*upper limit*"; the function "$f(x)$" is called the "*integrand*"; and the variable "x" is called the (dummy) "*variable of integration.*"

(v) If $f(x) \geq 0$ for each x in $[a, b]$, then the area, A, bounded by the curves $y = f(x)$, $y = 0$, $x = a$ and $x = b$, is defined to be the definite integral of $f(x)$ from $x = a$ to $x = b$. That is,

$$A = \int_a^b f(x)dx.$$

(vi) For convenience, we define

$$\int_a^a f(x)dx = 0, \quad \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Theorem 5.2.1 *If a function f is continuous on a closed and bounded interval $[a, b]$, then f is* integrable *on $[a, b]$.*

Proof. See the proof of Theorem 5.6.3.

Theorem 5.2.2 (Linearity) *Suppose that f and g are continuous on $[a, b]$ and c_1 and c_2 are two arbitrary constants. Then* *(i)* $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(ii) $\int_a^b (f(x)-g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(iii) $\int_a^b c_1 f(x)dx = c_1 \int_a^b f(x)dx, \int_a^b c_2 g(x)dx = c_2 \int_a^b g(x)dx$ *and*

$$\int_a^b (c_1 f(x) + c_2 g(x))dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx$$

Proof.
Part (i) Since f and g are continuous, $f+g$ is continuous and hence by Theorem 5.2.1 each of the following integrals exist:

$$\int_a^b f(x)dx, \int_a^b g(x)dx, \quad \text{and} \quad \int_a^b (f(x)+g(x))dx.$$

Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$. For each i, there exist number c_1, c_2, c_3, d_1, d_2, and d_3 on $[x_{i-1}, x_i]$ such that

$f(c_1)$ = absolute minimum of f on $[x_{i-1}, x_i]$,
$g(c_2)$ = absolute minimum of f on $[x_{i-1}, x_i]$,
$f(c_3) + g(c_3)$ = absolute minimum of $f+g$ on $[x_{i-1}, x_i]$,
$f(d_1)$ = absolute maximum of f on $[x_{i-1}, x_i]$,
$g(d_2)$ = absolute maximum of g on $[x_{i-1}, x_i]$,
$f(d_3) + g(d_3)$ = absolute maximum of $f+g$ on $[x_{i-1}, x_i]$.

It follows that

$$f(c_1) + g(c_2) \leq f(c_3) + g(c_3) \leq f(d_3) + g(d_3) \leq f(d_1) + g(d_2)$$

Consequently,

$$L_f + L_g \leq L_{(f+g)} \leq U_{(f+g)} \leq U_f + U_g \qquad \text{(Why?)}$$

Since f and g are integrable,

$$L_f = U_f = \int_a^b f(x)dx; \quad L_g = U_g = \int_a^b g(x)dx.$$

By the squeeze principle,

$$L_{(f+g)} = U_{(f+g)} = \int_a^b (f(x)+g(x))dx$$

and

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

This completes the proof of Part (i) of this theorem.

Part (iii) Let k be a positive constant and let F be a function that is continuous on $[a, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$ be any partition of $[a, b]$. Then for each i there exist numbers c_i and d_i such that $F(c_i)$ is the absolute minimum of F on $[x_{i-1}, x_i]$ and $F(d_i)$ is absolute maximum of F on $[x_{i-1}, x_i]$. Since k is a positive constant,

$kF(c_i)$ = absolute minimum of kF on $[x_{i-1}, x_i]$,
$kF(d_i)$ = absolute maximum of kF on $[x_{i-1}, x_i]$,
$-kF(d_i)$ = absolute minimum of $(-k)F$ on $[x_{i-1}, x_i]$,
$-kF(c_i)$ = absolute maximum of $(-k)F$ on $[x_{i-1}, x_i]$.

Then

$$\begin{aligned}
L(P) &= F(c_1)\Delta x_1 + F(c_2)\Delta x_2 + \cdots + F(c_n)\Delta x_n,\\
U(P) &= F(d_1)\Delta x_1 + F(d_2)\Delta x_2 + \cdots + F(d_n)\Delta x_n,\\
kL(P) &= (kF)(c_1)\Delta x_1 + (kF)(c_2)\Delta x_2 + \cdots + (kF)(c_n)\Delta x_n,\\
kU(P) &= (kF)(d_1)\Delta x_1 + (kF)(d_2)\Delta x_2 + \cdots + (kF)(d_n)\Delta x_n,\\
-kU(P) &= (-kF)(d_1)\Delta x_1 + (-kF)(d_2)\Delta x_2 + \cdots + (-kF)(d_n)\Delta x_n,\\
-kL(P) &= (-kF)(c_1)\Delta x_1 + (-kF)(c_2)\Delta x_2 + \cdots + (-kF)(c_n)\Delta x_n.
\end{aligned}$$

Since F is continuous, kF and $(-k)F$ are both continuous and

$$\begin{aligned}
L_f = U_p &= \int_a^b F(x)dx,\\
L_{(kF)} = U_{(kF)} &= k(L_F) = k(U_F) = k\int_a^b F(x)dx\\
L_{(-kF)} &= (-k)U_F, U_{(-kF)} = -kL_F,
\end{aligned}$$

and hence

$$L_{(-kF)} = U_{(-kF)} = (-k)\int_a^b F(x)dx.$$

Therefore,

$$\int_a^b (c_1 f(x) + c_2 g(x)) = \int_a^b c_1 f(x)dx + \int_a^b c_2 g(x)dx \qquad \text{(Part (i))}$$
$$= c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx \qquad \text{(Why?)}$$

This completes the proof of Part (iii) of this theorem.
Part (ii) is a special case of Part (iii) where $c_1 = 1$ and $c_2 = -1$. This completes the proof of the theorem.

Theorem 5.2.3 (Additivity) *If f is continuous on $[a, b]$ and $a < c < b$, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof. Suppose that f is continuous on $[a, b]$ and $a < c < b$. Then f is continuous on $[a, c]$ and on $[c, b]$ and, hence, f is integrable on $[a, b], [a, c]$ and $[c, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots x_n = b\}$. Suppose that $x_{i-1} \leq c \leq x_i$ for some i. Let $P_1 = \{a = x_0 < x_1 < x_2 < \cdots < x_{i-1} \leq c\}$ and $P_2 = \{c \leq x_i < x_{i+1} < \cdots < x_n = b\}$. Then there exist numbers c_1, c_2, c_3, d_1, d_2, and d_3 such that
$f(c_1)$ = absolute minimum of f on $[x_{i-1}, c]$,
$f(d_1)$ = absolute maximum of f on $[x_{i-1}, c]$,
$f(c_2)$ = absolute minimum of f on $[c, x_i]$,
$f(d_2)$ = absolute maximum of f on $[c, x_i]$,
$f(c_3)$ = absolute minimum of f on $[x_{i-1}, x_i]$,
$f(d_3)$ = absolute maximum of f on $[x_{i-1}, x_i]$,
Also,

$$f(c_3) \leq f(c_1),\ f(c_3) \leq f(c_2),\ f(d_1) \leq f(d_3) \quad \text{and} \quad f(d_2) \leq f(d_3).$$

It follows that

$$L(P) \leq L(P_1) + L(P_2) \leq U(P_1) + U(P_2) \leq U(P).$$

It follows that

$$\int_a^b f(x) = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

This completes the proof of the theorem.

Theorem 5.2.4 (Order Property) *If f and g are continuous on $[a, b]$ and $f(x) \le g(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

Proof. Suppose that f and g are continuous on $[a, b]$ and $f(x) \le g(x)$ for all x in $[a, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$. For each i there exists numbers c_i, c_i^*, d_i and d_i^* such that

$f(c_i)$ = absolute minimum of f on $[x_{i-1}, x_i]$,
$f(d_i)$ = absolute maximum of f on $[x_{i-1}, x_i]$,
$g(c_i^*)$ = absolute minimum of g on $[x_{i-1}, x_i]$,
$g(d_i^*)$ = absolute maximum of g on $[x_{i-1}, x_i]$.

By the assumption that $f(x) \le g(x)$ on $[a, b]$, we get

$$f(c_i) \le g(c_i^*) \quad \text{and} \quad f(d_i) \le g(d_i^*).$$

Hence

$$L_f \le L_g \quad \text{and} \quad U_f \le U_g.$$

It follows that

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

This completes the proof of this theorem.

Theorem 5.2.5 (Mean Value Theorem for Integrals) *If f is continuous on $[a, b]$, then there exists some point c in $[a, b]$ such that*

$$\int_a^b f(x)dx = f(c)(b - a).$$

Proof. Suppose that f is continuous on $[a, b]$, and $a < b$. Let

m = absolute minimum of f on $[a, b]$, and
M = absolute maximum of f on $[a, b]$.

Then, by Theorem 5.2.4,

$$m(b - a) \le \int_a^b m\ dx \le \int_a^b f(x)dx \le \int_a^b M\ dx = M(b - a)$$

and

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M.$$

By the intermediate value theorem for continuous functions, there exists some c such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\int_a^b f(x)dx = f(c)(b-a).$$

For $a = b$, take $c = a$. This completes the proof of this theorem.

Definition 5.2.2 The number $f(c)$ given in Theorem 5.2.6 is called the *average* value of f on $[a, b]$, denoted $f_{av}[a, b]$. That is

$$f_{av}[a,b] = \frac{1}{b-a}\int_a^b f(x)dx.$$

Theorem 5.2.6 (Fundamental Theorem of Calculus, First Form) *Suppose that f is continuous on some closed and bounded interval $[a, b]$ and*

$$g(x) = \int_a^x f(t)dt$$

for each x in $[a, b]$. Then $g(x)$ is continuous on $[a, b]$, differentiable on (a, b) and for all x in $(a, b), g'(x) = f(x)$. That is

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

Proof. Suppose that f is continuous on $[a, b]$ and $a < x < b$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{1}{h}[g(x+h) - g(x)] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x} f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt\right] \qquad \text{(Why?)} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_x^{x+h} f(t)dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}[f(c)(x+h-x)] \qquad \text{by Theorem 5.2.5)} \\ &= \lim_{h\to 0} f(c) \end{aligned}$$

for some c between x and $x + h$.

Since f is continuous on $[a, b]$ and c is between x and $x + h$, it follows that

$$g'(x) = \lim_{h\to 0} f(c) = f(x)$$

for all x such that $a < x < b$.

At the end points a and b, a similar argument can be used for one sided derivatives, namely,

$$\begin{aligned} g'(a^+) &= \lim_{h\to 0^+} \frac{g(x+h) - g(x)}{h} \\ g'(b^-) &= \lim_{h\to 0^-} \frac{g(x+h) - g(x)}{h}. \end{aligned}$$

We leave the end points as an exercise. This completes the proof of this theorem.

Theorem 5.2.7 (Fundamental Theorem of Calculus, Second Form) *If f and g are continuous on a closed and bounded interval $[a, b]$ and $g'(x) = f(x)$ on $[a, b]$, then*

$$\int_a^b f(x)dx = g(b) - g(a).$$

We use the notation: $[g(x)]_a^b = g(b) - g(a)$.

Proof. Let f and g be continuous on the closed and bounded interval $[a, b]$ and for each x in $[a, b]$, let

$$G(x) = \int_a^x f(t)dt.$$

Then, by Theorem 5.2.6, $G'(x) = f(x)$ on $[a, b]$. Since $G'(x) = g(x)$ for all x on $[a, b]$, there exists some constant C such that

$$G(x) = g(x) + C$$

for all x on $[a, b]$. Since $G(a) = 0$, we get $C = -g(a)$. Then

$$\begin{aligned} \int_a^b f(x)dx &= G(b) \\ &= g(b) + C \\ &= g(b) - g(a). \end{aligned}$$

This completes the proof of Theorem 5.2.7.

Theorem 5.2.8 (Leibniz Rule) *If $\alpha(x)$ and $\beta(x)$ are differentiable for all x and f is continuous for all x, then*

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} f(t)dt\right] = f(\beta(x)) \cdot \beta'(x) - f(\alpha(x)) \cdot \alpha'(x).$$

Proof. Suppose that f is continuous for all x and $\alpha(x)$ and $\beta(x)$ are differentiable for all x. Then

$$\begin{aligned} \frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} f(t)dt\right] &= \frac{d}{dx}\left[\int_{\alpha(x)}^{0} f(t)dt + \int_0^{\beta(x)} f(t)dt\right] \\ &= \frac{d}{dx}\left[\int_0^{\beta(x)} f(t)dt - \int_0^{\alpha(x)} f(t)dt\right] \\ &= \frac{d}{d(\beta(x))}\left(\int_0^{\beta(x)} f(t)dt\right) \cdot \frac{d(\beta(x))}{dx} - \frac{d}{d(\alpha(x))}\left(\int_0^{\alpha(x)} f(x)dt\right) \frac{d(\alpha(x))}{dx} \\ &= f(\beta(x))\ \beta'(x) - f(\alpha(x))\alpha'(x) \quad \text{(by Theorem 5.2.6)} \end{aligned}$$

This completes the proof of Theorem 5.2.8.

Example 5.2.1 Compute each of the following definite integrals and sketch the area represented by each integral:

(i) $\int_0^4 x^2 dx$ (ii) $\int_0^{\pi} \sin x \; dx$

(iii) $\int_{-\pi/2}^{\pi/2} \cos x \; dx$ (iv) $\int_0^{10} e^x dx$

(v) $\int_0^{\pi/3} \tan x \; dx$ (vi) $\int_{\pi/6}^{\pi/2} \cot x \; dx$

(vii) $\int_{-\pi/4}^{\pi/4} \sec x \; dx$ (viii) $\int_{\pi/4}^{3\pi/4} \csc x \; dx$

(xi) $\int_0^1 \sinh x \; dx$ (x) $\int_0^1 \cosh x \; dx$

We note that each of the functions in the integrand is positive on the respective interval of integration, and hence, represents an area. In order to compute these definite integrals, we use the Fundamental Theorem of Calculus, Theorem 5.2.2. As in Chapter 4, we first determine an anti-derivative $g(x)$ of the integrand $f(x)$ and then use

$$\int_a^b f(x)dx = g(b) - g(a) = [g(x)]_a^b .$$

graph

(i) $\int_0^4 x^2 dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3}$

graph

(ii) $\displaystyle\int_0^{\pi} \sin x \; dx = [-\cos x]_0^{\pi} = 1 - (-1) = 2$

graph

(iii) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \; dx = [\sin x]_{-\pi/2}^{\pi/2} = 1 - (-1) = 2$

graph

(iv) $\displaystyle\int_0^{10} e^x \; dx = [e^x]_0^{10} = e^{10} - e^0 = e^{10} - 1$

graph

(v) $\displaystyle\int_0^{\pi/3} \tan x \; dx = [\ln|\sec x|]_0^{\pi/3} = \ln\left|\sec\left(\frac{\pi}{3}\right)\right| = \ln 2$

graph

(vi) $\displaystyle\int_{\pi/6}^{\pi/2} \cot x \; dx = [\ln|\sin x|]_{\pi/6}^{\pi/2} = \ln(1) - \ln\left(\frac{1}{2}\right) = \ln 2$

graph

(vii) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec x \; dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln|\sqrt{2}+1| - \ln|\sqrt{2}-1|$

graph

(viii)
$$\int_{\pi/4}^{3\pi/4} \csc x \; dx = [-\ln|\csc x + \cot x|]_{\pi/4}^{3\pi/4}$$
$$= -\ln|\sqrt{2}-1| + \ln|\sqrt{2}+1|$$

graph

(ix) $\displaystyle\int_0^1 \sinh x \; dx = [\cosh x]_0^1 = \cosh 1 - \cosh 0 = \cosh 1 - 1$

graph

(x) $\int_0^1 \cosh x \; dx = [\sinh x]_0^1 = \sinh 1$

graph

Example 5.2.2 Evaluate each of the following integrals:

(i) $\int_1^{10} \frac{1}{x} \; dx$

(ii) $\int_0^{\pi/2} \sin(2x)dx$

(iii) $\int_0^{\pi/6} \cos(3x)dx$

(iv) $\int_0^2 (x^4 - 3x^2 + 2x - 1)dx$

(v) $\int_0^3 \sinh(4x)dx$

(vi) $\int_0^4 \cosh(2x)dx$

(i) Since $\frac{d}{dx} (\ln|x|) = \frac{1}{x}$,

$$\int_1^{10} \frac{1}{x} \; dx = [\ln|x|]_1^{10} = \ln(10)$$

(ii) Since $\frac{d}{dx} \left(\frac{-1}{2} \cos(2x) \right) = \sin(2x)$,

$$\int_0^{\pi/2} \sin 2x \; dx = \left[\frac{-1}{2} \cos(2x) \right]_0^{\pi/2} = \frac{1}{2} + \frac{1}{2} = 1.$$

(iii) $\int_0^{\pi/6} \cos(3x) = \left[\frac{1}{3} \sin(3x) \right]_0^{\pi/6} = \frac{1}{3} \sin \left(\frac{\pi}{2} \right) = \frac{1}{3}.$

(iv) $$\int_0^2 (x^4 - 3x^2 + 2x - 1)dx = \left[\frac{1}{5}x^5 - x^3 + x^2 - x\right]_0^2$$
$$= \left(\frac{32}{5} - 8 + 4 - 2\right) - 0$$
$$= \frac{2}{5}.$$

(v) $$\int_0^3 \sinh(4x)dx = \left[\frac{1}{4}\cosh(4x)\right]_0^3 = \frac{1}{4}\ \cosh(12) - \frac{1}{4}\ \cosh(0)$$
$$= \frac{1}{4}\ (\cosh(12) - 1)$$

(vi) $$\int_0^4 \cosh(2x)dx = \left[\frac{1}{2}\ \sinh(2x)\right]_0^4 = \frac{1}{2}\ \cosh(8)$$

Example 5.2.3 Verify each of the following:

(i) $\int_0^4 x^2dx = \int_0^3 x^2dx + \int_3^4 x^2dx$

(ii) $\int_1^4 x^2dx < \int_1^4 x^3dx$

(iii) $\frac{d}{dx}\ \left[\int_0^x (t^2 + 3t + 1)dt\right] = x^2 + 3x + 1$

(iv) $\frac{d}{dx}\ \left[\int_{x^2}^{x^3} \cos(t)dt\right] = 3x^2\cos(x^3) - 2x\cos(x^2).$

(v) If $f(x) = \sin x$, then $f_{av}[0, \pi] = \frac{2}{\pi}.$

(i) $$\int_0^4 x^2dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3}$$
$$\int_0^3 x^2dx + \int_3^4 x^2dx = \left[\frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3}\right]_3^4$$

$$= \left(\frac{27}{3} - 0\right) + \left(\frac{64}{3} - \frac{27}{3}\right) = \frac{64}{3}.$$

Therefore,

$$\int_1^4 x^2 dx = \int_0^3 x^2 dx + \int_3^4 x^2 dx.$$

(ii) $\displaystyle\int_1^4 x^2 dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = 21$

$\displaystyle\int_1^4 x^3 dx = \left[\frac{x^4}{4}\right]_1^4 = \left(64 - \frac{1}{4}\right)$

Therefore, $\displaystyle\int_1^4 x^2 dx < \int_1^4 x^3 dx$. We observe that $x^2 < x^3$ on $(1, 4]$.

(iii) $\displaystyle\int_0^x (t^2 + 3t + 1)dt = \left[\frac{t^3}{3} + 3\frac{t^2}{2} + t\right]_0^x$

$$= \frac{x^3}{3} + \frac{3}{2}x^2 + x$$

$$\frac{d}{dx}\left(\frac{x^3}{3} + \frac{3}{2}x^2 + x\right) = x^2 + 3x + 1.$$

(iv) $\displaystyle\frac{d}{dx}\left[\int_{x^2}^{x^3} \cos t dt\right] = \frac{d}{dx}\left[[\sin t]_{x^2}^{x^3}\right]$

$$= \frac{d}{dx}[\sin(x^3) - \sin(x^2)]$$

$$= \cos(x^3) \cdot 3x^2 - \cos(x^2) \cdot 2x$$

$$= 3x^2 \cos(x^3) - 2x \cos(x^2).$$

Using the Leibniz Rule, we get

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} \cos t dt\right) = \cos(x^3) \cdot 3x^2 - \cos(x^2) \cdot 2x$$

$$= 3x^2 \cos x^3 - 2x \cos x^2.$$

(v) The average value of $\sin x$ on $[0, \pi]$ is given by

$$\begin{aligned}\frac{1}{\pi - 0}\left(\int_0^{\pi} \sin x \; dx\right) &= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} \\ &= \frac{1}{\pi}[-(-1) + 1] \\ &= \frac{2}{\pi}.\end{aligned}$$

Basic List of Indefinite Integrals:

1. $\displaystyle\int x^3 dx = \frac{1}{4}x^4 + c$

2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c, \; n \neq 1$

3. $\displaystyle\int \frac{1}{x} \; dx = \ln|x| + c$

4. $\displaystyle\int \sin x \; dx = -\cos x + c$

5. $\displaystyle\int \sin(ax) dx = \frac{-1}{a} \; \cos(ax) + c$

6. $\displaystyle\int \cos x \; dx = \sin x + c$

7. $\displaystyle\int \cos(ax) \; dx = \frac{1}{a} \; \sin(ax) + c$

8. $\displaystyle\int \tan x dx = \ln|\sec x| + c$

9. $\displaystyle\int \tan(ax) \; dx = \frac{1}{a} \; \ln|\sec(ax)| + c$

10. $\displaystyle\int \cot x \; dx = \ln|\sin x| + c$

11. $\displaystyle\int \cot(ax) \; dx = \frac{1}{a} \; \ln|\sin(ax)| + c$

12. $\displaystyle\int e^x \; dx = e^x + c$

13. $\displaystyle\int e^{-x} \; dx = -e^{-x} + c$

14. $\displaystyle\int e^{ax} \; dx = \frac{1}{a} \; e^{ax} + c$

15. $\displaystyle\int \sinh x \; dx = \cosh x + c$

16. $\displaystyle\int \cosh x \; dx = \sinh x + c$

17. $\displaystyle\int \tanh x \; dx = \ln|\cosh x| + c$

18. $\displaystyle\int \coth x \; dx = \ln|\sinh x| + c$

19. $\displaystyle\int \sinh(ax) \; dx = \frac{1}{a} \; \cosh(ax) + c$

20. $\displaystyle\int \cosh(ax) \; dx = \frac{1}{a} \; \sinh(ax) + c$

21. $\int \tanh(ax)\ dx = \frac{1}{a}\ \ln|\cosh ax| + c$

22. $\int \coth(ax)\ dx = \frac{1}{a}\ \ln|\sinh(ax)| + c$

23. $\int \sec x\ dx = \ln|\sec x + \tan x| + c$

24. $\int \csc x\ dx = -\ln|\csc x + \cot x| + c$

25. $\int \sec(ax)\ dx = \frac{1}{a}\ \ln|\sec(ax) + \tan(ax)| + c$

26. $\int \csc(ax)\ dx = \frac{-1}{a}\ \ln|\csc(ax) + \cot(ax)| + c$

27. $\int \sec^2 x\ dx = \tan x + c$

28. $\int \sec^2(ax)\ dx = \frac{1}{a}\ \tan(ax) + c$

29. $\int \csc^2 x\ dx = -\cot x + c$

30. $\int \csc^2(ax)\ dx = \frac{-1}{a}\ \cot(ax) + c$

31. $\int \tan^2 x\ dx = \tan x - x + c$

32. $\int \cot^2 x\ dx = -\cot x - x + c$

33. $\int \sin^2 x\ dx = \frac{1}{2}\ (x - \sin x \cos x) + c$

34. $\int \cos^2 x\ dx = \frac{1}{2}\ (x + \sin x \cos x) + c$

35. $\int \sec x \tan x\ dx = \sec x + c$

36. $\int \csc x\ dx = -\csc x + c$

Exercises 5.2 Using the preceding list of indefinite integrals, evaluate the following:

1. $\int_1^5 \frac{1}{t}\ dt$

2. $\int_0^{3\pi/2} \sin x\ dx$

3. $\int_0^{3\pi/2} \cos x\ dx$

4. $\int_0^{10} e^x\ dx$

5. $\int_0^{\pi/10} \sin(5x)\ dx$

6. $\int_0^{\pi/6} \cos(5x)\ dx$

7. $\int_{\pi/12}^{\pi/6} \cot(3x)\ dx$

8. $\int_{-1}^{1} e^{-x} dx$

9. $\int_{0}^{2} e^{3x}\ dx$

10. $\int_{0}^{2} \sinh(2x)\ dx$

11. $\int_{0}^{4} \cosh(3x)\ dx$

12. $\int_{0}^{1} \tanh(2x)\ dx$

13. $\int_{1}^{2} \coth(3x)\ dx$

14. $\int_{\pi/12}^{\pi/6} \sec(2x) dx$

15. $\int_{\pi/12}^{\pi/6} \csc(2x)\ dx$

16. $\int_{0}^{\pi/8} \sec^2(2x)\ dx$

17. $\int_{\pi/12}^{\pi/6} \csc^2(2x)$

18. $\int_{0}^{\pi/4} \tan^2 x\ dx$

19. $\int_{\pi/6}^{\pi/4} \cot^2 x\ dx$

20. $\int_{0}^{\pi} \sin^2 x dx$

21. $\int_{-\pi/2}^{\pi/2} \cos^2 x\ dx$

22. $\int_{\pi/6}^{\pi/4} \sec x \tan x\ dx$

23. $\int_{\pi/6}^{\pi/4} \csc x \cot x\ dx$

24. $\int_{0}^{2} e^{-3x}\ dx$

Compute the average value of each given f on the given interval.

25. $f(x) = \sin x,\ \left[\frac{-\pi}{2}, \pi\right]$

26. $f(x) = x^{1/3}, [0, 8]$

27. $f(x) = \cos x,\ \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

28. $f(x) = \sin^2 x, [0, \pi]$

29. $f(x) = \cos^2 x, [0, \pi]$

30. $f(x) = e^{-x}, [-2, 2]$

Compute $g'(x)$ without computing the integrals explicitly.

31. $g(x) = \int_{0}^{x} (1 + t^2)^{2/3} dt$

32. $g(x) = \int_{x^2}^{4x^3} \arctan(x)\ dx$

33. $g(x) = \int_{x^3}^{x^2} (1 + t^3)^{1/3} dt$

34. $g(x) = \int_{\arcsin x}^{\operatorname{arcsinh} x} (1 + t^2)^{3/2} dt$

35. $g(x) = \int_1^x \left(\frac{1}{t}\right) dt$

36. $g(x) = \int_{\sin 2x}^{\sin 3x} (1+t^2)^{1/2} dt$

37. $g(x) = \int_{\sin(x^2)}^{\sin(x^3)} (1+t^3)^{1/3} dt$

38. $\int_x^{4x} \frac{1}{1+t^2}\, dt$

39. $\int_{x^2}^{x^3} \arcsin(x)\, dx$

40. $\int_{\ln x}^{e^x} 2^t dt$

5.3 Integration by Substitution

Many functions are formed by using compositions. In dealing with a composite function it is useful to change variables of integration. It is convenient to use the following differential notation:

If $u = g(x)$, then $du = g'(x)\, dx$.

The symbol "du" represents the "differential of u," namely, $g'(x)dx$.

Theorem 5.3.1 (Change of Variable) *If f, g and g' are continuous on an open interval containing $[a, b]$, then*

(i) $\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u) du$

(ii) $\int f(g(x))g'(x)\, dx = \int f(u) du,$

where $u = g(x)$ and $du = g'(x)\, dx$.

Proof. Let f, g, and g' be continuous on an open interval containing $[a, b]$. For each x in $[a, b]$, let

$$F(x) = \int_a^x f(g(x))g'(x)dx$$

and

$$G(x) = \int_{g(a)}^{g(x)} f(u)du.$$

Then, by Leibniz Rule, we have

$$F'(x) = f(g(x))g'(x),$$

and

$$G'(x) = f(g(x))g'(x)$$

for all x on $[a, b]$.

It follows that there exists some constant C such that

$$F(x) = G(x) + C$$

for all x on $[a, b]$. For $x = a$ we get

$$0 = F(a) = G(a) + C = 0 + C$$

and, hence,

$$C = 0.$$

Therefore, $F(x) = G(x)$ for all x on $[a, b]$, and hence

$$\begin{aligned} \int_a^b f(g(x))g'(x)dx &= F(b) \\ &= G(b) \\ &= \int_{g(a)}^{g(b)} f(u)du. \end{aligned}$$

This completes the proof of this theorem.

Remark 18 We say that we have changed the variable from x to u through the substitution $u = g(x)$.

Example 5.3.1

(i) $\displaystyle\int_0^2 \sin(3x)\ dx = \int_0^6 \frac{1}{3}\ \sin u du = \frac{1}{3}\ [-\cos u]_0^6 = \frac{1}{3}\ (1 - \cos 6),$

where $u = 3x,\ du = 3\ dx,\ dx = \dfrac{1}{3}\ du.$

(ii) $$\int_0^2 3x\cos(x^2)\ dx = \int_0^4 \cos u \left(\frac{3}{2}\ du\right) = \frac{3}{2}\ [\sin u]_0^4 = \frac{3}{2}\ \sin 4,$$

where $u = x^2,\ du = 2x\ dx,\ 3x\ dx = \frac{3}{2}\ du.$

(iii) $$\int_0^3 e^{x^2} x\ dx = \int_0^9 e^u \frac{1}{2}\ du = \frac{1}{2}\ [e^u]_0^9 = \frac{1}{2}\ (e^9 - 1),$$

where $u = x^2,\ du = 2x\ dx,\ x\ dx = \frac{1}{2}\ dx.$

Definition 5.3.1 Suppose that f and g are continuous on $[a, b]$. Then the area bounded by the curves $y = f(x),\ y = g(x),\ y = a$ and $x = b$ is defined to be A, where

$$A = \int_a^b |f(x) - g(x)|\ dx.$$

If $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$A = \int_a^b (f(x) - g(x))\ dx.$$

If $g(x) \geq f(x)$ for all x in $[a, b]$, then

$$A = \int_a^b (g(x) - f(x))\ dx.$$

Example 5.3.2 Find the area, A, bounded by the curves $y = \sin x,\ y = \cos x,\ x = 0$ and $x = \pi$.

graph

We observe that $\cos x \geq \sin x$ on $\left[0, \dfrac{\pi}{4}\right]$ and $\sin x \geq \cos x$ on $\left[\dfrac{\pi}{4}, \pi\right]$. Therefore, the area is given by

$$\begin{aligned}
A &= \int_0^{\pi} |\sin x - \cos x| \ dx \\
&= \int_0^{\pi/4} (\cos x - \sin x) \ dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\
&= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi} \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1\right) + \left[1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] \\
&= 2\sqrt{2}.
\end{aligned}$$

Example 5.3.3 Find the area, A, bounded by $y = x^2,\ y = x^3,\ x = 0$ and $x = 2$.

graph

We note that $x^3 \leq x^2$ on $[0, 1]$ and $x^3 \geq x^2$ on $[1, 2]$. Therefore, by definition,

$$\begin{aligned}
A &= \int_0^1 (x^2 - x^3) \ dx + \int_1^2 (x^3 - x^2) \ dx \\
&= \left[\frac{1}{3} \ x^3 - \frac{1}{4} \ x^4\right]_0^1 + \left[\frac{1}{4} \ x^4 - \frac{1}{3} \ x^3\right]_1^2 \\
&= \left(\frac{1}{3} - \frac{1}{4}\right) + \left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] \\
&= \frac{1}{12} + \frac{4}{3} + \frac{1}{12} \\
&= \frac{3}{2}.
\end{aligned}$$

Example 5.3.4 Find the area bounded by $y = x^3$ and $y = x$. To find the interval over which the area is bounded by these curves, we find the points of intersection.

graph

$$x^3 = x \leftrightarrow x^3 - x = 0 \leftrightarrow x(x^2 - 1) = 0$$
$$\leftrightarrow x = 0,\ x = 1,\ x = -1.$$

The curve $y = x$ is below $y = x^3$ on $[-1, 0]$ and the curve $y = x^3$ is below the curve $y = x$ on $[0, 1]$. The required area is A, where

$$\begin{aligned} A &= \int_{-1}^{0} (x^3 - x)\ dx + \int_0^1 (x - x^3)\ dx \\ &= \left[\frac{1}{4}\ x^4 - \frac{1}{2}\ x^2\right]_{-1}^{0} + \left[\frac{1}{2}\ x^2 - \frac{x^4}{4}\right]_0^1 \\ &= \left[\frac{1}{2} - \frac{1}{4}\right] + \left[\frac{1}{2} - \frac{1}{4}\right] \\ &= \frac{1}{2} \end{aligned}$$

Exercises 5.3 Find the area bounded by the given curves.

1. $y = x^2,\ y = x^3$
2. $y = x^4,\ y = x^3$
3. $y = x^2,\ y = \sqrt{x}$
4. $y = 8 - x^2,\ y = x^2$
5. $y = 3 - x^2,\ y = 2x$
6. $y = \sin x,\ y = \cos x,\ x = \frac{-\pi}{2}, x = \frac{\pi}{2}$
7. $y = x^2 + 4x,\ y = x$
8. $y = \sin 2x,\ y = x,\ x = \frac{\pi}{2}$
9. $y^2 = 4x,\ x - y = 0$
10. $y = x + 3,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{2}$

Evaluate each of the following integrals:

11. $\int \sin 3x \, dx$

12. $\int \cos 5x \, dx$

13. $\int e^{x^2} x \, dx$

14. $\int x \sin(x^2) \, dx$

15. $\int x^2 \tan(x^3+1) \, dx$

16. $\int \sec^2(3x+1) \, dx$

17. $\int \csc^2(2x-1) \, dx$

18. $\int x \sinh(x^2) \, dx$

19. $\int x^2 \cosh(x^3+1) \, dx$

20. $\int \sec(3x+5) \, dx$

21. $\int \csc(5x-7) \, dx$

22. $\int x \tanh(x^2+1) \, dx$

23. $\int x^2 \coth(x^3) \, dx$

24. $\int \sin^3 x \cos x \, dx$

25. $\int \tan^5 x \sec^2 x \, dx$

26. $\int \cot^3 x \csc^2 x \, dx$

27. $\int \sec^3 x \tan x \, dx$

28. $\int \csc^3 x \cot x \, dx$

29. $\int \frac{(\arcsin x)^4}{\sqrt{1-x^2}} \, dx$

30. $\int \frac{(\arctan x)^3}{1+x^2} \, dx$

31. $\int_0^1 x e^{x^2} \, dx$

32. $\int_0^{\pi/6} \sin(3x) dx$

33. $\int_0^{\pi/4} \cos(4x) \, dx$

34. $\int_0^3 \frac{1}{(3x+1)} \, dx$

35. $\int_0^{\pi/2} \sin^3 x \cos x \, dx$

36. $\int_0^{\pi/6} \cos^3(3x) \sin 3x \, dx$

5.4 Integration by Parts

The product rule of differentiation yields an integration technique known as integration by parts. Let us begin with the product rule:

$$\frac{d}{dx}\,(u(x)v(x)) = \frac{du(x)}{dx}\,v(x) + u(x)\frac{dv(x)}{dx}\ .$$

On integrating each term with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b \frac{d}{dx}\,(u(x)v(x))\ dx = \int_a^b v(x)\left(\frac{du(x)}{dx}\right)\ dx + \int_a^b u(x)\left(\frac{dv(x)}{dx}\right)\ dx.$$

By using the differential notation and the fundamental theorem of calculus, we get

$$[u(x)v(x)]_a^b = \int_a^b v(x)u'(x)\ dx + \int_a^b u(x)v'(x)\ dx.$$

The standard form of this integration by parts formula is written as

(i) $\displaystyle\int_a^b u(x)v'(x)\ dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)\ dx$

and

(ii) $\displaystyle\int udv = uv - \int vdu$

We state this result as the following theorem:

Theorem 5.4.1 (Integration by Parts) *If $u(x)$ and $v(x)$ are two functions that are differentiable on some open interval containing $[a, b]$, then*

(i) $\displaystyle\int_a^b u(x)v'(x)\ dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)\ dx$

for definite integrals and

(ii) $\displaystyle\int udv = uv - \int vdu$

for indefinite integrals.

Proof. Suppose that u and v are differentiable on some open interval containing $[a, b]$. For each x on $[a, b]$, let

$$F(x) = \int_a^x u(x)v'(x)dx + \int_a^x v(x)u'(x)dx.$$

Then, for each x on $[a, b]$,

$$\begin{aligned} F'(x) &= u(x)v'(x) + v(x)u'(x) \\ &= \frac{d}{dx}\,(u(x)v(x)). \end{aligned}$$

Hence, there exists some constant C such that for each x on $[a, b]$,

$$F(x) = u(x)v(x) + C.$$

For $x = a$, we get

$$F(a) = 0 = u(a)v(a) + C$$

and, hence,

$$C = -u(a)v(a).$$

Then,

$$\begin{aligned} \int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx &= F(b) \\ &= u(b)v(b) + C \\ &= u(b)v(b) - u(a)v(a). \end{aligned}$$

Consequently,

$$\int_a^b u(x)v'(x)dx = [u(b)v(b) - u(a)v(a)] - \int_a^b v(x)u'(x)dx.$$

This completes the proof of Theorem 5.4.1.

Remark 19 The "two parts" of the integrand are "$u(x)$" and "$v'(x)dx$" or "u" and "dv". It becomes necessary to compute $u'(x)$ and $v(x)$ to make the integration by parts step.

Example 5.4.1 Evaluate the following integrals:

(i) $\int x \sin x \ dx$ (ii) $\int xe^{-x} \ dx$ (iii) $\int (\ln x) \ dx$

(iv) $\int \arcsin x \ dx$ (v) $\int \arccos x \ dx$ (vi) $\int x^2 e^x \ dx$

(i) We let $u = x$ and $dv = \sin x \ dx$. Then $du = dx$ and

$$\begin{aligned} v(x) &= \int \sin x \ dx \\ &= -\cos x + c. \end{aligned}$$

We drop the constant c, since we just need one $v(x)$. Then, by the integration by parts theorem, we get

$$\begin{aligned} \int x \sin x \ dx &= \int u dv \\ &= uv - \int v du \\ &= x(-\cos x) - \int (-\cos x) \ dx \\ &= -x \cos x + \sin x + c. \end{aligned}$$

(ii) We let $u = x, \ du = \ dx, \ dv = e^{-x} dx, \ v = \int e^{-x} \ dx = -e^{-x}$. Then,

$$\begin{aligned} \int xe^{-x} \ dx &= x(-e^{-x}) - \int (-e^{-x}) \ dx \\ &= -xe^{-x} - e^{-x} + c. \end{aligned}$$

(iii) We let $u = (\ln x), \ du = \frac{1}{x} \ dx, \ dv = \ dx, \ v = x$. Then,

$$\begin{aligned} \int \ln x \ dx &= x \ln x - \int x \cdot \frac{1}{x} \ dx \\ &= x \ln x - x + c. \end{aligned}$$

(iv) We let $u = \arcsin x,\ du = \dfrac{1}{\sqrt{1-x^2}}\ dx,\ dv = \ dx,\ v = x$. Then,

$$\int \arcsin x\ dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\ dx.$$

To evaluate the last integral, we make the substitution $y = 1-x^2$. Then, $dy = -2xdx$ and $x\ dx = (-1/2)du$ and hence

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\ dx &= \int \frac{(-1/2)du}{u^{1/2}}\\ &= -\frac{1}{2}\int u^{-1/2}du\\ &= -u^{1/2} + c\\ &= -\sqrt{1-x^2} + c.\end{aligned}$$

Therefore,

$$\int \arcsin x\ dx = x \arcsin x - \sqrt{1-x^2} + c.$$

(v) Part (v) is similar to part (iv) and is left as an exercise.

(vi) First we let $u = x^2,\ du = 2x\ dx,\ dv = e^x\ dx,\ v = \int e^x dx = e^x$. Then,

$$\begin{aligned}\int x^2 e^x\ dx &= x^2 e^x - \int 2xe^x\ dx\\ &= x^2e^x - 2\int xe^x\ dx.\end{aligned}$$

To evaluate the last integral, we let $u = x,\ du = \ dx,\ dv = e^x dx,\ v = e^x$. Then

$$\begin{aligned}\int xe^x\ dx &= xe^x - \int e^x\ dx\\ &= xe^x - e^x + c.\end{aligned}$$

Therefore,

$$\begin{aligned}\int x^2e^x\ dx &= x^2e^x - 2(xe^x - e^x + c)\\ &= x^2e^x - 2xe^x + 2e^x - 2c\\ &= e^x(x^2 - 2x + 2) + D.\end{aligned}$$

Example 5.4.2 Evaluate the given integrals in terms of integrals of the same kind but with a lower power of the integrand. Such formulas are called the reduction formulas. Apply the reduction formulas for $n = 3$ and $n = 4$.

(i) $\displaystyle\int \sin^n x \; dx$ (ii) $\displaystyle\int \csc^{m+2} x \quad dx$ (iii) $\displaystyle\int \cos^n x \; dx$ (iv) $\displaystyle\int \sec^{m+2} x \quad dx$

(i) We let

$$u = (\sin x)^{n-1}, \;\; du = (n-1)(\sin x)^{n-2} \cos x \; dx$$
$$dv = \sin x \; dx, \;\; v = \int \sin x \; dx = -\cos x.$$

Then

$$\begin{aligned}\int \sin^n x \; dx &= \int (\sin x)^{n-1}(\sin x \; dx)\\ &= (\sin x)^{n-1}(-\cos x) - \int (-\cos x)(n-1)(\sin x)^{n-2} \cos x \; dx\\ &= -(\sin x)^{n-1} \cos x + (n-1) \int (\sin x)^{n-2}(1 - \sin^2 x) \; dx\\ &= -(\sin x)^{n-1} \cos x + (n-1) \int (\sin x)^{n-2} dx\\ &\qquad - (n-1) \int \sin^n x \; dx.\end{aligned}$$

We now use algebra to solve the integral as follows:

$$\int \sin^n x \; dx + (n-1) \int \sin^n x \; dx = -(\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x \; dx$$

$$n \int \sin^n x \; dx = -(\sin x)^{n-1} \cos x + (n-1) \int \sin^{n-2} x \; dx$$

$$\boxed{\int \sin^n x \; dx = \frac{-1}{n} \; (\sin x)^{n-1} \cos x + \frac{n-1}{n} \int \sin^{n-2} x \; dx}\,. \tag{1}$$

We have reduced the exponent of the integrand by 2. For $n = 3$, we get

$$\begin{aligned}\int \sin^3 x \; dx &= \frac{-1}{3} \; (\sin x)^2 \cos x + \frac{2}{3} \int \sin x \; dx\\ &= \frac{-1}{3} \; (\sin x)^2 \cos x \frac{-2}{3} \; \cos x + c.\end{aligned}$$

For $n = 2$, we get

$$\begin{aligned}\int \sin^2 x\ dx &= \frac{-1}{2}\ (\sin x)\cos x + \frac{1}{2}\int 1\ dx\\ &= \frac{-1}{2}\ \sin x\cos x + \frac{x}{2} + c\\ &= \frac{1}{2}\ (x - \sin x\cos x) + c.\end{aligned}$$

For $n = 4$, we get

$$\begin{aligned}\int \sin^4 x\ dx &= \frac{-1}{4}\ (\sin x)^3\cos x + \frac{3}{4}\int \sin^2 x\ dx\\ &= \frac{-1}{4}\ (\sin x)^3\cos x + \frac{3}{4}\cdot\frac{1}{2}\ (x - \sin x\cos x) + c.\end{aligned}$$

In this way, we have a reduction formula by which we can compute the integral of any positive integral power of $\sin x$. If n is a negative integer, then it is useful to go in the direction as follows:

Suppose $n = -m$, where m is a positive integer. Then, from equation (1) we get

$$\begin{aligned}\frac{n-1}{n}\ \int \sin^{n-2} x\ dx &= \frac{1}{n}\ (\sin x)^{n-1}\cos x + \int (\sin x)^n\ dx\\ \int \sin^{n-2} x\ dx &= \frac{1}{n-1}\ (\sin x)^{n-1}\cos x + \frac{n}{n-1}\int (\sin x)^n\ dx\\ \int \sin^{-m-2} x\ dx &= \frac{1}{-m-1}\ (\sin x)^{-m-1}\cos x\\ &\quad + \frac{-m}{-m-1}\int (\sin x)^{-m}\ dx\end{aligned}$$

$$\boxed{\int \csc^{m+2} x\ dx = \frac{-1}{m+1}\ (\csc x)^m\cot x + \frac{m}{m+1}\int (\csc x)^m\ dx}. \quad (2)$$

This gives us the reduction formula for part (iii). Also,

$$\int \csc^n x\ dx = \frac{-1}{n-1}(\csc x^{n-2})\cot x + \frac{n-2}{n-1}\int (\csc x^{n-2})\ dx.$$

(iii) We can derive a formula by a method similar to part (i). However, let us make use of a trigonometric reduction formula to get it. Recall that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin x$. Then

$$\begin{aligned}
\int \cos^n x\ dx &= \int \sin^n\left(\frac{\pi}{2} - x\right)\ dx \qquad\qquad \left(\text{let } u = \frac{\pi}{2} - x,\ du = -dx\right)\\
&= \int \sin^n(u)(-du)\\
&= -\int \sin^n u du\\
&= -\left[\frac{-1}{n}\ (\sin u)^{n-1}\cos u + \frac{n-1}{n}\int \sin^{n-2} u du\right] \qquad \text{(by (1))}\\
&= \frac{1}{n}\ \left(\sin\left(\frac{\pi}{2} - x\right)\right)^{n-1}\cos\left(\frac{\pi}{2} - x\right)\\
&\quad - \frac{n-1}{n}\int\left(\sin\left(\frac{\pi}{2} - x\right)\right)^{n-2} d\left(\frac{\pi}{2} - x\right)
\end{aligned}$$

$$\boxed{\int \cos^n x\ dx = \frac{1}{n}\ (\cos x)^{n-1}\sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx}\ . \qquad (3)$$

To get part (iv) we replace n by $-m$ and get

$$\begin{aligned}
\int \cos^{-m} x\ dx &= \frac{1}{-m}\ (\cos x)^{-m-1}\sin x + \frac{-m-1}{-m}\int \cos^{-m-2} x\ dx\\
\int \sec^m x\ dx &= \frac{-1}{m}\ (\sec x)^m \tan x + \frac{m+1}{m}\int \sec^{m+2} x\ dx.
\end{aligned}$$

On solving for the last integral, we get

$$\boxed{\int \sec^{m+2} x\ dx = \frac{1}{m+1}\ (\sec x)^m \tan x + \frac{m}{m+1}\int \sec^m x\ dx}\ . \qquad (4)$$

Also, $\displaystyle\int \sec^n x\ dx = \frac{1}{n-1}\ \sec^{n-2} x\ \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx.$

In parts (ii), (iii) and (vi) we leave the cases for $n = 3$ and 4 as an exercise. These are handled as in part (i).

Example 5.4.3 Develop the reduction formulas for the following integrals:

(i) $\displaystyle\int \tan^n x\,dx$ (ii) $\displaystyle\int \cot^n x\,dx$ (iii) $\displaystyle\int \sinh^n x\,dx$ (iv) $\displaystyle\int \cosh^n x\,dx$

(i) First, we break $\tan^2 x = \sec^2 x - 1$ away from the integrand:

$$\begin{aligned}\int \tan^n x\ dx &= \int \tan^{n-2} x \cdot \tan^2 x\ dx \\ &= \int \tan^{n-2} x(\sec^2 x - 1)\ dx \\ \int \tan^n x\ dx &= \int \tan^{n-2} x \sec^2 x\ dx - \int \tan^{n-2} x\ dx.\end{aligned}$$

For the middle integral, we let $u = \tan x$ as a substitution.

$$\begin{aligned}\int \tan^n x\ dx &= \int u^{n-2} du - \int \tan^{n-2} x\ dx \\ &= \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\ dx \\ &= \frac{(\tan x)^{n-1}}{n-1} - \int \tan^{n-2} x\ dx.\end{aligned}$$

Therefore,

$$\boxed{\int \tan^n x\ dx = \frac{(\tan x)^{n-1}}{n-1} - \int \tan^{n-2} x\ dx \quad n \neq 1}. \tag{5}$$

$\displaystyle\int \tan x\ dx = \ln|\sec x| + c$ for $n = 1$.

(ii) We use the reduction formula $\tan\left(\frac{\pi}{2} - x\right) = \cot x$ in (5).

$$\int \cot^n x\ dx = \int \tan^n\left(\frac{\pi}{2} - x\right)\ dx; \qquad \left(\text{let } u = \frac{\pi}{2} - x,\ du = -dx\right)$$
$$= -\int \tan^n u(-du)$$
$$= -\int \tan^n u\ du$$
$$= -\left[\frac{\tan^{n-1}(u)}{n-1} - \int \tan^{n-2} u\ du\right], n \neq 1$$
$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x(-dx),\ n \neq 1$$
$$= -\frac{\cot^{n-1} x}{n-1} + \int \cot^{n-2} x\ dx,\ n \neq 1$$
$$\int \cot x\ dx = \ln|\sin x| + c,\ \text{for } n = 1.$$

Therefore,

$$\boxed{\int \cot^n(x)\ dx = -\frac{\cot^{n-1} x}{n-1} + \int \cot^{n-2} x\ dx, n \neq 1} \tag{6}$$
$$\int \cot x\ dx = \ln|\sin x| + c.$$

(iii) $$\int \sinh^n x\ dx = \int (\sinh^{n-1} x)(\sinh x\ dx); u = \sinh^{n-1} x,\ dv = \sinh x\ dx$$
$$= \sinh^{n-1} x \cosh x - \int \cosh x \cdot (n-1)\sinh^{n-2} x \cosh x dx$$
$$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x(\cosh^2 x)\ dx$$
$$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x(1 + \sinh^2 x)\ dx$$
$$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x\ dx - (n-1)\int \sinh^n x\ dx.$$

On bringing the last integral to the left, we get

$$n\int \sinh^n x\ dx = \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x\ dx$$

$$\boxed{\int \sinh^n x\ dx = \frac{1}{n}\ \sinh^{n-1} \cosh x - \frac{n-1}{n}\ \int \sinh^{n-2} x\ dx}. \qquad (7)$$

(iv) $\int \cosh^n x\ dx = \int (\cosh^{n-1} x)(\cosh x\ dx);\quad u = \cosh^{n-1} x,\ dv = \cosh x\ dx, v = \sinh x$

$$= \cosh^{n-1}(x) \sinh x - \int \sinh x (n-1) \cosh^{n-2} x \sinh x dx$$

$$= \cosh^{n-1} x \sinh x - (n-1)\int \cosh^{n-2} x \sinh^2 x\ dx$$

$$= \cosh^{n-1} x \sinh x - (n-1)\int \cosh^{n-2} x (\cosh^2 x - 1)\ dx$$

$$= \cosh^{n-1} x \sinh x - (n-1)\int \cosh^n x\ dx$$

$$+(n-1)\textstyle\int \cosh^{n-2} x\ dx$$

$$\int \cosh^n x\ dx\ + (n-1)\int \cosh^n x\ dx = \cosh^{n-1} x \sinh x$$

$$+(n-1)\textstyle\int \cosh^{n-2} x\ dx$$

$$n\int \cosh^n x\ dx\ = \cosh^{n-1} x \sinh x + (n-1)\int \cosh^{n-2} x\ dx$$

$$\boxed{\int \cosh^n x\ dx = \frac{1}{n}\ \cosh^{n-1} x \sinh x + \frac{n-1}{n}\ \int \cosh^{n-2} x\ dx} \qquad (8)$$

Example 5.4.4 Develop reduction formulas for the following:

(i) $\int x^n e^x \ dx$ (ii) $\int x^n \ln x \ dx$ (iii) $\int (\ln x)^n \ dx$

(iv) $\int x^n \sin x$ (v) $\int x^n \cos x \ dx$ (vi) $\int e^{ax} \sin(\ln x) \ dx$

(vii) $\int e^{ax} \cos(\ln x) \ dx$

(i) We let $u = x^n,\ dv = e^x \ dx,\ du = nx^{n-1}dx,\ v = e^x$. Then

$$\begin{aligned}\int x^n e^x \ dx &= x^n e^x - \int e^x (nx^{n-1}) \ dx \\ &= x^n e^x - n \int x^{n-1} e^x \ dx.\end{aligned}$$

Therefore,

$$\boxed{\int x^n e^x \ dx = x^n e^x - n \int x^{n-1} e^x \ dx}\ . \tag{9}$$

(ii) We let $u = \ln x,\ du = (1/x) \ dx,\ dv = x^n \ dx,\ v = x^{n+1}/(n+1)$. Then,

$$\begin{aligned}\int x^n \ln x \ dx &= (\ln x)\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \ dx \\ &= \frac{x^{n+1}(\ln x)}{n+1} - \frac{1}{n+1} \int x^n \ dx \\ &= \frac{x^{n+1}(\ln x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c.\end{aligned}$$

Therefore,

$$\boxed{\int x^n \ln x \ dx = \frac{x^{n+1}}{(n+1)^2} \ [(n+1)\ln(x) - 1] + c}\ . \tag{10}$$

(iii) We let $u = (\ln x)^n,\ du = n(\ln x)^{n-1}\frac{1}{x} \ dx,\ dv = \ dx,\ v = x$. Then,

$$\begin{aligned}\int (\ln x)^n \ dx &= x(\ln x)^n - \int x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} \ dx \\ &= x(\ln x)^n - n \int (\ln x)^{n-1} \ dx\end{aligned}$$

Therefore,

$$\boxed{\int (\ln x)^n \; dx = x(\ln x)^n - n \int (\ln x)^{n-1} \; dx}\,. \tag{11}$$

(iv) We let $u = x^n, \; du = nx^{n-1}dx, \; dv = \sin x \; dx, \; v = -\cos x$. Then,

$$\boxed{\begin{aligned} \int x^n \sin x \; dx &= x^n(-\cos x) - \int (-\cos x) nx^{n-1} \; dx \\ &= -x^n \cos x = n \int x^{n-1} \cos x \; dx. \end{aligned}} \tag{$*$}$$

Again in the last integral we let $u = x^{n-1}, \; du = (n-1)x^{n-2}dx, \; dv = \cos x \; dx, \; v = \sin x$. Then

$$\boxed{\begin{aligned} \int x^{n-1} \cos x \; dx &= x^{n-1} \sin x - \int \sin x (n-1) x^{n-2} dx \\ &= x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x \; dx. \end{aligned}} \tag{$**$}$$

By substitution, we get the reduction formula

$$\int x^n \sin x \; dx = -x^n \cos x + n \left[x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x \; dx \right]$$

$$\boxed{\int x^n \sin x \; dx = -x^n \cos x + nx^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x \; dx} \tag{12}$$

(v) We can use $(**)$ and $(*)$ in part (iv) to get the following:

$$\int x^{n-1} \cos x \; dx = x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x \; dx \qquad \text{by } (**)$$

$$= x^{n-1} \sin x - (n-1) \left[-x^{n-2} \cos x + (n-2) \int x^{n-3} \cos x \; dx \right] \qquad \text{by } (*)$$

$$\int x^{n-1} \cos x \, dx = x^{n-1}x + (n-1)x^{n-2} \cos x - (n-1)(n-2) \int x^{n-3} \cos x \, dx.$$

If we replace n by $n+1$ throughout the last equation, we get

$$\boxed{\int x^n \cos x \; dx = x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \; dx} \tag{13}$$

(vi) We let $dv = e^{ax} \; dx, \; v = \frac{1}{a} \; e^{ax}, \; u = \sin(bx), \; du = b \cos(bx) \; dx$. Then

$$\int e^{ax} \sin(bx) \; dx = \frac{1}{a} \; e^{ax} \sin(bx) - \frac{b}{a} \int e^{ax} \cos(bx) \; dx. \qquad (***)$$

In the last integral, we let $dv = e^{ax} dx, \; v = \frac{1}{a} \; e^{ax}, \; u = \cos bx$. Then

$$\int e^{ax} \cos(bx) \; dx = \frac{1}{a} \; e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \; dx \qquad (****)$$

First we substitute $(****)$ into $(***)$ and then solve for

$$\int e^{ax} \sin bx \; dx.$$

$$\begin{aligned}
\int e^{ax} \sin bx \; dx &= \frac{1}{a} \; e^{ax} \sin bx - \frac{b}{a} \; \left[\frac{1}{a} \; e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \; dx \right] \\
&= \frac{e^{ax}}{a^2} \; (a \sin bx - b \cos bx) - \frac{b^2}{a^2} \int ax \sin bx \; dx \\
\left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx \; dx &= \frac{e^{ax}}{a^2} \; (a \sin bx - b \cos bx \; dx)
\end{aligned}$$

$$\boxed{\int e^{ax} \sin bx \; dx = \frac{e^{ax}}{a^2 + b^2} \; (a \sin bx - b \cos bx) + c}. \tag{14}$$

(vii) We start with $(* * **)$ and substitute in (14) without the constant c and get

$$\begin{aligned}\int e^{ax}\cos bx\ dx &= \frac{1}{a}\ e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\ dx\\ &= \frac{1}{a}\ e^{ax}\cos bx + \frac{b}{a}\left[\frac{e^{ax}}{a^2b^2}\ (a\sin bx - b\cos bx)\right] + c\\ &= e^{ax}\left[\frac{1}{a}\ \cos bx + \frac{1}{a^2+b^2}\ \left(b\sin bx - \frac{b^2}{a}\ \cos bx\right)\right] + c\\ &= \frac{e^{ax}}{a^2+b^2}\ [b\sin b + a\cos bx] + c.\end{aligned}$$

Therefore,

$$\boxed{\int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}\ [b\sin bx + a\cos bx] + c}\ . \tag{15}$$

Exercises 5.4 Evaluate the following integrals and check your answers by differentiation. You may use the reduction formulas given in the examples.

1. $\int xe^{-2x}\ dx$
2. $\int x^3\ln x$
3. $\int \frac{dx}{x(\ln x)4}$
4. $\int (\ln x)^3\ dx$
5. $\int e^{2x}\sin 3x\ dx$
6. $\int e^{3x}\cos 2x\ dx$
7. $\int x^2\sin 2x\ dx$
8. $\int x^2\cos 3x\ dx$
9. $\int x\ln(x+1)\ dx$
10. $\int \arcsin(2x)\ dx$
11. $\int \arccos(2x)\ dx$
12. $\int \arctan(2x)\ dx$
13. $\int \sec^3 x\ dx$
14. $\int \sec^5 x\ dx$
15. $\int \tan^5 x\ dx$
16. $\int x^2\ln x\ dx$
17. $\int x^3\sin x\ dx$
18. $\int x^3\cos x\ dx$

19. $\int x \sinh x \, dx$

20. $\int x \cosh x \, dx$

21. $\int x(\ln x)^3 dx$

22. $\int x \arctan x \, dx$

23. $\int x \text{arccot}\, x \, dx$

24. $\int \sin^3 x \, dx$

25. $\int \cos^3 x \, dx$

26. $\int \sin^4 x \, dx$

27. $\int \cos^4 x \, dx$

28. $\int \sinh^2 x \, dx$

29. $\int \cosh^2 x \, dx$

30. $\int \sinh^3 x \, dx$

31. $\int x^2 \sinh x \, dx$

32. $\int x^2 \cosh x \, dx$

33. $\int x^3 \sinh x \, dx$

34. $\int x^3 \cosh x \, dx$

35. $\int x^2 e^{2x} dx$

36. $\int x^3 e^{-x} dx$

37. $\int x \sin(3x) \, dx$

38. $\int x \cos(x+1) dx$

39. $\int x \ln(x+1) dx$

40. $\int x \; 2^x dx$

41. $\int x \; 10^{2x} dx$

42. $\int x^2 \; 10^{3x} dx$

43. $\int x^2 (\ln x)^3 dx$

44. $\int \text{arcsinh}\,(3x) dx$

45. $\int \text{arccosh}\,(2x) dx$

46. $\int \text{arctanh}\,(2x) dx$

47. $\int \text{arccoth}\,(3x) dx$

48. $\int x \text{arcsec}\, x \, dx$

50. $\int x \text{arccsc}\, x \, dx$

5.5 Logarithmic, Exponential and Hyperbolic Functions

With the Fundamental Theorems of Calculus it is possible to rigorously develop the logarithmic, exponential and hyperbolic functions.

Definition 5.5.1 For each $x > 0$ we define the *natural logarithm of* x, denoted $\ln x$, by the equation

$$\ln(x) = \int_1^x \frac{1}{t}\, dt \quad , \quad x > 0.$$

Theorem 5.5.1 (Natural Logarithm) *The natural logarithm,* $\ln x$*, has the following properties:*

(i) $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$ *for all* $x > 0$.
The natural logarithm is an increasing, continuous and differentiable function on $(0, \infty)$.

(ii) If $a > 0$ *and* $b > 0$*, then* $\ln(ab) = \ln(a) + \ln(b)$.

(iii) If $a > 0$ *and* $b > 0$*, then* $\ln(a/b) = \ln(a) + \ln(b)$.

(iv) If $a > 0$ *and* n *is a natural number, then* $\ln(a^n) = n \ln a$.

(v) The range of $\ln x$ *is* $(-\infty, \infty)$.

(vi) $\ln x$ *is one-to-one and has a unique inverse, denoted* e^x.

Proof.

(i) Since $1/t$ is continuous on $(0, \infty)$, (i) follows from the Fundamental Theorem of Calculus, Second Form.

(ii) Suppose that $a > 0$ and $b > 0$. Then

$$\begin{aligned}
\ln(ab) &= \int_1^{ab} \frac{1}{t}\, dt \\
&= \int_1^{a} \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt \\
&= \ln a + \int_1^{b} \frac{1}{au}\, adu \; ; \; \left(u = \frac{1}{a}\, t, \; du = \frac{1}{a}\, dt\right) \\
&= \ln a + \ln b.
\end{aligned}$$

(iii) If $a > 0$ and $b > 0$, then

$$\begin{aligned}
\ln\left(\frac{a}{b}\right) &= \int_1^{\left(\frac{a}{b}\right)} \frac{1}{t}\ dt \\
&= \int_1^a \frac{1}{t}\ dt + \int_a^{\frac{a}{b}} \frac{1}{t}\ dt; \left(u = \frac{b}{a}\ t,\ du = \frac{b}{a}\ dt\right) \\
&= \int_1^a \frac{1}{t}\ dt + \int_b^1 \frac{1}{\left(\frac{au}{b}\right)}\ \left(\frac{a}{b}\ dt\right) \\
&= \int_1^a \frac{1}{t}\ dt - \int_1^b \frac{1}{u}\ du \\
&= \ln a - \ln b.
\end{aligned}$$

(iv) If $a > 0$ and n is a natural number, then

$$\begin{aligned}
\ln(a^n) &= \int_1^{a^n} \frac{1}{t}\ dt\ ;\ t = u^n,\ dt = nu^{n-1}du \\
&= \int_1^a \frac{1}{u^n} \cdot nu^{n-1}du \\
&= n\int_1^a \frac{1}{u}\ du \\
&= n \ln a
\end{aligned}$$

as required.

(v) From the partition $\{1, 2, 3, 4, \cdots\}$, we get the following inequality using upper and lower sum approximations:

graph

$$\frac{13}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} < \ln 4 < 1 + \frac{1}{2} + \frac{1}{3}.$$

Hence, $\ln 4 > 1$. $\ln(4^n) = n \ln 4 > n$ and $\ln 4^{-n} = -n \ln 4 < -n$. By the intermediate value theorem, every interval $(-n, n)$ is contained in the range of $\ln x$. Therefore, the range of $\ln x$ is $(-\infty, \infty)$, since the derivative of $\ln x$ is always positive, $\ln x$ is increasing and hence one-to-one. The inverse of $\ln x$ exists.

(vi) Let e denote the number such that $\ln(e) = 1$. Then we define $y = e^x$ if and only if $x = \ln(y)$ for $x \in (-\infty, \infty), y > 0$.

This completes the proof.

Definition 5.5.2 If x is any real number, we define $y = e^x$ if and only if $x = \ln y$.

Theorem 5.5.2 (Exponential Function) *The function $y = e^x$ has the following properties:*

(i) $e^0 = 1$, $\ln(e^x) = x$ *for every real x and* $\frac{d}{dx}(e^x) = e^x$.

(ii) $e^a \cdot e^b = e^{a+b}$ *for all real numbers a and b.*

(iii) $\frac{e^a}{e^b} = e^{a-b}$ *for all real numbers a and b.*

(iv) $(e^a)^n = e^{na}$ *for all real numbers a and natural numbers n.*

Proof.

(i) Since $\ln(1) = 0$, $e^0 = 1$. By definition $y = e^x$ if and only if $x = \ln(y) = \ln(e^x)$. Suppose $y = e^x$. Then $x = \ln y$. By implicit differentiation, we get

$$1 = \frac{1}{y}\frac{dy}{dx}, \quad \frac{dy}{dx} = y = e^x.$$

Therefore,

$$\frac{d}{dx}(e^x) = e^x.$$

(ii) Since $\ln x$ is increasing and, hence, one-to-one,

$$\begin{aligned} e^a \cdot e^b = e^{a+b} &\leftrightarrow \\ \ln(e^a \cdot e^b) = \ln(e^{a+b}) &\leftrightarrow \\ \ln(e^a) + \ln(e^b) = a + b &\leftrightarrow \\ a + b = a + b. \end{aligned}$$

It follows that for all real numbers a and b,

$$e^a \cdot e^b = e^{a+b}.$$

(iii)

$$\frac{e^a}{e^b} = e^{a-b} \leftrightarrow$$

$$\ln\left(\frac{e^a}{e^b}\right) = \ln(e^{a-b}) \leftrightarrow$$

$$\ln(e^a) - \ln(e^b) = a - b \leftrightarrow$$

$$a - b = a - b.$$

It follows that for all real numbers a and b,

$$\frac{e^a}{e^b} = e^{a-b}.$$

(iv)

$$(e^a)^n = e^{na} \leftrightarrow$$

$$\ln((e^a)^n) = \ln(e^{na}) \leftrightarrow$$

$$n\ln(e^a) = na \leftrightarrow$$

$$na = na.$$

Therefore, for all real numbers a and natural numbers n, we have

$$(e^a)^n = e^{na}.$$

Definition 5.5.3 Suppose $b > 0$ and $b \neq 1$. Then we define the following:

(i) For each real number x, $b^x = e^{x \ln b}$.

(ii) $y = \log_b x = \dfrac{\ln x}{\ln b}$.

Theorem 5.5.3 (General Exponential Function) *Suppose $b > 0$ and $b \neq 1$. Then*

(i) $\ln(b^x) = x \ln b$, *for all real numbers x.*

(ii) $\dfrac{d}{dx}\,(b^x) = b^x \ln b$, *for all real numbers x.*

(iii) $b^{x_1} \cdot b^{x_2} = b^{x_1 + x_2}$, *for all real numbers x_1 and x_2.*

(iv) $\dfrac{b^{x_1}}{b^{x_2}} = b^{x_1 - x_2}$, *for all real numbers x_1 and x_2.*

(v) $(b^{x_1})^{x_2} = b^{x_1 x_2}$, *for all real numbers x_1 and x_2.*

(vi) $\displaystyle\int b^x \; dx = \frac{b^x}{\ln b} + c.$

Proof.

(i) $\ln(b^x) = \ln(e^{x \ln b}) = x \ln b$

(ii) $$\frac{d}{dx}\,(b^x) = \frac{d}{dx}\,(e^{x \ln b}) = e^{x \ln b} \cdot (\ln b) \qquad \text{(by the chain rule)}$$
$$= b^x \ln b.$$

(iii) $$b^{x_1} \cdot b^{x_2} = e^{x_1 \ln b} \cdot e^{x_2 \ln b}$$
$$= e^{(x_1 \ln b + x_2 \ln b)}$$
$$= e^{(x_1 + x_2) \ln b}$$
$$= b^{(x_1 + x_2)}$$

(iv)
$$\begin{aligned}\frac{b^{x_1}}{b^{x_2}} &= \frac{e^{x_1 \ln b}}{e^{x_2 \ln b}} \\ &= e^{x_1 \ln b - x_2 \ln b} \\ &= e^{(x_1 - x_2) \ln b} \\ &= b^{(x_1 - x_2)}.\end{aligned}$$

(v) By Definition 5.5.3 (i), we get

$$\begin{aligned}(b^{x_1})^{x_2} &= e^{x_2 \ln(b^{x_1})} \\ &= e^{x_2 \ln(e^{x_1 \ln b})} \\ &= e^{x_2 \cdot x_1 \ln b} \\ &= e^{(x_1 x_2) \ln b} \\ &= b^{x_1 x_2}.\end{aligned}$$

(vi) Since

$$\frac{d}{dx}\,(b^x) = b^x \ln b,$$

we get

$$\begin{aligned}\int b^x (\ln b)\; dx &= b^x + c, \\ \ln b \int b^x \; dx &= b^x + c, \\ \int e^x \; dx &= \frac{b^x}{\ln b} + D,\end{aligned}$$

where D is some constant. This completes the proof.

Theorem 5.5.4 *If $u(x) > 0$ for all x, and $u(x)$ and $v(x)$ are differentiable functions, then we define*

$$y = (u(x))^{v(x)} = e^{v(x) \ln(u(x))}.$$

Then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{d}{dx}\,(u(x))^{v(x)} = (u(x))^{v(x)}\left[v'(x)\ln(u(x)) + v(x)\,\frac{u'(x)}{u(x)}\right].$$

Proof. This theorem follows by the chain rule and the product rule as follows

$$\frac{d}{dx}\,[u^v] = \frac{d}{dx}\,[e^{v\ln u}] = e^{v\ln u}\left[v'\ln u + v\,\frac{u'}{u}\right] = u^v\left[v'\ln u + v\,\frac{u1}{u}\right].$$

Theorem 5.5.5 *The following differentiation formulas for the hyperbolic functions are valid.*

(i) $\frac{d}{dx}\,(\sinh x) = \cosh x$ *(ii)* $\frac{d}{dx}\,(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}\,(\tanh x) = \operatorname{sech}^2 x$ *(iv)* $\frac{d}{dx}\,(\coth x) = -\operatorname{csch}^2 x$

(v) $\frac{d}{dx}\,(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ *(vi)* $\frac{d}{dx}\,(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Proof. We use the definitions and properties of hyperbolic functions given in Chapter 1 and the differentiation formulas of this chapter.

(i) $$\frac{d}{dx}\,(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

(ii) $$\frac{d}{dx}\,(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

(iii) $$\frac{d}{dx}\,(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{(\cosh x)(\cosh x) - \sinh(\sinh x)}{(\cosh x)^2}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{(\cosh x)^2} = \frac{1}{\cosh x)^2} = \operatorname{sech}^2 x$$

(iv) $$\frac{d}{dx}(\coth x) = \frac{d}{dx}(\tanh x)^{-1} = -1(\tanh x)^{-2} \cdot \operatorname{sech}^2 x$$
$$= -\frac{\cosh^2 x}{\sinh^2 x} \cdot \frac{1}{\cosh^2 x} = -\frac{1}{\sinh^2 x}$$
$$= -\operatorname{csch}^2 x.$$

(v) $$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}(\cosh x)^{-1} = -1(\cosh x)^{-2} \cdot \sinh x$$
$$= -\operatorname{sech} x \tanh x.$$

(vi) $$\frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx}(\sinh x)^{-1} = -1(\sinh x)^{-2} \cdot \cosh x$$
$$= -\coth x \operatorname{csch} x.$$

This completes the proof.

Theorem 5.5.6 *The following integration formulas are valid:*

(i) $$\int \sinh x \, dx = \cosh x + c$$

(ii) $$\int \cosh x \, dx = \sinh x + c$$

(iii) $$\int \tanh x \, dx = \ln(\cosh x) + c$$

(iv) $$\int \coth x dx = \ln|\sinh x| + c$$

(v) $$\int \operatorname{sech} x \, dx = 2\arctan(e^x) + c$$

(vi) $$\int \operatorname{csch} x \, dx = \ln\left|\tanh\left(\frac{x}{2}\right)\right| + c$$

Proof. Each formula can be easily verified by differentiating the right-hand side to get the integrands on the left-hand side. This proof is left as an exercise.

Theorem 5.5.7 *The following differentiation and integration formulas are valid:*

(i) $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{1+x^2}}$ *(ii)* $\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arcsinh} x + c$

(iii) $\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}}$ *(iv)* $\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x + c$

(v) $\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2}, |x| < 1$ *(vi)* $\int \frac{1}{1-x^2}\, dx = \operatorname{arctanh} x + c$

Proof. This theorem follows directly from the following definitions:

(1) $\operatorname{arcsinh} x = \ln(x + \sqrt{1+x^2})$ (2) $\operatorname{arccosh} x = \ln(x + \sqrt{x^2-1})$

(3) $\operatorname{arctanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), |x| < 1.$

The proof is left as an exercise.

Exercises 5.5

1. Prove Theorem 5.5.6.
2. Prove Theorem 5.5.7.
3. Show that $\sinh mx$ and $\cosh mx$ are linearly independent if $m \neq 0$. (Hint: Show that the Wronskian $W(\sinh mx, \cosh mx)$ is not zero if $m \neq 0$.)
4. Show that e^{mx} and e^{-mx} are linearly independent if $m \neq 0$.
5. Show that solution of the equation $y'' - m^2 y = 0$ can be expressed as $y = c_1 e^{mx} + c_2 e^{-mx}$.
6. Show that every solution of $y'' - m^2 y = 0$ can be written as $y = A \sinh mx + B \cosh mx$.
7. Determine the relation between c_1 and c_2 in problem 5 with A and B in problem 6.
8. Prove the basic identities for hyperbolic functions:

(i) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$

(ii) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y.$

(iii) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$

(iv) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y.$

(v) $\sinh 2x = 2 \sinh x \cosh x.$

(vi) $\cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \cosh 2x.$

(vii) $\cosh^2 x - \sinh^2 x = 1,\ 1 - \tanh^2 x = \operatorname{sech}^2 x,\ \coth^2 x - 1 = \operatorname{csch}^2 x.$

9. Eliminate the radical sign using the given substitution:

(i) $\sqrt{a^2+x^2},\ x = a\sinh t$ (ii) $\sqrt{a^2-x^2},\ x = \tanh t$

(iii) $\sqrt{x^2-a^2},\ x = a\cosh t.$

10. Compute y' in each of the following:

(i) $y = 2\sinh(3x) + 4\cosh(2x)$ (ii) $y = 4\tanh(5x) - 6\coth(3x)$

(iii) $y = x \operatorname{sech}(2x) + x^2 \operatorname{csch}(5x)$ (iv) $y = 3\sinh^2(4x+1)$

(v) $y = 4\cosh^2(2x-1)$ (vi) $y = \sinh(2x)\cosh(3x)$

11. Compute y' in each of the following:

(i) $y = x^2 e^{-x^3}$ (ii) $y = 2^{x^2}$ (iii) $y = (x^2+1)^{\sin(2x)}$

(iv) $y = \log_{10}(x^2+1)$ (v) $y = \log_2(\sec x + \tan x)$(vi) $y = 10^{(x^3+1)}$

12. Compute y' in each of the following:

(i) $y = x\ln x - x$ (ii) $y = \ln(x + \sqrt{x^2-4})$ (iii) $y = \ln(x + \sqrt{4+x^2})$

(iv) $y = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$ (v) $y = \operatorname{arcsinh}(3x)$ (vi) $y = \operatorname{arccosh}(3x)$

13. Evaluate each of the following integrals:

(i) $\int \sinh(3x)\ dx$ (ii) $\int x^3 e^{x^2}\ dx$ (iii) $\int x^2 \ln(x+1)\ dx$

(iv) $\int x \sinh 2x\ dx$ (v) $\int x \cosh 3x\ dx$ (vi) $\int x 4^{x^2}\ dx$

14. Evaluate each of the following integrals:

(i) $\int \operatorname{arcsinh} x\ dx$ (ii) $\int \operatorname{arccosh} x\ dx$ (iii) $\int \operatorname{arctanh} x\ dx$

(iv) $\int \frac{dx}{\sqrt{4-x^2}}$ (v) $\int \frac{dx}{\sqrt{4+x^2}}$ (vi) $\int \frac{dx}{\sqrt{x^2-4}}$

15. Logarithmic Differentiation is a process of computing derivatives by first taking logarithms and then using implicit differentiation. Find y' in each of the following, using logarithmic differentiation.

(i) $y = \frac{(x^2+1)^3(x^2+4)^{10}}{(x^2+2)^5(x^2+3)^4}$ (ii) $y = (x^2+4)^{(x^3+1)}$

(iii) $y = (\sin x + 3)^{(4\cos x + 7)}$ (iv) $y = (3\sinh x + \cos x + 5)^{(x^3+1)}$

(v) $y = (e^{x^2}+1)^{(2x+1)}$ (vi) $y = x^2(x^2+1)^{(x^3+1)}$

In problems 16–30, compute $f'(x)$ each $f(x)$.

16. $f(x) = \int_1^x \sinh^3(t) dt$

17. $f(x) = \int_x^{x^2} \cosh^5(t) dt$

18. $f(x) = \int_{\sinh x}^{\cosh x} (1+t^2)^{3/2} dt$

19. $f(x) = \int_{\tanh x}^{\operatorname{sech} x} (1+t^3)^{1/2} dt$

20. $f(x) = \int_{\ln x}^{(\ln x)^2} (4+t^2)^{5/2} dt$

21. $f(x) = \int_{e^{x^2}}^{e^{x^2}} (1+4t^2)^{\pi} dt$

22. $f(x) = \int_{e^{\sin x}}^{e^{\cos x}} \frac{1}{(1+t^2)^{3/2}}\, dt$

23. $f(x) = \int_{2^x}^{3^x} \frac{1}{(4+t^2)^{5/2}}\, dt$

24. $f(x) = \int_{4^{2x}}^{5^{3x}} (1+2t^2)^{3/2} dt$

25. $f(x) = \int_{\log_2 x}^{\log_3 x} (1+5t^3)^{1/2} dt$

26. $f(x) = \int_{\operatorname{arcsinh} x}^{\operatorname{arccosh} x} \frac{1}{(1+t^2)^{3/2}}\, dt$

27. $f(x) = \int_{2^{x^2}}^{4^{x^3}} e^{t^2} dt$

28. $f(x) = \int_{4^{\sin x}}^{5^{\cos x}} e^{-t^2} dt$

29. $f(x) = \int_{\sinh(x^2)}^{\cosh(x^3)} e^{-t^3} dt$

30. $f(x) = \int_{\operatorname{arctanh} x}^{\operatorname{arccoth} x} \sin(t^2) dt$

In problems 31–40, evaluate the given integrals.

31. $\int \frac{e^{\arctan x}}{1+x^2}\, dx$

32. $\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\, dx$

33. $\int e^{\sin 2x} \cos 2x\, dx$

34. $\int x^2 e^{x^3} dx$

35. $\int \frac{e^{2x}}{1+e^{2x}}\, dx$

36. $\int e^x \cos(1+2e^x) dx$

37. $\int e^{3x} \sec^2(2+e^{3x}) dx$

38. $\int 10^{\cos x} \sin x\, dx$

39. $\int \frac{4^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\, dx$

40. $\int x\ 10^{x^2+3}\, dx$

5.6 The Riemann Integral

In defining the definite integral, we restricted the definition to continuous functions. However, the definite integral as defined for continuous functions is a special case of the general Riemann Integral defined for bounded functions that are not necessarily continuous.

Definition 5.6.1 Let f be a function that is defined and bounded on a closed and bounded interval $[a, b]$. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Let $C = \{c_i : x_{i-1} \leq c_i \leq x_i, i = 1, 2, \cdots, n\}$ be any arbitrary selection of points of $[a, b]$. Then the *Riemann Sum* that is associated with P and C is denoted $R(P)$ and is defined by

$$\begin{aligned} R(P) &= f(c_1)(x_1 - x_0) + f(c_2)(x_2 - x_1) + \cdots + f(c_n)(x_n + x_{n-1}) \\ &= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}). \end{aligned}$$

Let $\Delta x_i = x_i - x_{i-1},\ i = 1, 2, \cdots, n$. Let $||\Delta|| = \max\limits_{1 \leq i \leq n}\{\Delta x_i\}$. We write

$$R(P) = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

We say that

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

if and only if for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(c_i)\Delta x_i - I \right| < \epsilon$$

whenever $||\Delta|| < \delta$ for all partitions P and all selections C that define the Riemann Sum.

If the limit I exists as a finite number, we say that f is (Riemann) integrable and write

$$I = \int_a^b f(x)\ dx.$$

Next we will show that if f is continuous, the Riemann integral of f is the definite integral defined by lower and upper sums and it exists. We first prove two results that are important.

Definition 5.6.2 A function f is said to be *uniformly continuous* on its domain D if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$, for any x_1 and x_2 in D, then

$$|f(x_1) - f(x_2)| < \epsilon.$$

Definition 5.6.3 A collection $C = \{U_\alpha : U_\alpha \text{ is an open interval}\}$ is said to *cover* a set D if each element of D belongs to some element of C.

Theorem 5.6.1 *If $C = \{U_\alpha : U_\alpha$ is an open interval$\}$ covers a closed and bounded interval $[a, b]$, then there exists a finite subcollection $B = \{U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n}\}$ of C that covers $[a, b]$.*

Proof. We define a set A as follows:

$$A = \{x : x \in [a, b] \text{ and } [a, x] \text{ can be covered by a finite subcollection of } C\}.$$

Since $a \in A$, A is not empty. A is bounded from above by b. Then A has a least upper bound, say $\text{lub}(A) = p$. Clearly, $p \leq b$. If $p < b$, then some U_α in C contains p. If $U_\alpha = (a_\alpha, b_\alpha)$, then $a_\alpha < p < b_\alpha$. Since $p = \ell\text{ub}(A)$, there exists some point a^* of A between a_α and p. There exists a subcollection $B = \{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ that covers $[a, a^*]$. Then the collection $B_1 = \{U_{\alpha_1}, \cdots, U_{\alpha_n}, U_\alpha\}$ covers $[a, b_\alpha)$. By the definition of A, A must contain all points of $[a, b]$ between p and b_α. This contradicts the assumption that $p = \ell\text{ub}(A)$. So, $p = b$ and $b \in A$. It follows that some finite subcollection of C covers $[a, b]$ as required.

Theorem 5.6.2 *If f is continuous on a closed and bounded interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Let $\epsilon > 0$ be given. If $p \in [a, b]$, then there exists $\delta_p > 0$ such that $|f(x) - f(p)| < \epsilon/3$, whenever $p - \delta_p < x < p + \delta_p$. Let $U_p = \left(p - \frac{1}{3}\,\delta_p,\ p + \frac{1}{3}\,\delta_p\right)$. Then $C = \{U_p : p \in [a, b]\}$ covers $[a, b]$. By Theorem 5.6.1, some finite subcollection $B = \{U_{p_1}, U_{p_2}, \ldots, U_{p_n}\}$ of C covers $[a, b]$. Let $\delta = \frac{1}{3}\min\{\delta_{p_i} : i = 1, 2, \cdots, n\}$. Suppose that $|x_1 - x_2| < \delta$ for any two points x_1 and x_2 of $[a, b]$. Then $x_1 \in U_{p_i}$ and $x_2 \in U_{p_j}$ for some p_i and p_j. We note that

$$\begin{aligned}
|p_i - p_j| &= |(p_i - x_1) + (x_1 - x_2) + (x_2 - p_j)| \\
&\leq |p_i - x_i| + |x_1 - x_2| + |x_2 - p_j| \\
&< \frac{1}{3}\,\delta_{p_i} + \delta + \frac{1}{3}\,\delta_{p_j} \\
&\leq \max\{\delta_{p_i}, \delta_{p_j}\}.
\end{aligned}$$

It follows that both p_i and p_j are either in U_{p_i} or U_{p_j}. Suppose that p_i and p_j are both in U_{p_i}. Then

$$\begin{aligned}|x_2 - p_i| &= |(x_2 - x_1)| + (x_1 - p_i)| \\ &\leq |x_2 - x_1| + |x_1 - p_i| \\ &< \delta + \frac{1}{3}\,\delta_{p_i} \\ &< \delta_{p_i}.\end{aligned}$$

So, x_1, x_2, p_i and p_j are all in U_{p_i}. Then

$$\begin{aligned}|f(x_1) - f(x_2)| &= |(f(x_1) - f(p_i)) + (f(p_i) - f(x_2))| \\ &\leq |f(x_1) - f(p_i)| + |f(p_i) - f(x_2)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon.\end{aligned}$$

By Definition 5.6.2, f is uniformly continuous on $[a, b]$.

Theorem 5.6.3 *If f is continuous on $[a, b]$, then f is (Riemann) integrable and the definite integral and the Riemann integral have the same value.*

Proof. Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of $[a, b]$ and $C = \{c_i : x_{i-1} \leq c_i \leq x_i, i = 1, 2, \ldots, n\}$ be an arbitrary selection. For each $i = 1, 2, \ldots, n$ let

m_i = absolute minimum of f on $[x_{i-1}, x_i]$ obtained at c_i^*, $f(c_i^*) = m_i$;
M_i = absolute maximum of f on $[x_{i-1}, x_i]$ obtained at $c_i^{**}) = M_i$;
m = absolute minimum of f on $[a, b]$;
M = absolute maximum of f on $[a, b]$;
$R(P) = \displaystyle\sum_{i=1}^{n} f(c_i)\Delta x_i$,

Then for each $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}m(b-a) &\leq \sum_{i=1}^{n} f(c_i^*)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} f(c_i^{**})(x_i - x_{i-1}) \leq M(b-a).\end{aligned}$$

We recall that

$$L(P) = \sum_{i=1}^{n} f(c_i^*)\Delta x_i, \;\; R(P) = \sum_{i=1}^{n} f(c_i)\Delta x_i, U(P) = \sum_{i=1}^{n} f(c_i^{**})\Delta x_i.$$

We note that $L(P)$ and $U(P)$ are also Riemann sums and for every partition P, we have

$$L(P) \leq R(P) \leq U(P).$$

To prove the theorem, it is sufficient to show that

$$\text{lub}\{L(P)\} = \text{glb}\{U(P)\}.$$

Since f is uniformly continuous, by Theorem 5.6.2, for each $\epsilon > 0$ there is some $\delta > 0$ such that $|f(x)-f(y)| < \dfrac{\epsilon}{b-a}$ whenever $|x-y| < \delta$ for x and y in $[a, b]$. Consider all partitions P, selections $C = \{c_i\}, C^* = \{c_i^*\}, C^{**} = \{c_i^{**}\}$ such that

$$||\Delta|| = \max_{1\leq i\leq n}(x_i - x_{i-1}) < \frac{\delta}{3}.$$

Then, for each $i = 1, 2, \ldots, n$

$$\begin{aligned}
|f(c_i^{**}) - f(c_i^*)| &< \frac{\epsilon}{b-a} \\
|f(c_i^*) - f(c_i)| &< \frac{\epsilon}{b-a} \\
|f(c_i^{**}) - f(c_i)| &< \frac{\epsilon}{b-a} \\
|U(P) - L(P)| &= \left|\sum_{i=1}^{n}(f(c_i^{**}) - f(c_i^*))\Delta x_i\right| \\
&\leq \sum_{i=1}^{m} |f(c_i^{**}) - f(c_i^*)|\Delta x_i \\
&< \frac{\epsilon}{b-a} \;\sum_{i=1}^{n} \Delta x_i \\
&= \epsilon.
\end{aligned}$$

It follows that

$$\text{lub}\{L(P)\} = \lim_{||\Delta||\to 0} R(P)p = \text{glb}\{U(P)\} = I.$$

By definition of the definite integral, I equals the definite integral of $f(x)$ from $x = a$ to $x = b$, which is also the Riemann integral of f on $[a, b]$. We write

$$I = \int_a^b f(x)\ dx.$$

This proves Theorem 5.6.2 as well as Theorem 5.2.1.

Exercises 5.6

1. Prove Theorem 5.2.3. (Hint: For each partition $P = \{a = x_0 < x_1 < \ldots < x_n = b]$ of $[a, b]$,

$$\begin{aligned} g(b) - g(a) &= [g(x_n) - g(x_{n-1})] + [g(x_{n-1}) - g(x_{n-2})] + \ldots + [g(x_1) - g(x_0)] \\ &= \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})] \\ &= \sum_{i=1}^{n} g'(c_i)(x_i - x_{i-1}) \qquad \text{(by Mean Value Theorem)} \\ &= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\ &= R(P) \end{aligned}$$

for some selection $C = \{c_i : x_{i-1} < c_i < x_i, i = 1, 2, \cdots, n\}$.)

2. Prove Theorem 5.2.3 on the linearity property of the definite integral. (Hint:

$$\begin{aligned} \int_a^b [Af(x) + bg(x)]\ dx &= \lim_{||\Delta|| \to 0} \left\{ \sum_{i=1}^{n} [Af(c_i) + Bg(c_i)] \cdot [x_i - x_{i-1}) \right\} \\ &= \lim_{||\Delta|| \to 0} \left(A \sum_{i=1}^{n} f(c_i)\Delta x_i + B \sum_{i=1}^{n} g(c_i)\Delta x_i \right) \\ &= A \left(\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \right) + B \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} g(c_i)\Delta x_i \\ &= A \int_a^b f(x)dx + B \int_a^b g(x)dx.) \end{aligned}$$

3. Prove Theorem 5.2.4.

   (Hint: $[a, b] = [a, c] \cup [c, b]$. If $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ is a partition of $[a, b]$, then for some i, $P_1 = \{a = x_0 < \ldots < x_{i-1} < c < x_i < \ldots < x_n = b\}$ yields a partition of $[a, b]$; $\{a < x_0 < \cdots < x_{i-1} < c\}$ is a partition of $[a, c]$ and $\{c < x_i < \cdots < x_n = b\}$ is a partition of $[c, b]$. The addition of c to the partition does not increase $||\Delta||$.)

4. Prove Theorem 5.2.5.

   (Hint: For each partition P and selection C we have

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}).)$$

5. Prove that if f is continuous on $[a, b]$ and $f(x) > 0$ for each $x \in [a, b]$, then

$$\int_a^b f(x)\ dx > 0.$$

   (Hint: There is some c in $[a, b]$ such that $f(c)$ is the absolute minimum of f on $[a, b]$ and $f(c) > 0$. Then argue that

$$0 < f(c)(b - a) \leq L(P) \leq U(P)$$

   for each partition P.)

6. Prove that if f and g are continuous on $[a, b]$, $f(x) > g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)\ dx > \int_a^b g(x)\ dx.$$

   (Hint: By problem 5,

$$\int_a^b (f(x) - g(x))\ dx > 0.$$

   Use the linearity property to prove the statement.)

7. Prove that if f is continuous on $[a, b]$, then

$$\left| \int_a^b f(x)\ dx \right| \leq \int_a^b |f(x)| dx.$$

(Hint: Recall that $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. Use problem 5 to conclude the result.)

8. Prove the Mean Value Theorem, Theorem 5.2.6.

   (Hint: Let

   $m =$ absolute minimum of f on $[a, b]$;

   $M =$ absolute minimum of f on $[a, b]$;

$$f_{av}[a, b] = \frac{1}{b-a} \int_a^b f(x)\ dx;$$
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

   Then $m \leq f_{av}[a, b] \leq M$. By the intermediate value theorem for continuous functions, there exists some c on $[a, b]$ such that $f(c) = f_{av}[a, b]$.)

9. Prove the Fundamental Theorem of Calculus, First Form, Theorem 5.2.6.

   (Hint:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right] \\ &= \lim_{h\to 0} \frac{1}{h} \left[\int_a^x f(t)\ dx + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \right] \\ &= \lim_{h\to 0} \frac{1}{h} \left[\int_x^{x+h} f(t)dt \right] \\ &= \lim_{h\to 0} f(c), (\text{for some } c,\ x \leq c \leq x+h;) \\ &= f(x) \end{aligned}$$

   where $x \leq c \leq x + h$, by Theorem 5.2.6.)

10. Prove the Leibniz Rule, Theorem 5.2.8.

    (Hint:

$$\int_{\alpha(x)}^{\beta(x)} f(t)dt = \int_a^{\beta(x)} f(t)dt - \int_a^{\alpha(x)} f(t)dt$$

    for some a. Now use the chain rule of differentiation.)

11. Prove that if f and g are continuous on $[a,b]$ and g is nonnegative, then there is a number c in (a,b) for which

$$\int_a^b f(x)g(x)\ dx = f(c)\int_a^b g(x)\ dx.$$

(Hint: If m and M are the absolute minimum and absolute maximum of f on $[a,b]$, then $mg(x) \le f(x)g(x) \le Mg(x)$. By the Order Property,

$$m\int_a^b g(x)\ dx \le \int_a^b f(x)g(x) \le M\int_a^b g(x)\ dx$$
$$m \le \frac{\int_a^b f(x)g(x)\ dx}{\int_a^b g(x)\ dx} \le M \quad \left(\text{if } \int_0^b g(x)\ dx \ne 0\right).$$

By the Intermediate Value Theorem, there is some c such that

$$f(x) = \frac{\int_a^b f(x)g(x)\ dx}{\int_a^b g(x)\ dx} \text{ or}$$
$$\int_a^b f(x)g(x)\ dx = f(c)\int_a^b g(x)\ dx.$$

If $\displaystyle\int_a^b g(x)\ dx = 0$, then $g(x) \not\equiv 0$ on $[a,b]$ and all integrals are zero.)

Remark 20 The number $f(c)$ is called the weighted average of f on $[a,b]$ with respect to the weight function g.

5.7 Volumes of Revolution

One simple application of the Riemann integral is to define the volume of a solid.

Theorem 5.7.1 *Suppose that a solid is bounded by the planes with equations $x = a$ and $x = b$. Let the cross-sectional area perpendicular to the x-axis at x be given by a continuous function $A(x)$. Then the volume V of the solid is given by*

$$V = \int_a^b A(x)\ dx.$$

Proof. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$. For each $i = 1, 2, 3, \cdots, n$, let

V_i = volume of the solid between the planes with equations $x = x_{i-1}$ and $x = x_i$,

m_i = absolute minimum of $A(x)$ on $[x_{i-1}, x_i]$,

M_i = absolute maximum of $A(x)$ on $[x_{i-1}, x_i]$,

$\Delta x_i = x_i - x_{i-1}$.

Then

$$m_i \Delta x_i \leq V_i \leq M_i \Delta x_i, m_i \leq \frac{V_i}{\Delta x_i} \leq M_i.$$

Since $A(x)$ is continuous, there exists some c_i such that $x_{i-1} \leq c_i \leq M_i$ and

$$m_i \leq A(c_i) = \frac{V_i}{\Delta x_i} \leq M_i$$

$$V_i = A(c_i)\Delta x_i$$

$$V = \sum_{i=1}^{n} A(c_i)\Delta x_i.$$

It follows that for each partition P of $[a, b]$ there exists a Riemann sum that equals the volume. Hence, by definition,

$$V = \int_a^b A(x)\ dx.$$

Theorem 5.7.2 *Let f be a function that is continuous on $[a, b]$. Let R denote the region bounded by the curves $x = a,\ x = b,\ y = 0$ and $y = f(x)$. Then the volume V obtained by rotating R about the x-axis is given by*

$$V = \int_a^b \pi(f(x))^2 dx.$$

Proof. Clearly, the volume of the rotated solid is between the planes with equations $x = a$ and $x = b$. The cross-sectional area at x is the circle generated by the line segment joining $(x, 0)$ and $(x, f(x))$ and has area $A(x) = \pi(f(x))^2$. Since f is continuous, $A(x)$ is a continuous function of x. Then by Theorem 5.7.1, the volume V is given by

$$V = \int_a^b \pi(f(x))^2\ dx.$$

Theorem 5.7.3 *Let f and R be defined as in Theorem 5.7.2. Assume that $f(x) > 0$ for all $x \in [a, b]$, either $a \geq 0$ or $b \leq 0$, so that $[a, b]$ does not contain 0. Then the volume V generated by rotating the region R about the y-axis is given by*

$$V = \int_a^b (2\pi x f(x))\ dx.$$

Proof. The line segment joining $(x, 0)$ and $(x, f(x))$ generates a cylinder whose area is $A(x) = 2\pi x f(x)$. We can see this if we cut the cylinder vertically at $(-x, 0)$ and flattening it out. By Theorem 5.7.1, we get

$$V = \int_a^b 2\pi x f(x)\ dx.$$

Theorem 5.7.4 *Let f and g be continuous on $[a, b]$ and suppose that $f(x) > g(x) > 0$ for all x on $[a, b]$. Let R be the region bounded by the curves $x = a,\ x = b,\ y = f(x)$ and $y = g(x)$.*

(i) The volume generated by rotating R about the x-axis is given by

$$\int_a^b \pi[(f(x))^2 - (g(x))^2]\ dx.$$

(ii) If we assume R does not cross the y-axis, then the volume generated by rotating R about the y-axis is given by

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx.$$

(iii) If, in part (ii), R does not cross the line $x = c$, then the volume generated by rotating R about the line $x = c$ is given by

$$V = \int_a^b 2\pi|c - x|[f(x) - g(x)]dx.$$

Proof. We leave the proof as an exercise.

Remark 21 There are other various horizontal or vertical axes of rotation that can be considered. The basic principles given in these theorems can be used. Rotations about oblique lines will be considered later.

Example 5.7.1 Suppose that a pyramid is 16 units tall and has a square base with edge length of 5 units. Find the volume of V of the pyramid.

graph

We let the y-axis go through the center of the pyramid and perpendicular to the base. At height y, let the cross-sectional area perpendicular to the y-axis be $A(y)$. If $s(y)$ is the side of the square $A(y)$, then using similar triangles, we get

$$\begin{aligned}
\frac{s(y)}{5} = \frac{16-y}{16}\ , s(y) &= \frac{5}{16}\ (16-y) \\
A(y) &= \frac{25}{256}\ (16-y)^2.
\end{aligned}$$

Then the volume of the pyramid is given by

$$\begin{aligned}
\int_0^{16} A(y)dy &= \int_0^{16} \frac{25}{256}\ (16-y)^2 dy \\
&= \frac{25}{256}\ \left[\frac{(16-y)^3}{-3}\right]_0^{16} \\
&= \frac{25}{256}\ \left[\frac{(16)^3}{3}\right] = \frac{(25)(16)}{3} \\
&= \frac{400}{3} \text{ cubic units.} \\
Check:\ V &= \frac{1}{3}\ (\text{base side})^2 \cdot\ \text{height} \\
&= \frac{1}{3}\ (25)\cdot 16 \\
&= \frac{400}{3}.
\end{aligned}$$

Example 5.7.2 Consider the region R bounded by $y = \sin x,\ y = 0,\ x = 0$ and $x = \pi$. Find the volume generated when R rotated about

(i) x-axis (ii) y-axis (iii) $y = -2$ (iv) $y = 1$
(v) $x = \pi$ (vi) $x = 2\pi$.

(i) By Theorem 5.7.2, the volume V is given by

$$\begin{aligned} V &= \int_0^\pi \pi \sin^2 x\ dx \\ &= \pi \cdot \left[\frac{1}{2}\ (x - \sin x \cos x)\right]_0^\pi \\ &= \frac{\pi^2}{2}. \end{aligned}$$

graph

(ii) By Theorem 5.7.3, the volume V is given by (integrating by parts)

$$\begin{aligned} V &= \int_0^\pi 2\pi x \sin x\ dx \quad ; \quad (u = x,\ dv = \sin x\ dx) \\ &= 2\pi[-x \cos x + \sin x]_0^\pi \\ &= 2\pi[\pi] \\ &= 2\pi^2. \end{aligned}$$

graph

(iii) In this case, the volume V is given by

$$\begin{aligned} V &= \int_0^{\pi} \pi(\sin x + 2)^2 \ dx \\ &= \int_0^{\pi} \pi[\sin^2 x + 4\sin x + 4] \ dx \\ &= \pi\left[\frac{1}{2}\ (x - \sin x\cos x) - 4\cos x + 4x\right]_0^{\pi} \\ &= \pi\left[\frac{1}{2}\ \pi + 8 + 4\pi\right] \\ &= \frac{9}{2}\pi^2 + 8\pi. \end{aligned}$$

graph

(iv) In this case,

$$V = \int_0^{\pi} \pi[1^2 - (1 - \sin x)^2] \ dx.$$

graph

$$\begin{aligned}
V &= \int_0^\pi \pi[1 - 1 + 2\sin x - sin^2 x]dx \\
&= \pi\left[-2\cos x - \frac{1}{2}\ (x - \sin x \cos x)\right]_0^\pi \\
&= \pi\left[4 - \frac{1}{2}\ (\pi)\right] \\
&= \frac{\pi(8-\pi)}{2}.
\end{aligned}$$

(v)

$$\begin{aligned}
V &= \int_0^\pi (2\pi(\pi - x)\sin x]\ dx \\
&= 2\pi \int_0^\pi [\pi \sin x - x \sin x]\ dx \\
&= 2\pi[-\pi\cos x + x\cos x - \sin x]_0^\pi \\
&= 2\pi[2\pi - \pi] \\
&= 2\pi^2.
\end{aligned}$$

graph

(vi)

$$\begin{aligned}
V &= \int_0^\pi 2\pi(2\pi - x)\sin x\ dx \\
&= 2\pi[-2\pi\cos x + x\cos x - \sin x]_0^\pi \\
&= 2\pi[4\pi - \pi] \\
&= 6\pi^2.
\end{aligned}$$

graph

Example 5.7.3 Consider the region R bounded by the circle $(x-4)^2+y^2 = 4$. Compute the volume V generated when R is rotated around

(i) $y = 0$ (ii) $x = 0$ (iii) $x = 2$

graph

(i) Since the area crosses the x-axis, it is sufficient to rotate the top half to get the required solid.

$$V = \int_2^6 \pi y^2 \, dx = \pi \int_2^6 [4-(x-4)^2] \, dx$$
$$= \pi \left[4x - \frac{1}{3}\,(x-4)^3\right]_2^6 = \pi \left[16 - \frac{8}{3} - \frac{8}{3}\right] = \frac{32}{3}\pi.$$

This is the volume of a sphere of radius 2.

(ii) In this case,

$$V = \int_2^6 2\pi x(2y) \, dx = 4\pi \int_2^6 x[\sqrt{4-(x-4)^2}]dx \; ; x - 4 = 2\sin t$$
$$dx = 2\cos t dt$$
$$= 4\pi \int_{-\pi/2}^{\pi/2} (4+2\sin t)(2\cos t)(2\cos t)dt$$
$$= 4\pi \int_{-\pi/2}^{\pi/2} (16\cos^2 t + 8\cos^2 t \sin t) \, dx$$
$$= 4\pi \left[16 \cdot \frac{1}{2}\,(t+\sin t\cos t) - \frac{8}{3}\,\cos^3 t\right]_{-\pi/2}^{\pi/2}$$
$$= 4\pi[8(\pi)]$$
$$= 32\pi^2$$

(iii) In this case,

$$\begin{aligned}
V &= \int_2^6 2\pi(x-2)2y\ dx \\
&= 4\pi \int_2^6 (x-2)\sqrt{4-(x-4)^2}\ dx\ ; x-4 = 2\sin t \\
&\qquad\qquad\qquad\qquad\qquad\qquad dx = 2\cos t dt \\
&= 4\pi \int_{-\pi/2}^{\pi/2} (2+2\sin t)(2\cos t)(2\cos t)dt \\
&= 4\pi \int_{-\pi/2}^{\pi/2} (8\cos^2 t + 8\cos^2 t \sin t)dt \\
&= 4\pi \left[4(t+\sin t\cos t) - \frac{8}{3}\ \cos^3 t\right]_{-\pi/2}^{\pi/2} \\
&= 4\pi[4\pi] \\
&= 16\pi^2
\end{aligned}$$

Exercises 5.7

1. Consider the region R bounded by $y = x$ and $y = x^2$. Find the volume generated when R is rotated around the line with equation

   (i) $x = 0$ (ii) $y = 0$ (iii) $y = 1$ (iv) $x = 1$
   (v) $x = 4$ (vi) $x = -1$ (vii) $y = -1$ (viii) $y = 2$

2. Consider the region R bounded by $y = \sin x,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{2}$. Find the volume generated when R is rotated about the line with equation

   (i) $x = 0$ (ii) $y = 0$ (iii) $y = 1$ (iv) $x = \frac{\pi}{2}$

3. Consider the region R bounded by $y = e^x,\ x = 0,\ x = \ln 2,\ y = 0$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = \ln 2$ (iv) $y = -2$
   (v) $y = 2$ (iv) $x = 2$

4. Consider the region R bounded by $y = \ln x,\ y = 0,\ x = 1,\ x = e$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = 1$ (v) $x = e$
   (v) $y = 1$ (vi) $y = -1$

5. Consider the region R bounded by $y = \cosh x,\ y = 0,\ x = -1,\ x = 1$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 2$ (iii) $x = 1$ (iv) $y = -1$
   (v) $y = 6$ (vi) $x = 0$

6. Consider the region R bounded by $y = x,\ y = x^3$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = -1$ (iv) $x = 1$
   (v) $y = 1$ (vi) $y = -1$

7. Consider the region R bounded by $y = x^2,\ y = 8 - x^2$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $y = -4$ (iv) $y = 8$
   (v) $x = -2$ (vi) $x = 2$

8. Consider the region R bounded by $y = \sinh x,\ y = 0,\ x = 0,\ x = 2$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = 2$ (iv) $x = -2$
   (v) $y = -1$ (vi) $y = 10$

9. Consider the region R bounded by $y = \sqrt{x},\ y = 4,\ x = 0$. Find the volume generated when R is rotated about the line with equation

   (i) $y = 0$ (ii) $x = 0$ (iii) $x = 16$ (iv) $y = 4$

10. Compute the volume of a cone with height h and radius r.

5.8 Arc Length and Surface Area

The Riemann integral is useful in computing the length of arcs. Let f and f' be continuous on $[a, b]$. Let C denote the arc

$$C = \{(x, f(x)) : a \leq x \leq b\}.$$

Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of $[a, b]$. For each $i = 1, 2, \ldots, n$, let

graph

$$\begin{aligned}
\Delta x_i &= x_i - x_{i-1} \\
\Delta y_i &= f(x_i) - f(x_{i-1}) \\
\Delta s_i &= \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_i)^2} \\
||\Delta|| &= \max_{1 \leq i \leq n} \{\Delta x_n\}.
\end{aligned}$$

Then Δs_i is the length of the line segment joining the two points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$. Let

$$A(P) = \sum_{i=1}^{n} \Delta s_i.$$

Then $A(P)$ is called the polygonal approximation of C with respect to the portion P.

Definition 5.8.1 Let $C = \{(x, f(x)) : x \in [a, b]\}$ where f and f' are continuous on $[a, b]$. Then the arc length L of the arc C is defined by

$$L = \lim_{||\Delta|| \to 0} Ap = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_{i-1})^2}.$$

Theorem 5.8.1 *The arc length L defined in Definition 5.8.1 is given by*

$$L = \int_a^b \sqrt{(f'(x))^2 + 1}\ dx.$$

Proof. By the Mean Value Theorem, for each $i = 1, 2, \ldots, n$,

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$$

for some c_i such that $x_{i-1} < c_i < x_i$. Therefore, each polynomial approximation Ap is a Riemann Sum of the continuous function

$$\sqrt{(f'(x))^2 + 1}$$

$$A(P) = \sum_{i=1}^{n} \sqrt{(f'(c_i))^2 + 1}\ \Delta x_i,$$

for some c_i such that $x_{i-1} < c_i < x_i$.

By the definition of the Riemann integral, we get

$$L = \int_a^b \sqrt{(f'(x))^2 + 1}\ dx.$$

Example 5.8.1 Let $C = \{(x, \cosh x) : 0 \le x \le 2\}$. Then the arc length L of C is given by

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \sinh^2 x}\ dx \\ &= \int_0^2 \cosh x\ dx \\ &= [\sinh x]_0^2 \\ &= \sinh 2. \end{aligned}$$

Example 5.8.2 Let $C = \left\{\left(x, \frac{2}{3}\, x^{3/2}\right) : 0 \le x \le 4\right\}$. Then the arc length L of the curve C is given by

$$\begin{aligned} L &= \int_0^4 \sqrt{1 + \left(\frac{2}{3} \cdot \frac{3}{2}\, x^{1/2}\right)^2}\ dx \\ &= \int_0^4 (1 + x)^{1/2}\ dx \\ &= \left[\frac{2}{3}\, (1 + x)^{3/2}\right]_0^4 \\ &= \frac{2}{3}\, [5\sqrt{5} - 1]. \end{aligned}$$

Definition 5.8.2 Let C be defined as in Definition 5.8.1.

(i) The surface area S_x generated by rotating C about the x-axis is given by

$$S_x = \int_a^b 2\pi |f(x)| \sqrt{(f'(x))^2 + 1}\ dx.$$

(ii) The surface area S_y generated by rotating C about the y-axis

$$S_y = \int_a^b 2\pi |x| \sqrt{(f'(x))^2 + 1}\ dx.$$

Example 5.8.3 Let $C = \{(x, \cosh x) : 0 \leq x \leq 4\}$.

(i) Then the surface area S_x generated by rotating C around the x-axis is given by

$$\begin{aligned} S_x &= \int_0^4 2\pi \cosh x \sqrt{1 + \sinh^2 x}\ dx \\ &= 2\pi \int_0^4 \cosh^2 x\ dx \\ &= 2\pi \left[\frac{1}{2}\ (x + \sinh x \cosh x) \right]_0^4 \\ &= \pi[4 + \sinh 4 \cosh 4]. \end{aligned}$$

(ii) The surface area S_y generated by rotating the curve C about the y-axis is given by

$$\begin{aligned} S_y &= \int_0^4 2\pi x \sqrt{1 + \sinh^2 x}\ dx \\ &= 2\pi \int_0^4 x \cosh x\ dx\ ;\ (u = x,\ dv = \cosh x\ dx) \\ &= 2\pi [x \sinh x - \cosh x]_0^4 \\ &= 2\pi [4 \sinh 4 - \cosh 4 + 1] \end{aligned}$$

Theorem 5.8.2 *Let $C = \{(x(t), y(t)) : a \leq t \leq b\}$. Suppose that $x'(t)$ and $y'(t)$ are continuous on $[a, b]$.*

(i) The arc length L of C is given by

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt.$$

(ii) The surface area S_x generated by rotating C about the x-axis is given by

$$S_x = \int_a^b 2\pi|y(t)|\sqrt{(x'(t))^2 + (y'(t))^2}\ dt.$$

(iii) The surface area S_y generated by rotating C about the y-axis is given by

$$S_y = \int_a^b 2\pi|x(t)|\sqrt{(x'(t))^2 + (y'(t))^2}\ dt.$$

Proof. The proof of this theorem is left as an exercise.

Example 5.8.4 Let $C = \left\{(e^t \sin t, e^t \cos t) : 0 \leq t \leq \dfrac{\pi}{2}\right\}$. Then

$$\begin{aligned} ds &= \sqrt{(x'(t))^2 + (y'(t))^2}\ dt \\ &= \sqrt{(e^t(\sin t + \cos t))^2 + (e^t(\cos t - \sin t))^2}\ dt \\ &= \{e^{2t}(\sin^2 t + \cos^2 t + 2\sin t\cos t + \cos^2 t + \sin^2 t - 2\cos t \sin t)\}^{1/2}\ dt \\ &= e^t\sqrt{2}\ dt. \end{aligned}$$

(i) The arc length L of C is given by

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{2}e^t dt \\ &= \sqrt{2}\left[e^t\right]_0^{\pi/2} \\ &= \sqrt{2}\left(e^{\pi/2} - 1\right). \end{aligned}$$

(ii) The surface area S_x obtained by rotating C about the x-axis is given by

$$\begin{aligned} S_x &= \int_0^{\pi/2} 2\pi(e^t \cos t)(\sqrt{2}e^t dt) \\ &= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \cos t dt \\ &= 2\sqrt{2}\pi \left[\frac{e^{2t}}{5} (2\cos t + \sin t) \right]_0^{\pi/2} \\ &= 2\sqrt{2}\pi \left[\frac{e^{\pi}}{5} (1) - \frac{2}{5} \right] \\ &= \frac{2\sqrt{2}\pi}{5} (e^{\pi} - 2). \end{aligned}$$

(iii) The surface area S_y obtained by rotating C about the y-axis is given by

$$\begin{aligned} S_y &= \int_0^{\pi/2} 2\pi(e^t \sin t)(\sqrt{2}e^t dt) \\ &= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \sin t dt \\ &= 2\sqrt{2}\pi \left[\frac{e^{2t}}{5} [2\sin t - \cos t] \right]_0^{\pi/2} \\ &= 2\sqrt{2}\pi \left[\frac{2e^{\pi}}{5} + \frac{1}{5} \right] = \frac{2\sqrt{2}\pi}{5} (2e^{\pi} + 1). \end{aligned}$$

Exercises 5.8 Find the arc lengths of the following curves:

1. $y = x^{3/2},\ 0 \le x \le 4$

2. $y = \frac{1}{3} (x^2 + 2)^{3/2},\ 0 \le x \le 1$

3. $C = \left\{ (4(\cos t + t\sin t),\ 4(\sin t - t\cos t)) : 0 \le t \le \frac{\pi}{2} \right\}$

4. $x(t) = a(\cos t + t\sin t),\ y(t) = a(\sin t - t\cos t),\ 0 \le t \le \frac{\pi}{2}$

5. $x(t) = \cos^3 t, y(t) = \sin^3 t,\ 0 \le t \le \pi/2$

6. $y = \dfrac{1}{2}\,x^2,\ 0 \le t \le 1$

7. $x(t) = t^3,\ y(t) = t^2,\ 0 \le t \le 1$

8. $x(t) = 1 - \cos t,\ y(t) = t - \sin t,\ 0 \le t \le 2\pi$

9. In each of the curves in exercises 1-8, set up the integral that represents the surface area generated when the given curve is rotated about

   (a) the x-axis

   (b) the y-axis

10. Let $C = \{(x, \cosh x) : -1 \le x \le 1\}$

    (a) Find the length of C.

    (b) Find the surface area when C is rotated around the x-axis.

    (c) Find the surface area when C is rotated around the y-axis.

In exercises 11–20, consider the given curve C and the numbers a and b. Determine the integral that represents:

(a) Arc length of C

(b) Surface area when C is rotated around the x-axis.

(c) Surface area when C is rotated around the y-axis.

(d) Surface area when C is rotated around the line $x = a$.

(e) Surface area when C is rotated around the line $y = b$.

11. $C = \{(x, \sin x) : 0 \le x \le \pi\};\ a = \pi, b = 1$

12. $C = \left\{(x, \cos x) : 0 \le x \le \dfrac{\pi}{3}\right\};\ a = \pi,\ b = 2$

13. $C = \{(t, \ln t) : 1 \le t \le e\};\ a = 4,\ b = 3$

14. $C = \{(2 + \cos t, \sin t) : 0 \le t \le \pi\};\ a = 4,\ b = -2$

15. $C = \left\{(t, \ln \sec t) : 0 \le t \le \dfrac{\pi}{3}\right\};\ a = \pi,\ b = -3$

16. $C = \{(2x, \cosh 2x) : 0 \le x \le 1\};\ a = -2,\ b = \dfrac{1}{2}$

17. $C = \left\{ (\cos t, 3 + \sin t) : -\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2} \right\};\ a = 2,\ b = 5$

18. $C = \left\{ (e^t \sin 2t, e^t \cos 2t) : 0 \leq t \leq \dfrac{\pi}{4} \right\};\ a = -1,\ b = 3$

19. $C = \{ (e^{-t}, e^t) : 0 \leq t \leq \ln 2 \};\ a = -1,\ b = -4$

20. $C = \{ (4^{-t}, 4^t) : 0 \leq t \leq 1 \};\ a = -2,\ b = -3$

Chapter 6

Techniques of Integration

6.1 Integration by formulae

There exist many books that contain extensive lists of integration, differentiation and other mathematical formulae. For our purpose we will use the list given below.

1. $\int af(u)du = a\int f(u)du$

2. $\int \left(\sum_{i=1}^{n} a_i f_i(u)\right) du = \sum_{i=1}^{n}\left(\int a_i f_i(u)du\right)$

3. $\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

4. $\int u^{-1} du = \ln|u| + C$

5. $\int e^{au} du = \frac{e6au}{a} + C$

6. $\int a^{bu} du = \frac{a^{bu}}{b\ln a} + C, \ a > 0, \ a \neq 1$

7. $\int \ln|u| du = u\ln|u| - u + C$

8. $\displaystyle\int \sin(au)du = \frac{-\cos(au)}{a} + C$

9. $\displaystyle\int \cos(au)du = \frac{\sin(au)}{a} + C$

10. $\displaystyle\int \tan(au)du = \frac{\ln|\sec(au)|}{a} + C$

11. $\displaystyle\int \cot(au)du = \frac{\ln|\sin(au)|}{a} + C$

12. $\displaystyle\int \sec(au)du = \frac{\ln|\sec(au) + \tan(au)|}{a} + C$

13. $\displaystyle\int \csc(au)du = \frac{\ln|\csc(au) - \cot(au)|}{a} + C$

14. $\displaystyle\int \sinh(au)du = \frac{\cosh(au)}{a} + C$

15. $\displaystyle\int \cosh(au)du = \frac{\sinh(au)}{a} + C$

16. $\displaystyle\int \tanh(au)du = \frac{\ln|\cosh(au)|}{a} + C$

17. $\displaystyle\int \coth(au)du = \frac{\ln|\sinh(au)|}{a} + C$

18. $\displaystyle\int \operatorname{sech}(au)du = \frac{2}{a}\ \arctan(e^{au}) + C$

19. $\displaystyle\int \operatorname{csch}(au)\ du = \frac{2}{a}\ \operatorname{arctanh}(e^{au}) + C$

20. $\displaystyle\int \sin^2(au)du = \frac{u}{2} - \frac{\sin(au)\cos(au)}{2a} + C$

21. $\displaystyle\int \cos^2(au)du = \frac{u}{2} + \frac{\sin(au)\cos(au)}{2a} + C$

22. $\displaystyle\int \tan^2(au)du = \frac{\tan(au)}{a} - u + C$

23. $\displaystyle\int \cot^2(au)du = -\frac{\cot(au)}{a} - u + C$

24. $\displaystyle\int \sec^2(au)du = \frac{\tan(au)}{a} + C$

25. $\displaystyle\int \csc^2(au)du = -\frac{\cot(au)}{a} + C$

26. $\displaystyle\int \sinh^2(au)du = -\frac{u}{2} + \frac{\sinh(2au)}{4a} + C$

27. $\displaystyle\int \cosh^2(au)du = \frac{u}{2} + \frac{\sinh(2au)}{4a} + C$

28. $\displaystyle\int \tanh^2(au)du = u - \frac{\tanh(au)}{a} + C$

29. $\displaystyle\int \coth^2(au)du = u - \frac{\coth(au)}{a} + C$

30. $\displaystyle\int \operatorname{sech}^2(au)du = \frac{\tanh(au)}{a} + C$

31. $\displaystyle\int \operatorname{csch}^2(au)du = \frac{-\coth(au)}{a} + C$

32. $\displaystyle\int \sec(au)\tan(au)du = \frac{\sec(au)}{a} + C$

33. $\displaystyle\int \csc(au)\cot(au)du = -\frac{\csc(au)}{a} + C$

34. $\displaystyle\int \operatorname{sech}(au)\tanh(au)du = -\frac{\operatorname{sech}(au)}{a} + C$

35. $\displaystyle\int \operatorname{csch}(au)\coth(au)du = -\frac{\operatorname{csch}(au)}{a} + C$

36. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + C$

37. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{a}\ \operatorname{arctanh}\left(\frac{u}{a}\right) + C = \frac{1}{2a}\ \ln\left|\frac{a+u}{a-u}\right| + C$

38. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \text{arcsinh}\left(\frac{u}{a}\right) + C$

39. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C,\ |a| > |u|$

40. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \text{arccosh}\left(\frac{u}{a}\right) + C,\ |u| > |a|$

41. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ \text{arcsec}\left(\frac{u}{a}\right) + C,\ |u| > |a|$

42. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ \text{arcsech}\left(\frac{u}{a}\right) + C,\ |a| > |u|$

43. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ \text{arccsch}\left(\frac{u}{a}\right) + C$

44. $\displaystyle\int \frac{u\ du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

45. $\displaystyle\int \frac{u\ du}{a^2-u^2} = -\ln\sqrt{a^2-u^2} + C,\ |a| > |u|$

46. $\displaystyle\int \frac{u\ du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + C$

47. $\displaystyle\int \frac{u\ du}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2} + C,\ |a| > |u|$

48. $\displaystyle\int \frac{u\ du}{\sqrt{u^2-a^2}} = \sqrt{u^2-a^2} + C,\ |u| > |a|$

49. $\displaystyle\int \arcsin(au)du = u\arcsin(au) + \frac{1}{a}\ \sqrt{1-a^2u^2} + C,\ |a||u| < 1$

50. $\displaystyle\int \arccos(au)du = u\arccos(au) - \frac{1}{a}\ \sqrt{1-a^2u^2} + C,\ |a||u| < 1$

51. $\displaystyle\int \arctan(au)du = u\arctan(au) - \frac{1}{2a}\ \ln(1+a^2u^2) + C$

52. $\displaystyle\int \text{arccot}\,(au)du = u\text{arccot}\,(au) + \frac{1}{2a}\ \ln(1+a^2u^2) + C$

53. $\int \operatorname{arcsec}(au)du = u\operatorname{arcsec}(au) - \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2 - 1}\right| + C,\ au > 1$

54. $\int \operatorname{arccsc}(au)du = u\operatorname{arccsc}(au) + \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2 - 1}\right| + C,\ au > 1$

55. $\int \operatorname{arcsinh}(au)du = u\operatorname{arcsinh}(au) - \frac{1}{a}\ \sqrt{1 + a^2u^2} + C$

56. $\int \operatorname{arccosh}(au)du = u\operatorname{arccosh}(au) - \frac{1}{a}\ \sqrt{-1 + a^2u^2} + C,\ |a||u| > 1$

57. $\int \operatorname{arctanh}(au)du = u\operatorname{arctanh}(au) + \frac{1}{2a}\ \ln(-1 + a^2u^2) + C,\ |a||u| \neq 1$

58. $\int \operatorname{arccoth}(au)du = u\operatorname{arccoth}(au) + \frac{1}{2a}\ \ln(-1 + a^2u^2) + C,\ |a||u| \neq 1$

59. $\int \operatorname{arcsech}(au)du = u\operatorname{arcsech}(au) + \frac{1}{a}\ \arcsin(au) + C,\ |a||u| < 1$

60. $\int \operatorname{arccsch}(au)du = u\operatorname{arccsch}(au) + \frac{1}{a}\ \ln\left|au + \sqrt{a^2u^2 + 1}\right| + C$

61. $\int e^{au}\sin(bu)du = \frac{e^{au}[a\sin(bu) - b\cos(bu)]}{a^2 + b^2} + C$

62. $\int e^{au}\cos(bu)du = \frac{e^{au}[a\cos(bu) + b\sin(bu)]}{a^2 + b^2} + C$

63. $\int \sin^n(u)du = \frac{-1}{n}\ \left[\sin^{n-1}(u)\cos(u)\right] + \frac{n-1}{n}\ \int \sin^{n-2}(u)du$

64. $\int \cos^n(u)du = \frac{1}{n}\ \left[\cos^{n-1}(u)\sin(u)\right] + \frac{n-1}{n}\int \cos^{n-2}(u)du$

65. $\int \tan^n(u)du = \frac{\tan^{n-1}(u)}{n-1} - \int \tan^{n-2}(u)du$

66. $\int \cot^n(u)du = -\frac{\cot^{n-1}(u)}{n-1} - \int \cot^{n-2}(u)du$

67. $\int \sec^n(u)du = \frac{1}{n-1}\ \left[\sec^{n-2}(u)\tan(u)\right] + \frac{n-2}{n-1}\int \sec^{n-2}(u)du$

68. $\displaystyle\int \csc^n(u)du = \frac{-1}{n-1}\left[\csc^{n-2}(u)\cot(u)\right] + \frac{n-2}{n-1}\int \csc^{n-2}(u)du$

69. $\displaystyle\int \sin(mu)sin(nu)du = \frac{\sin[(m-n)u]}{2(m-n)} - \frac{\sin[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

70. $\displaystyle\int \cos(mu)\cos(nu)du = \frac{\sin[(m-n)u]}{2(m-n)} + \frac{\sin[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

71. $\displaystyle\int \sin(mu)\cos(nu)du = \frac{\cos[(m-n)u]}{2(m-n)} - \frac{\cos[(m+n)u]}{2(m+n)} + C,\ m^2 \neq n^2$

Exercises 6.1

1. Define the statement that $g(x)$ is an antiderivative of $f(x)$ on the closed interval $[a, b]$

2. Prove that if $g(x)$ and $h(x)$ are any two antiderivatives of $f(x)$ on $[a, b]$, then there exists some constant C such that $g(x) = \ln(x) + C$ for all x on $[a, b]$.

In problems 3–30, evaluate each of the indefinite integrals.

3. $\displaystyle\int x^5 dx$

4. $\displaystyle\int \frac{4}{x^3}\,dx$

5. $\displaystyle\int x^{-3/5}dx$

6. $\displaystyle\int 3x^{2/3}dx$

7. $\displaystyle\int \frac{2}{\sqrt{x}}\,dx$

8. $\displaystyle\int t^2\sqrt{t}\ dt$

9. $\displaystyle\int \left(t^{-1/2} + t^{3/2}\right)dt$

10. $\displaystyle\int (1+x^2)^2 dx$

11. $\displaystyle\int t^2(1+t)^2 dt$

12. $\displaystyle\int (1+t^2)(1-t^2)dt$

13. $\displaystyle\int \left(\frac{1}{t^{1/2}} + \sin t\right)dt$

14. $\displaystyle\int (2\sin t + 3\cos t)dt$

15. $\displaystyle\int 3\sec^2 t\ dt$

16. $\displaystyle\int 2\csc^2 x\ dx$

17. $\displaystyle\int 4\sec t\tan t\ dt$

18. $\displaystyle\int 2\csc t\cot t\ dt$

19. $\displaystyle\int \sec t(\sec t + \tan t)dt$

20. $\int \csc t(\csc t - \cot t)dt$

21. $\int \frac{\sin x}{\cos^2 x}\, dx$

22. $\int \frac{\cos x}{\sin^2 x}\, dx$

23. $\int \frac{\sin^3 t - 3}{\sin^2 t}\, dt$

24. $\int \frac{\cos^3 t + 2}{\cos^2 t}\, dt$

25. $\int \tan^2 t\ dt$

26. $\int \cot^2 t\ dt$

27. $\int (2\sec^2 t + 1)dt$

28. $\int \frac{2}{t}\, dt$

29. $\int \sinh t\ dt$

30. $\int \cosh t\ dt$

31. Determine $f(x)$ if $f'(x) = \cos x$ and $f(0) = 2$.

32. Determine $f(x)$ if $f''(x) = \sin x$ and $f(0) = 1,\ f'(0) = 2$.

33. Determine $f(x)$ if $f''(x) = \sinh x$ and $f(0) = 2,\ f'(0) = -3$.

34. Prove each of the integration formulas 1–77.

6.2 Integration by Substitution

Theorem 6.2.1 *Let $f(x), g(x), f(g(x))$ and $g'(x)$ be continuous on an interval $[a, b]$. Suppose that $F'(u) = f(u)$ where $u = g(x)$. Then*

(i) $\int f(g(x))g'(x)dx = \int f(u)du = F(g(x)) + C$

(ii) $\int_a^b f(g(x))g'(x)dx = \int_{u=g(u)}^{u=g(b)} f(u)du = F(g(b)) - F(g(a))$.

Proof. See the proof of Theorem 5.3.1.

Exercises 6.2 In problems 1–39, evaluate the integral by making the given substitution.

1. $\displaystyle\int 3x(x^2+1)^{10}dx,\ u=x^2+1$

2. $\displaystyle\int x\sin(1+x^2)dx,\ u=1+x^2$

3. $\displaystyle\int \frac{\cos(\sqrt{t})}{\sqrt{t}}\,dt,\ x=\sqrt{t}$

4. $\displaystyle\int \frac{3x^2}{(1+x^3)^{3/2}}\,dx,\ u=1+x^3$

5. $\displaystyle\int \frac{2e^{\arcsin x}}{\sqrt{1-x^2}}\,dx,\ u=\arcsin x$

6. $\displaystyle\int \frac{3e^{\arccos x}}{\sqrt{1-x^2}}\,dx$

7. $\displaystyle\int x\ 4^{x^2}dx,\ u=4^{x^2}$

8. $\displaystyle\int 10^{\sin x}\cos x\ dx,\ u=\sin x$

9. $\displaystyle\int \frac{4^{\arctan x}}{1+x^2}\,dx,\ u=4^{\arctan x}$

10. $\displaystyle\int \frac{(1+\ln x)^{10}}{x}\,dx,\ u=1+\ln x$

11. $\displaystyle\int \frac{5^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\,dx,\ u=\operatorname{arcsec} x$

12. $\displaystyle\int (\tan 2x)^3\sec^2 2x\ dx,\ u=\tan 2x$

13. $\displaystyle\int (\cot 3x)^5\csc^2 3x\ dx,\ u\ =\cot 3x$

14. $\displaystyle\int \sin^{21} x\cos x\ dx,\ u=\sin x$

15. $\displaystyle\int \cos^5 x\sin x\ dx,\ u=\cos x$

16. $\displaystyle\int (1+\sin x)^{10}\cos x\ dx,\ u=1+\sin x$

17. $\displaystyle\int \sin^3 x\ dx,\ u=\cos x$

18. $\displaystyle\int \cos^3 x\ dx,\ u=\sin x$

19. $\displaystyle\int \tan^3 x\ dx,\ u=\tan x$

20. $\displaystyle\int \cot^3 x\ dx,\ u=\cot x$

21. $\displaystyle\int \sec^4 x\ dx,\ u=\tan x$

22. $\displaystyle\int \csc^4 x\ dx,\ u=\cot x$

23. $\displaystyle\int \sin^3 x\cos^3 x\ dx,\ u=\sin x$

24. $\displaystyle\int \sin^3 x\cos^3 x\ dx,\ u=\cos x$

25. $\displaystyle\int \tan^4 x\ dx,\ u=\tan x$

26. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx,\ u=\ln x$

27. $\displaystyle\int \frac{x\cos(\ln(1+x^2))}{1+x^2}\,dx,\ u=\ln(1+x)^2$

28. $\displaystyle\int \tan^3 x\sec^4 x\ dx,\ u=\sec x$

29. $\displaystyle\int \cot^3 x\csc^4 x\ dx,\ u=\csc x$

30. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}, x=2\sin t$

31. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}, x=3\cos t$

32. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}, x=2\sinh t$

33. $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}, x=3\cosh t$

34. $\displaystyle\int \frac{dx}{4+x^2}, x=2\tan t$

35. $\displaystyle\int \frac{dx}{4-x^2}, x=2\tanh t$

36. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}, x=2\sec t$

37. $\displaystyle\int 4e^{\sin(3x)}\cos(3x)dx,\ u=\sin 3x$

38. $\displaystyle\int x\ 3^{(x^2+4)}dx,\ u=3^{x^2+4}$

39. $\displaystyle\int 3\ e^{\tan 2x}\sec^2 x\ dx,\ u=\tan 2x$

40. $\displaystyle\int x\sqrt{x+2}\ dx,\ u=x+2$

Evaluate the following definite integrals.

41. $\displaystyle\int_0^1 (x+1)^{30}dx$

42. $\displaystyle\int_1^2 x(4-x^2)^{1/2}dx$

43. $\displaystyle\int_0^{\pi/4} \tan^3 x\sec^2 x\ dx$

44. $\displaystyle\int_0^1 x^3(x^2+1)^3dx$

45. $\displaystyle\int_0^2 (x+1)(x-2)^{10}dx$

46. $\displaystyle\int_0^8 x^2(1+x)^{1/2}dx$

47. $\displaystyle\int_0^{\pi/6} \sin(3x)dx$

48. $\displaystyle\int_0^{\pi/4} \cos(2x)dx$

49. $\displaystyle\int_0^{\pi/4} \sin^3 2x\cos 2x\ dx$

50. $\displaystyle\int_0^{\pi/6} \cos^4 3x\sin 3x\ dx$

51. $\displaystyle\int_0^1 \frac{e^{\arctan x}}{1+x^2}\ dx$

52. $\displaystyle\int_0^{1/2} \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\ dx$

53. $\displaystyle\int_2^3 \frac{e^{\operatorname{arcsec} x}}{x\sqrt{x^2-1}}\,dx$

54. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

6.3 Integration by Parts

Theorem 6.3.1 *Let $f(x), g(x), f'(x)$ and $g'(x)$ be continuous on an interval $[a,b]$. Then*

(i) $\displaystyle\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

(ii) $\displaystyle\int_a^b f(x)g'(x)dx = (f(b)g(b) - f(a)g(a)) - \int_a^b g(x)f'(x)dx$

(iii) $\displaystyle\int udv = uv - \int vdu$

where $u = f(x)$ and $dv = g'(x)dx$ are the parts of the integrand.

Proof. See the proof of Theorem 5.4.1.

Exercises 6.3 Evaluate each of the following integrals.

1. $\displaystyle\int x\sin x\ dx$

2. $\displaystyle\int x\cos x\ dx$

3. $\displaystyle\int x\ln x\ dx$

4. $\displaystyle\int x\ e^x\ dx$

5. $\displaystyle\int x\ 4^x\ dx$

6. $\displaystyle\int x^2\ln x\ dx$

7. $\displaystyle\int x^2\sin x\ dx$

8. $\displaystyle\int x^2\cos x\ dx$

9. $\displaystyle\int x^2e^x\ dx$

10. $\displaystyle\int x^2\ 10^x\ dx$

11. $\int e^x \sin x \; dx$ (Let $u = e^x$ twice and solve.)

12. $\int e^x \cos x \; dx$ (Let $u = e^x$ twice and solve.)

13. $\int e^{2x} \sin 3x \; dx$ (Let $u = e^{2x}$ twice and solve.)

14. $\int x \sin(3x) dx$

15. $\int x^2 \cos(2x) dx$

16. $\int x^2 e^{4x} dx$

17. $\int x^3 \ln(2x) dx$

18. $\int x \sec^2 x \; dx$

19. $\int x \csc^2 x \; dx$

20. $\int x \sinh(4x) dx$

21. $\int x^2 \cosh x \; dx$

22. $\int x \cos(5x) dx$

23. $\int \sin(\ln x) dx$

24. $\int \cos(\ln x) dx$

25. $\int x \arcsin x \; dx$

26. $\int x \arccos x \; dx$

27. $\int x \arctan x \; dx$

28. $\int x \; \text{arcsec}\, x \; dx$

29. $\int \arcsin x \; dx$

30. $\int \arccos x \; dx$

31. $\int \arctan x \; dx$

32. $\int \text{arcsec}\, x \; dx$

Verify the following integration formulas:

33. $\int \sin^n(ax)dx = -\frac{\sin^{n-1}(ax)\cos(ax)}{na} + \frac{n-1}{n}\int (\sin^{n-2} ax)dx$

34. $\int \cos^n(ax)dx = \frac{1}{na}\ \cos^{n-1}(ax)\sin(ax) + \frac{n-1}{n}\int (\cos^{n-2} ax)dx$

35. $\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

36. $\int x^n \sin x\ dx = -x^n \cos x + n\int x^{n-1}\cos x\ dx$

37. $\int x^n \cos x\ dx = x^n \sin x - n\int x^{n-1}\sin x\ dx$

38. $\int e^{ax}\sin(bx)dx = \frac{1}{a^2+b^2}\ e^{ax}[a\sin(bx) - b\cos(bx)] + C$

39. $\int e^{ax}\cos(bx)\ dx = \frac{1}{a^2+b^2}\ e^{ax}[a\cos(bx) + b\sin(bx)] + C$

40. $\int x^n \ln x\ dx = \frac{1}{n+1}\ x^{n+1}\ln x - \frac{1}{(n+1)^2}\ x^{n+1} + C,\ n \neq -1,\ x > 0$

41. $\int \sec^n x\ dx = \frac{1}{n-1}\ \sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx,\ n \neq 1,\ n > 0$

42. $\int \csc^n x\ dx = \frac{-1}{n-1}\ \csc^{n-2} x\cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\ dx,\ n \neq 1, n > 0$

Use the formulas 33–42 to evaluate the following integrals:

43. $\int \sin^4 x\ dx$

44. $\int \cos^5 x\ dx$

45. $\int x^3 e^x dx$

46. $\int x^4 \sin x\ dx$

47. $\int x^3 \cos x\ dx$

48. $\int e^{2x}\sin 3x\ dx$

49. $\int e^{3x}\cos 2x\ dx$

50. $\int x^5 \ln x\ dx$

51. $\int \sec^3 x \; dx$

52. $\int \csc^3 x \; dx$

Prove each of the following formulas:

53. $\int \tan^n x \; dx = \frac{1}{n-1} \; \tan^{n-1} x - \int \tan^{n-2} x \; dx, \; n \neq 1$

54. $\int \cot^n x \; dx = \frac{1}{n-1} \; \cot^{n-1} x - \int \cot^{n-2} x \; dx, \; n \neq 1$

55. $\int \sin^{2n+1} x \; dx = -\int (1-u^2)^n du, \; u = \cos x$

56. $\int \cos^{2n+1} x \; dx = -\int (1-u^2)^n du, \; u = \sin x$

57. $\int \sin^{2n+1} x \; \cos^m x \; dx = -\int (1-u^2)^n u^m du, \; u = \cos x$

58. $\int \cos^{2n+1} x \; \sin^m x \; dx = \int (1-u^2)^n u^m du, \; u = \sin x$

59. $\int \sin^{2n} x \; \cos^{2m} x \; dx = \int (\sin x)^{2n} (1-\sin^2 x)^m dx$

60. $\int \tan^n x \; \sec^{2m} x \; dx = \int u^n (1+u^2)^{m-1} du, \; u = \tan x$

61. $\int \cot^n x \; \csc^{2m} x \; dx = -\int u^n (1+u^2)^{m-1} du, \; u = \cot x$

62. $\int \tan^{2n+1} x \; \sec^m x \; dx = \int (u^2-1)^n u^{m-1} du, \; u = \sec x$

63. $\int \cot^{2n+1} x \; \csc^m x \; dx = -\int (u^2-1)^n u^{m-1} du, \; u = \csc x$

64. $\int \sin mx \cos nx \; dx = -\frac{1}{2} \left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right] + C; \; m^2 \neq n^2$

65. $\displaystyle\int \sin mx \sin nx \; dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] + C; \; m^2 \neq n^2$

66. $\displaystyle\int \cos mx \cos nx \; dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right] + C; \; m^2 \neq n^2$

6.4 Trigonometric Integrals

The trigonometric integrals are of two types. The integrand of the first type consists of a product of powers of trigonometric functions of x. The integrand of the second type consists of $\sin(nx)\cos(mx), \sin(nx)\sin(mx)$ or $\cos(nx)\cos(mx)$. By expressing all trigonometric functions in terms of sine and cosine, many trigonometric integrals can be computed by using the following theorem.

Theorem 6.4.1 *Suppose that m and n are integers, positive, negative, or zero. Then the following reduction formulas are valid:*

1. $\displaystyle\int \sin^n x \; dx = \frac{-1}{n} \; \sin^{n-1} x \cos x + \frac{(n-1)}{n} \int \sin^{n-2} x \; dx, \; n > 0$

2. $\displaystyle\int \sin^{n-2} x \; dx = \frac{1}{n-1} \; \sin^{n-1} x \cos x + \frac{n}{n-1} \int \sin^n x \; dx, \; n \leq 0$

3. $\displaystyle\int (\sin x)^{-1} \; dx = \int \csc x \; dx = \ln|\csc x - \cot x| + c \text{ or } -\ln|\csc x + \cot x| + c$

4. $\displaystyle\int \cos^n x \; dx = \frac{1}{n} \; \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \; dx, \; n > 0$

5. $\displaystyle\int \cos^{n-2} x \; dx = \frac{-1}{n-1} \; \cos^{n-1} x \sin x + \frac{n}{n-1} \int \cos^n x \; dx, \; n \leq 0$

6. $\displaystyle\int (\cos x)^{-1} \; dx = \int \sec x \; dx = \ln|\sec x + \tan x| + c$

7. $$\begin{aligned}\int sin^n x \cos^{2m+1} x \; dx &= \textstyle\int \sin^n x(1-\sin^2 x)^m \cos x \; dx \\ &= \textstyle\int u^n(1-u^2)^m du, \; u = \sin x, \; du = \cos x \; dx\end{aligned}$$

8. $$\int sin^{2n+1} x \cos^m x \; dx = \int \cos^m x(1-\cos^2 x)^n \sin x \; dx$$
$$= -\int u^m(1-u^2)^n du, \; u = \cos x, \; du = -\sin x \; dx$$

9. $$\int \sin^{2n} x \cos^{2m} x \; dx = \int (1-\cos^2 x)^n \cos^{2m} x \; dx$$
$$= \int (1-\sin^2 x)^m \sin^{2n} x \; dx$$

10. $$\int \sin(nx)\cos(mx)dx = \frac{-1}{2}\left[\frac{\cos(m+n)x}{m-n} + \frac{\cos(m-n)x}{m-n}\right] + c, \; m^2 \neq n^2$$

11. $$\int \sin(mx)\sin(mx) \; dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] + c, \; m^2 \neq n^2$$

12. $$\int \cos(mx)\cos(mx) \; dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right] + c, \; m^2 \neq n^2$$

Corollary. The following integration formulas are valid:

13. $$\int \tan^n u \; du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u \; d$$

14. $$\int \sec^n u \; du = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \; dx$$

15. $$\int \csc^n u \; du = \frac{-1}{n-1} \csc^{n-2} x \cot x + \frac{n-2}{n-1} \int \csc^{n-2} x \; dx$$

Exercises 6.4 Evaluate each of the following integrals.

1. $\int \sin^5 x \; dx$
2. $\int \cos^4 x \; dx$
3. $\int \tan^5 x \; dx$
4. $\int \cot^4 x \; dx$
5. $\int \sec^5 x \; dx$
6. $\int \csc^4 x \; dx$

7. $\int \sin^5 x \cos^4 x \ dx$

8. $\int \sin^3 x \cos^5 x \ dx$

9. $\int \sin^4 x \cos^3 x \ dx$

10. $\int \sin^2 x \cos^4 x \ dx$

11. $\int \tan^5 x \sec^4 x \ dx$

12. $\int \cot^5 x \csc^4 x \ dx$

13. $\int \tan^4 x \sec^5 x \ dx$

14. $\int \cot^4 x \csc^5 x \ dx$

15. $\int \tan^4 x \sec^4 x \ dx$

16. $\int \cot^4 x \csc^4 x \ dx$

17. $\int \tan^3 x \sec^3 x \ dx$

18. $\int \cot^3 x \csc^3 x \ dx$

19. $\int \sin 2x \cos 3x \ dx$

20. $\int \sin 4x \cos 4x \ dx$

21. $\int \sin 3x \cos 3x \ dx$

22. $\int \sin 2x \sin 3x \ dx$

23. $\int \sin 4x \sin 6x \ dx$

24. $\int \sin 3x \sin 5x \ dx$

25. $\int \cos 3x \cos 5x \ dx$

26. $\int \cos 2x \cos 4x \ dx$

27. $\int \cos 3x \cos 4x \ dx$

28. $\int \sin 4x \cos 4x \ dx$

6.5 Trigonometric Substitutions

Theorem 6.5.1 ($a^2 - u^2$ Forms). *Suppose that* $u = a \sin t, \ a > 0$. *Then*

$$du = a\cos t dt,\ a^2 - u^2 = a^2\cos^2 t,\ \sqrt{a^2-u^2} = a\cos t,\ t = \arcsin(u/a),$$

$$\sin t = \frac{u}{a},\ \cos t = \frac{\sqrt{a^2-u^2}}{a},\ \tan t = \frac{u}{\sqrt{a^2-u^2}},$$

$$\cot t = \frac{\sqrt{a^2-u^2}}{u},\ \sec t = \frac{a}{\sqrt{a^2-u^2}},\ \csc t\ \frac{a}{u}.$$

graph

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{a^2-u^2} = -\frac{1}{2}\ \ln|a^2-u^2| + c$

2. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ \ln\left|\frac{a-u}{a+u}\right| + c = \frac{1}{a}\ \text{arctanh}\left(\frac{u}{a}\right) + c$

3. $\displaystyle\int \frac{udu}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2} + c$

4. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + c$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a}\ \ln\left|\frac{a}{u} - \frac{\sqrt{a^2-u^2}}{u}\right| + c$

6. $\displaystyle\int \sqrt{a^2-u^2}\ du = \frac{a^2}{2}\ \arcsin\left(\frac{u}{a}\right) + \frac{1}{2}\ u\sqrt{a^2-u^2} + c$

Proof. The proof of this theorem is left as an exercise.

Theorem 6.5.2 ($a^2 + u^2$ Forms). *Suppose that* $u = a\tan t,\ a > 0$. *Then*

$$du = a\sec^2 t dt, a^2 + u^2 = a^2\sec^2 t, \sqrt{a^2+u^2} = a\sec t, t = \arctan\left(\frac{u}{a}\right),$$

$$\sin t = \frac{u}{\sqrt{a^2+u^2}},\ \cos t = \frac{a}{\sqrt{a^2+u^2}},\ \tan t = \frac{u}{a}$$

$$\csc t = \frac{\sqrt{a^2+u^2}}{u},\ \sec t = \frac{\sqrt{a^2+u^2}}{a},\ \cot t = \frac{a}{u}.$$

graph

Proof. The proof of this theorem is left as an exercise.

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{a^2+u^2} = \frac{1}{2}\ \ln\left|a^2+u^2\right| + c$
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + c$
3. $\displaystyle\int \frac{udu}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2} + c$
4. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left|u + \sqrt{a^2+u^2}\right| + c$
5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{1}{a}\ \ln\left|\frac{\sqrt{a^2+u^2}}{u} - \frac{a}{u}\right| + c$
6. $\displaystyle\int \sqrt{a^2+u^2}\ du = \frac{1}{2}\ u\sqrt{a^2+u^2} + \frac{a^2}{2}\ \ln\left|u + \sqrt{a^2+u^2}\right| + c$

Theorem 6.5.3 *($u^2 - a^2$ Forms) Suppose that* $u = a\sec t,\ a > 0$. *Then*

$$du = a\sec t\tan t\ dt,\ u^2 - a^2 = a^2\tan^2 t,\ \sqrt{u^2-a^2} = a\tan t, t = arcsec\left(\frac{u}{a}\right),$$

$$\sin t = \frac{\sqrt{u^2-a^2}}{u},\ \cos t = \frac{a}{u},\ \tan t = \frac{\sqrt{u^2-a^2}}{a},$$

$$\csc t = \frac{u}{\sqrt{u^2-a^2}},\ \sec t = \frac{u}{a},\ \cot t = \frac{a}{\sqrt{u^2-a^2}}.$$

graph

Proof. The proof of this theorem is left as an exercise.

The following integration formulas are valid:

1. $\displaystyle\int \frac{udu}{u^2 - a^2} = \frac{1}{2}\ \ln\left|u^2 - a^2\right| + c$

2. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ \ln\left|\frac{u-a}{u+a}\right| + c$

3. $\displaystyle\int \frac{udu}{\sqrt{u^2 - a^2}} = \sqrt{u^2 - a^2} + c$

4. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + c$

5. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\ \operatorname{arcsec}\left(\frac{u}{a}\right) + c$

6. $\displaystyle\int \sqrt{u^2 - a^2}\ du = \frac{1}{2}\ u\sqrt{u^2 - a^2} - \frac{a^2}{2}\ \ln\left|u + \sqrt{u^2 - a^2}\right| + c$

Exercises 6.5 Prove each of the following formulas:

1. $\displaystyle\int \frac{u\ du}{a^2 - u^2} = -\frac{1}{2}\ \ln|a^2 - u^2| + C$

2. $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ \ln\left|\frac{a-u}{a+u}\right| + C$

3. $\displaystyle\int \frac{u\ du}{\sqrt{a^2 - u^2}} = -\sqrt{a^2 - u^2} + C$

4. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C,\ \ a > 0$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a}\ \ln\left|\frac{a}{u} - \frac{\sqrt{a^2 - u^2}}{u}\right| + C$

6. $\displaystyle\int \sqrt{a^2 - u^2}\ du = \frac{a^2}{2}\ \arcsin\left(\frac{u}{a}\right) + \frac{1}{2}\ u\sqrt{a^2 - u^2} + C,\ a > 0$

7. $\displaystyle\int \frac{u\ du}{a^2 + u^2} = \frac{1}{2}\ \ln\left|a^2 + u^2\right| + C$

8. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\ \arctan\left(\frac{u}{a}\right) + C$

9. $\displaystyle\int \frac{u\ du}{\sqrt{a^2 + u^2}} = \sqrt{a^2 + u^2} + C$

10. $\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left|u + \sqrt{a^2 + u^2}\right| + C$

11. $\displaystyle\int \frac{du}{u\sqrt{a^2 + u^2}} = \frac{1}{a}\ \ln\left|\frac{\sqrt{a^2 + u^2}}{u} - \frac{a}{u}\right| + C$

12. $\displaystyle\int \sqrt{a^2 + u^2}\ du = \frac{1}{2}\ u\sqrt{a^2 + u^2} + \frac{a^2}{2}\ \ln\left|u + \sqrt{a^2 + u^2}\right| + C$

13. $\displaystyle\int \frac{u\ du}{u^2 - a^2} = \frac{1}{2}\ \ln\left|u^2 - a^2\right| + C$

14. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ \ln\left|\frac{u - a}{u + a}\right| + C$

15. $\displaystyle\int \frac{u\ du}{\sqrt{u^2 - a^2}} = \sqrt{u^2 - a^2} + C$

16. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

17. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\ \operatorname{arcsec}\left(\frac{u}{a}\right) + C$

18. $\displaystyle\int \sqrt{u^2 - a^2}\ du = \frac{1}{2}\ u\sqrt{u^2 - a^2} - \frac{a^2}{2}\ \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

Evaluate each of the following integrals:

19. $\displaystyle\int \frac{x\ dx}{\sqrt{4-x^2}}$ 20. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ 21. $\displaystyle\int \frac{x\ dx}{4-x^2}$

22. $\displaystyle\int \frac{dx}{4-x^2}$ 23. $\displaystyle\int \frac{x\ dx}{9+x^2}$ 24. $\displaystyle\int \frac{dx}{9+x^2}$

25. $\displaystyle\int \frac{x\ dx}{\sqrt{9+x^2}}$ 26. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$ 27. $\displaystyle\int \frac{x\ dx}{x^2-16}$

28. $\displaystyle\int \frac{dx}{x^2-16}$ 29. $\displaystyle\int \frac{x\ dx}{\sqrt{x^2-16}}$ 30. $\displaystyle\int \frac{dx}{\sqrt{x^2-16}}$

31. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$ 32. $\displaystyle\int \frac{dx}{x\sqrt{9-x^2}}$ 33. $\displaystyle\int \frac{dx}{x\sqrt{x^2+16}}$

34. $\displaystyle\int \sqrt{9-x^2}\ dx$ 35. $\displaystyle\int \sqrt{4-9x^2}$ 36. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\ dx$

37. $\displaystyle\int \frac{x^2}{\sqrt{4+x^2}}\ dx$ 38. $\displaystyle\int \frac{x^2}{\sqrt{x^2-16}}\ dx$ 39. $\displaystyle\int \frac{dx}{(9+x^2)^2}$

40. $\displaystyle\int \frac{dx}{(9-x^2)^2}$ 41. $\displaystyle\int \frac{dx}{(x^2-16)^2}$ 42. $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}}$

43. $\displaystyle\int \frac{\sqrt{4+x^2}}{x}$ 44. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\ dx$ 45. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

46. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$ 47. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$ 48. $\displaystyle\int \frac{dx}{x^2-2x+5}$

49. $\displaystyle\int \frac{dx}{x^2-4x+12}$ 50. $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$ 51. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+12}}$

52. $\displaystyle\int \frac{dx}{4x-x^2}$ 53. $\displaystyle\int \frac{dx}{\sqrt{x^2-2x+5}}$ 54. $\displaystyle\int \frac{x\ dx}{x^2-4x-12}$

55. $\displaystyle\int \frac{x\ dx}{\sqrt{x^2-2x+5}}$ 56. $\displaystyle\int \frac{x}{x^2+4x+13}\ dx$ 57. $\displaystyle\int (5-4x-x^2)^{1/2}dx$

58. $\displaystyle\int \frac{2x+7}{x^2+4+13}\,dx$ 59. $\displaystyle\int \frac{x+3}{\sqrt{x^2+2x+5}}\,dx$ 60. $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$

61. $\displaystyle\int \frac{x+4}{\sqrt{9x^2+16}}\,dx$ 62. $\displaystyle\int \frac{x+2}{\sqrt{16-9x^2}}\,dx$ 63. $\displaystyle\int \frac{e^{2x}dx}{(5-e^{2x}+e^{4x})^{1/2}}$

64. $\displaystyle\int \frac{e^{3x}dx}{(e^{6x}+4e^{3x}+3)^{1/2}}$

6.6 Integration by Partial Fractions

A polynomial with real coefficients can be factored into a product of powers of linear and quadratic factors. This fact can be used to integrate rational functions of the form $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials that have no factors in common. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then by long division we can express the rational function by

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

where $q(x)$ is the quotient and $r(x)$ is the remainder whose degree is less than the degree of $Q(x)$. Then $Q(x)$ is factored as a product of powers of linear and quadratic factors. Finally $r(x)/Q(x)$ is split into a sum of fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

and

$$\frac{B_1x+c_1}{ax^2+bx+c} + \frac{B_2x+c_2}{(ax^2+bxc)^2} + \cdots + \frac{B_mx+c_m}{(ax^2+bx+c)^m}.$$

Many calculators and computer algebra systems, such as Maple or Mathematica, are able to factor polynomials and split rational functions into partial fractions. Once the partial fraction split up is made, the problem of integrating a rational function is reduced to integration by substitution using linear or trigonometric substitutions. It is best to study some examples and do some simple problems by hand.

Exercises 6.6 Evaluate each of the following integrals:

1. $\displaystyle\int \frac{dx}{(x-1)(x-2)(x+4)}$

2. $\displaystyle\int \frac{dx}{(x-4)(10+x)}$

3. $\displaystyle\int \frac{dx}{(x-a)(x-b)}$

4. $\displaystyle\int \frac{dx}{(x-a)(b-x)}$

5. $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)}$

6. $\displaystyle\int \frac{dx}{(x-1)(x^2+1)}$

7. $\displaystyle\int \frac{2x\ dx}{x^2-5x+6}$

8. $\displaystyle\int \frac{x\ dx}{(x+3)(x+4)}$

9. $\displaystyle\int \frac{x+1}{(x+2)(x^2+4)}\ dx$

10. $\displaystyle\int \frac{(x+2)dx}{(x+3)(x^2+1)}$

11. $\displaystyle\int \frac{2\ dx}{(x^2+4)(x^2+9)}$

12. $\displaystyle\int \frac{dx}{(x^2-4)(x^2-9)}$

13. $\displaystyle\int \frac{x^2\ dx}{(x^2+4)(x^2+9)}$

14. $\displaystyle\int \frac{x\ dx}{(x^2-4)(x^2-9)}$

15. $\displaystyle\int \frac{dx}{x^4-16}$

16. $\displaystyle\int \frac{x\ dx}{x^4-81}$

6.7 Fractional Power Substitutions

If the integrand contains one or more fractional powers of the form $x^{s/r}$, then the substitution, $x = u^n$, where n is the least common multiple of the denominators of the fractional exponents, may be helpful in computing the integral. It is best to look at some examples and work some problems by hand.

Exercises 6.7 Evaluate each of the following integrals using the given substitution.

1. $\displaystyle\int \frac{4x^{3/2}}{1+x^{1/3}}\ dx;\ x = u^6$

2. $\displaystyle\int \frac{dx}{1+x^{1/3}};\ x = u^3$

3. $\displaystyle\int \frac{dx}{\sqrt{1+e^{2x}}}$; $u^2 = 1 + e^{2x}$

4. $\displaystyle\int \frac{dx}{x\sqrt{x^3-8}}$; $u^2 = x^3 - 8$

Evaluate each of the following by using an appropriate substitution:

5. $\displaystyle\int \frac{x\ dx}{\sqrt{x+2}}$

6. $\displaystyle\int \frac{x^2 dx}{\sqrt{x+4}}$

7. $\displaystyle\int \frac{1}{4+\sqrt{x}}\ dx$

8. $\displaystyle\int \frac{x\ dx}{1+\sqrt{x}}$

9. $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}$

10. $\displaystyle\int \frac{x^{2/3}}{8+x^{1/2}}$

11. $\displaystyle\int \frac{1}{x^{2/3}+1}\ dx$

12. $\displaystyle\int \frac{dx}{1+\sqrt{x}}$

13. $\displaystyle\int \frac{x\ dx}{1+x^{2/3}}$

14. $\displaystyle\int \frac{1+\sqrt{x}}{2+\sqrt{x}}\ dx$

15. $\displaystyle\int \frac{1-\sqrt{x}}{1+x^{3/2}}\ dx$

16. $\displaystyle\int \frac{1+\sqrt{x}}{1-x^{3/2}}\ dx$

6.8 Tangent $x/2$ Substitution

If the integrand contains an expression of the form $(a+b\sin x)$ or $(a+b\cos x)$, then the following theorem may be helpful in evaluating the integral.

Theorem 6.8.1 *Suppose that* $u = \tan(x/2)$. *Then*

$$\sin x = \frac{2u}{1+u^2},\ \cos x = \frac{1-u^2}{1+u^2}\ \ \textit{and}\ \ dx = \frac{2}{1+u^2}\ du.$$

Furthermore,

$$\int \frac{dx}{a+b\sin x} = \int \frac{(2/(1+u^2))du}{a+b\left(\frac{2u}{1+u^2}\right)} = \int \frac{2du}{a(1+u^2)+2bu}$$
$$\int \frac{dx}{a+b\cos x} = \int \frac{(2/(1+u^2))du}{a+b\left(\frac{1-u^2}{1+u^2}\right)} = \int \frac{2du}{a(1+u^2)+b(1-u^2)}.$$

Proof. The proof of this theorem is left as an exercise.

Exercises 6.8

1. Prove Theorem 6.8.1

Evaluate the following integrals:

2. $\displaystyle\int \frac{dx}{2+\sin x}$

3. $\displaystyle\int \frac{dx}{\sin x+\cos x}$

4. $\displaystyle\int \frac{dx}{\sin x-\cos x}$

5. $\displaystyle\int \frac{dx}{2\sin x+3\cos x}$

6. $\displaystyle\int \frac{dx}{2-\sin x}$

7. $\displaystyle\int \frac{dx}{3+\cos x}$

8. $\displaystyle\int \frac{dx}{3-\cos x}$

9. $\displaystyle\int \frac{\sin x\ dx}{\sin x+\cos x}$

10. $\displaystyle\int \frac{\cos x\ dx}{\sin x-\cos x}$

11. $\displaystyle\int \frac{(1+\sin x)dx}{(1-\sin x)}$

12. $\displaystyle\int \frac{1-\cos x}{1+\cos x}\ dx$

13. $\displaystyle\int \frac{2-\cos x}{2+\cos x}\ dx$

14. $\displaystyle\int \frac{2+\cos x}{2-\sin x}\ dx$

15. $\displaystyle\int \frac{2-\sin x}{3+\cos x}\ dx$

16. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

6.9 Numerical Integration

Not all integrals can be computed in the closed form in terms of the elementary functions. It becomes necessary to use approximation methods. Some of the simplest numerical methods of integration are stated in the next few theorems.

Theorem 6.9.1 (Midpoint Rule) *If f, f' and f'' are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{f''(c)}{24}\,(b-a)^3.$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.2 (Trapezoidal Rule) *If f, f' and f'' are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = (b-a)\left[\frac{1}{2}\,(f(a)+f(b))\right] - \frac{f''(c)}{12}\,(b-a)^3.$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.3 (Simpson's Rule) *If $f, f', f'', f^{(3)}$ and $f^{(4)}$ are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{b-a}{6}\,\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{f^{(4)}(c)}{2880}\,(b-a)^5.$$

These basic numerical formulas can be applied on each subinterval $[x_i, x_{i+1}]$ of a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of the interval $[a, b]$ to get composite numerical methods. We assume that $h = (b-a)/n,\ x_i = a + ih,\ i = 0, 1, 2, \cdots, n$.

Proof. The proof of this theorem is omitted.

Theorem 6.9.4 (Composite Trapezoidal Rule) *If f, f' and f'' are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] - \frac{b-a}{12}\,h^2 f''(c).$$

Proof. The proof of this theorem is omitted.

Theorem 6.9.5 (Composite Simpson's Rule) *If $f, f', f'', f^(3)$ and $f^{(4)}$ are continuous on $[a, b]$, then there exists some c such that $a < c < b$ and*

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(b)\right] - \frac{b-a}{180}\,h^4 f^{(4)}(c).$$

where n is an even natural number.

Proof. The proof of this theorem is omitted.

Remark 22 In practice, the composite Trapezoidal and Simpson's rules can be applied when the value of the function is known at the subdivision points $x_i, i = 0, 1, 2, \cdots, n$.

Exercises 6.9 Approximate the value of each of the following integrals for a given value of n and using

(a) Left-hand end point approximation: $\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$

(b) Right-hand end point approximation: $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$

(c) Mid point approximation: $\sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)(x_i - x_{i-1})$

(d) Composite Trapezoidal Rule

(e) Composite Simpson's Rule

1. $\int_1^3 \frac{1}{x}\, dx,\ n = 10$
2. $\int_2^4 \frac{1}{\sqrt{x}}\, dx,\ n = 10$
3. $\int_0^1 \frac{1}{1+\sqrt{x}}\, dx,\ n = 10$
4. $\int_1^2 \frac{1}{1+x^2}\, dx,\ n = 10$
5. $\int_0^1 \frac{1+\sqrt{x}}{1+x}\, dx,\ n = 10$
6. $\int_0^2 x^3\, dx,\ n = 10$
7. $\int_0^2 (x^2 - 2x)\, dx,\ n = 10$
8. $\int_0^1 (1+x^2)^{1/2} dx,\ n = 10$
9. $\int_0^1 (1+x^3)^{1/2} dx,\ n = 10$
10. $\int_0^1 (1+x^4)^{1/2} dx,\ n = 10$

Chapter 7

Improper Integrals and Indeterminate Forms

7.1 Integrals over Unbounded Intervals

Definition 7.1.1 Suppose that a function f is continuous on $(-\infty, \infty)$. Then we define the following improper integrals when the limits exist

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \tag{1}$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx \tag{2}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx \tag{3}$$

provided the integrals on the right hand side exist for some c. If these improper integrals exist, we say that they are convergent; otherwise they are said to be divergent.

Definition 7.1.2 Suppose that a function f is continuous on $[0, \infty)$. Then the Laplace transform of f, written $\mathcal{L}(f)$ or $F(s)$, is defined by

$$\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{-st} f(t)^{dt}.$$

Theorem 7.1.1 *The Laplace transform has the following properties:*

$$\mathcal{L}(c) = \frac{c}{s} \tag{4}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} \tag{5}$$

$$\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2} \tag{6}$$

$$\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2} \tag{7}$$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2} \tag{8}$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2} \tag{9}$$

$$\mathcal{L}(t) = \frac{1}{s^2} \tag{10}$$

Proof.

(i) $$\mathcal{L}(c) = \int_0^\infty ce^{-st}dt$$

$$= \left.\frac{ce^{-st}}{-s}\right|_0^\infty$$

$$= \frac{c}{s}.$$

(ii) $$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt$$

$$= \int_0^\infty e^{-(s-a)t}dt$$

$$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$

$$= \frac{1}{s-a}$$

provided $s > a$.

$$\text{(iii)}\quad \mathcal{L}(\cosh at) = \int_0^\infty \left(\frac{e^{at} + e^{-at}}{2}\right) e^{-st} dt$$

$$= \frac{1}{2}[\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2 - a^2}, s > |a|.$$

$$\text{(iv)}\quad \mathcal{L}(\sinh at) = \int_0^\infty \frac{1}{2}\,(e^{at} - e^{-at})e^{-st} dt$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right], s > |a|$$

$$= \frac{a}{s^2 - a^2}, s > |a|.$$

$$\text{(v)}\quad \mathcal{L}(\cos \omega t) = \int_0^\infty \cos \omega t e^{-st} dt$$

$$= \frac{1}{\omega^2 + s^2}\left[e^{-st}(-s\cos\omega t + \omega \sin \omega t)\right]_0^\infty$$

$$= \frac{s}{\omega^2 + s^2}.$$

$$\text{(vi)}\quad \mathcal{L}(\sin \omega t) = \int_0^\infty \sin \omega t e^{-st} dt$$

$$= \frac{1}{\omega^2 + s^2}\left[e^{-st}(-s\sin\omega t - \omega \cos \omega t)\right]_0^\infty$$

$$= \frac{\omega}{\omega^2 + s^2}.$$

$$\text{(vii)}\quad \mathcal{L}(t) = \int_0^\infty te^{-st}dt; \quad (u = t, dv = e^{-st}dt)$$

$$= \left.\frac{te^{-st}}{-s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\, dt$$

$$= \left.\frac{e^{-st}}{-s^2}\right|_0^\infty$$

$$= \frac{1}{s^2}.$$

This completes the proof of Theorem 7.1.1.

Theorem 7.1.2 *Suppose that f and g are continuous on $[a,\infty)$ and $0 \leq f(x) \leq g(x)$ on $[a,\infty)$.*

(i) If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

(ii) If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.

Proof. The proof of this follows from the order properties of the integral and is omitted.

Definition 7.1.3 For each $x > 0$, the Gamma function, denoted $\Gamma(x)$, is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Theorem 7.1.3 *The Gamma function has the following properties:*

$$\Gamma(1) = 1 \tag{11}$$

$$\Gamma(x+1) = x\Gamma(x) \tag{12}$$

$$\Gamma(n+1) = n!,\ n = \textit{ natural number} \tag{13}$$

Proof.

$$
\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-t} dt \\
&= -e^{-t}\Big|_0^\infty \\
&= 1 \\
\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt; \ (u = t^x, dv = e^{-t} dt) \\
&= -t^x e^{-t}\Big|_0^\infty + x \int_0^\infty t^{x-1} e^{-t} dt \\
&= x\Gamma(x), \ x > 0 \\
\Gamma(2) &= 1\Gamma(1) = 1 \\
\Gamma(3) &= 2\Gamma(2) = 1 \cdot 2 = 2! \\
\text{If } \Gamma(k) &= (k-1)!, \ \text{then} \\
\Gamma(k+1) &= k\Gamma(k) \\
&= k((k-1)!) \\
&= k!.
\end{aligned}
$$

By the principle of mathematical induction,

$$\Gamma(n+1) = n!$$

for all natural numbers n. This completes the proof of this theorem.

Theorem 7.1.4 *Let f be the normal probability distribution function defined by*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \ e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

where μ is the constant mean of the distribution and σ is the constant standard deviation of the distribution. Then the improper integral

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Let F be the normal distribution function defined by

$$F(x) = \int_{-\infty}^{x} f(x) dx.$$

Then $F(b) - F(a)$ represents the percentage of normally distributed data that lies between a and b. This percentage is given by

$$\int_a^b f(x)dx.$$

Furthermore,

$$\int_{\mu+a\sigma}^{\mu+b\sigma} f(x)dx = \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}dx.$$

Proof. The proof of this theorem is omitted.

Exercises 7.1 None available.

7.2 Discontinuities at End Points

Definition 7.2.1 (i) Suppose that f is continuous on $[a, b)$ and

$$\lim_{x\to b^-} f(x) = +\infty \quad \text{or} \quad -\infty.$$

Then, we define

$$\int_a^b f(x)dx = \lim_{x\to b^-} \int_a^x f(x)dx.$$

If the limit exists, we say that the improper integral converges; otherwise we say that it diverges.

(ii) Suppose that f is continuous on $(a, b]$ and

$$\lim_{x\to a^+} f(x) = +\infty \quad \text{or} \quad -\infty.$$

Then we define,

$$\int_a^b f(x)dx = \lim_{x\to a^+} \int_x^b f(x)dx.$$

If the limit exists, we say that the improper integral converges; otherwise we say that it diverges.

Exercises 7.2

1. Suppose that f is continuous on $(-\infty, \infty)$ and $g'(x) = f(x)$. Then define each of the following improper integrals:

(a) $\displaystyle\int_a^{+\infty} f(x)dx$

(b) $\displaystyle\int_{-\infty}^{b} f(x)dx$

(c) $\displaystyle\int_{-\infty}^{+\infty} f(x)dx$

2. Suppose that f is continuous on the open interval (a, b) and $g'(x) = f(x)$ on (a, b). Define each of the following improper integrals if f is not continuous at a or b:

   (a) $\displaystyle\int_a^{x} f(x)dx,\ a \le x < b$

   (b) $\displaystyle\int_x^{b} f(x)dx,\ a < x \le b$

   (c) $\displaystyle\int_a^{b} f(x)dx$

3. Prove that $\displaystyle\int_0^{+\infty} e^{-x}dx = 1$

4. Prove that $\displaystyle\int_0^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$

5. Prove that $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi$

6. Prove that $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1}$, if and only if $p > 1$.

7. Show that $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}dx = 2\int_0^{\infty} e^{-x^2}dx$. Use the comparison between e^{-x} and e^{-x^2}. Show that $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}dx$ exists.

8. Prove that $\displaystyle\int_0^{1} \frac{dx}{x^p}$ converges if and only if $p < 1$.

9. Evaluate $\int_0^{+\infty} e^{-x}\sin(2x)dx$.

10. Evaluate $\int_0^{+\infty} e^{-4x}\cos(3x)dx$.

11. Evaluate $\int_0^{+\infty} x^2 e^{-x}dx$.

12. Evaluate $\int_0^{+\infty} xe^{-x}dx$.

13. Prove that $\int_0^{\infty} \sin(2x)dx$ diverges.

14. Prove that $\int_0^{\infty} \cos(3x)dx$ diverges.

15. Compute the volume of the solid generated when the area between the graph of $y = e^{-x^2}$ and the x-axis is rotated about the y-axis.

16. Compute the volume of the solid generated when the area between the graph of $y = e^{-x},\ 0 \le x < \infty$ and the x-axis is rotated

    (a) about the x-axis

    (b) about the y-axis.

17. Let A represent the area bounded by the graph $y = \dfrac{1}{x},\ 1 \le x < \infty$ and the x-axis. Let V denote the volume generated when the area A is rotated about the x-axis.

    (a) show that A is $+\infty$

    (b) show that $V = \pi$

    (c) show that the surface area of V is $+\infty$.

    (d) Is it possible to fill the volume V with paint and not be able to paint its surface? Explain.

18. Let A represent the area bounded by the graph of $y = e^{-2x}, 0 \le x < \infty$, and $y = 0$.

(a) Compute the area of A.

(b) Compute the volume generated when A is rotated about the x-axis.

(c) Compute the volume generated when A is rotated about the y-axis.

19. Assume that $\int_0^{+\infty} \sin(x^2)dx = \sqrt{(\pi/8)}$. Compute $\int_0^{+\infty} \frac{\sin x}{\sqrt{x}}\, dx$.

20. It is known that $\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi}$.

(a) Compute $\int_0^{+\infty} e^{-x^2} dx$.

(b) Compute $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{x}}\, dx$.

(c) Compute $\int_0^{+\infty} e^{-4x^2} dx$.

Definition 7.2.2 Suppose that $f(t)$ is continuous on $[0, \infty)$ and there exist some constants $a > 0$, $M > 0$ and $T > 0$ such that $|f(t)| < Me^{at}$ for all $t \geq T$. Then we define the Laplace transform of $f(t)$, denoted $\mathcal{L}\{f(t)\}$, by

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

for all $s \geq s_0$. In problems 21–34, compute $\mathcal{L}\{f(t)\}$ for the given $f(t)$.

21. $f(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

22. $f(t) = t$

23. $f(t) = t^2$

24. $f(t) = t^3$

25. $f(t) = t^n, n = 1, 2, 3, \cdots$

26. $f(t) = e^{bt}$

27. $f(t) = te^{bt}$

28. $f(t) = t^n e^{bt}, n = 1, 2, 3, \cdots$

29. $f(t) = \dfrac{e^{at} - e^{bt}}{a - b}$

30. $f(t) = \dfrac{ae^{at} - be^{bt}}{a - b}$

31. $f(t) = \dfrac{1}{b} \sin(bt)$

32. $f(t) = \cos(bt)$

33. $f(t) = \dfrac{1}{b} \sinh(bt)$

34. $f(t) = \cosh(bt)$

Definition 7.2.3 For $x > 0$, we define the Gamma function $\Gamma(x)$ by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

In problems 35–40 assume that $\Gamma(x)$ exists for $x > 0$ and $\displaystyle\int_0^{+\infty} e^{-x^2} = \frac{1}{2}\sqrt{\pi}$.

35. Show that $\Gamma(1/2) = \sqrt{\pi}$

36. Show that $\Gamma(1) = 1$

37. Prove that $\Gamma(x+1) = x\Gamma(x)$

38. Show that $\Gamma\left(\dfrac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}$

39. Show that $\Gamma\left(\dfrac{5}{2}\right) = \dfrac{3}{4}\sqrt{\pi}$

40. Show that $\Gamma(n+1) = n!$

In problems 41–60, evaluate the given improper integrals.

41. $\displaystyle\int_0^{+\infty} 2xe^{-x^2} dx$

42. $\displaystyle\int_1^{+\infty} \frac{dx}{x^{3/2}}$

43. $\displaystyle\int_4^{+\infty} \frac{dx}{x^{5/2}}$

44. $\displaystyle\int_1^{+\infty} \frac{4x}{1+x^2}\, dx$

45. $\displaystyle\int_1^{+\infty} \frac{x}{(1+x^2)^{3/2}}\, dx$

46. $\displaystyle\int_{16}^{+\infty} \frac{4}{x^2-4}\, dx$

47. $\displaystyle\int_2^{+\infty} \frac{1}{x(\ln x)^2}\, dx$

48. $\displaystyle\int_2^{+\infty} \frac{1}{x(\ln x)^p}\, dx, p > 1$

49. $\displaystyle\int_{-\infty}^{1} 3xe^{-x^2}dx$

50. $\displaystyle\int_{-\infty}^{2} e^x\ dx$

51. $\displaystyle\int_{0}^{\infty} \frac{2}{e^x+e^{-x}}\ dx$

52. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+9}$

53. $\displaystyle\int_{0}^{2} \frac{dx}{\sqrt{4-x^2}}$

54. $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{16-x^2}}\ dx$

55. $\displaystyle\int_{0}^{5} \frac{x}{(25-x^2)^{2/3}}\ dx$

56. $\displaystyle\int_{2}^{+\infty} \frac{dx}{x\sqrt{x^2-4}}$

57. $\displaystyle\int_{0}^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$

58. $\displaystyle\int_{0}^{\infty} \frac{dx}{\sqrt{x}(x+25)}$

59. $\displaystyle\int_{0}^{\infty} \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}}\ dx$

60. $\displaystyle\int_{0}^{+\infty} x^2e^{-x^3}dx$

7.3

Theorem 7.3.1 (Cauchy Mean Value Theorem) *Suppose that two functions f and g are continuous on the closed interval $[a,b]$, differentiable on the open interval (a,b) and $g'(x) \neq 0$ on (a,b). Then there exists at least one number c such that $a < c < b$ and*

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. See the proof of Theorem 4.1.6.

Theorem 7.3.2 *Suppose that f and g are continuous and differentiable on an open interval (a,b) and $a < c < b$. If $f(c) = g(c) = 0,\ g'(x) \neq 0$ on (a,b) and*

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

Proof. See the proof of Theorem 4.1.7.

Theorem 7.3.3 (L'Hôpital's Rule) *Let* lim *represent one of the limits*

$$\lim_{x\to c}, \ \lim_{x\to c^+}, \ \lim_{x\to c^-}, \ \lim_{x\to +\infty}, \quad \text{or} \quad \lim_{x\to -\infty}.$$

Suppose that f and g are continuous and differentiable on an open interval (a,b) except at an interior point c, $a < c < b$. Suppose further that $g'(x) \neq 0$ on (a,b), $\lim f(x) = \lim g(x) = 0$ or $\lim f(x) = \lim g(x) = +\infty$ or $-\infty$. If

$$\lim \frac{f'(x)}{g'(x)} = L, +\infty \quad \text{or} \quad -\infty$$

then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

Proof. The proof of this theorem is omitted.

Definition 7.3.1 (Extended Arithmetic) For the sake of convenience in dealing with indeterminate forms, we define the following arithmetic operations with real numbers, $+\infty$ and $-\infty$. Let c be a real number and $c > 0$. Then we define

$$+\infty + \infty = +\infty, \ -\infty - \infty = -\infty, \ c(+\infty) = +\infty, \ c(-\infty) = -\infty$$

$$(-c)(+\infty) = -\infty, \ (-c)(-\infty) = +\infty, \ \frac{c}{+\infty} = 0, \ \frac{-c}{+\infty} = 0, \ \frac{c}{-\infty} = 0,$$

$$\frac{-c}{-\infty} = 0, (+\infty)^c = +\infty, \ (+\infty)^{-c} = 0, \ (+\infty)(+\infty) = +\infty, \ (+\infty)(-\infty) = -\infty,$$

$$(-\infty)(-\infty) = +\infty.$$

Definition 7.3.2 The following operations are indeterminate:

$$\frac{0}{0}, \ \frac{+\infty}{+\infty}, \ \frac{+\infty}{-\infty}, \ \frac{-\infty}{-\infty}, \ \frac{-\infty}{+\infty}, \ \infty - \infty, \ 0 \cdot \infty, 0^0, \ 1^\infty, \ \infty^0.$$

Remark 23 The L'Hôpital's Rule can be applied directly to the $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$ forms. The forms $\infty - \infty$ and $0 \cdot \infty$ can be changed to the $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ by using arithmetic operations. For the 0^0 and 1^∞ forms we use the following procedure:

$$\lim (f(x))^{g(x)} = \lim e^{g(x)\ln(f(x))} = e^{\lim \frac{\ln(f(x))}{(1/g(x))}}.$$

It is best to study a lot of examples and work problems.

Exercises 7.3

1. Prove the Theorem of the Mean: Suppose that a function f is continuous on a closed and bounded interval $[a, b]$ and f' exists on the open interval (a, b). Then there exists at least one number c such that $a < c < b$ and

(1) $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ (2) $f(b) = f(a) + f'(c)(b - a)$.

2. Prove the Generalized Theorem of the Mean: Suppose that f and g are continuous on a closed and bounded interval $[a, b]$ and f' and g' exist on the open interval (a, b) and $g'(x) \neq 0$ for any x in (a, b). Then there exists some c such that $a < c < b$ and

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

3. Prove the following theorem known as l'Hôpital's Rule: Suppose that f and g are differentiable functions, except possibly at a, such that

$$\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0, \quad \text{and} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

4. Prove the following theorem known as an alternate form of l'Hôpital's Rule: Suppose that f and g are differentiable functions, except possibly at a, such that

$$\lim_{x \to a} f(x) = \infty, \quad \lim_{x \to a} g(x) = \infty, \quad \text{and} \quad \lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

5. Prove that if f' and g' exist and

$$\lim_{x\to+\infty} f(x) = 0, \quad \lim_{x\to+\infty} g(x) = 0, \quad \text{and} \quad \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L.$$

6. Prove that if f' and g' exist and

$$\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to+\infty} g(0) = 0, \quad \text{and} \quad \lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = L.$$

7. Prove that if f' and g' exist and

$$\lim_{x\to+\infty} f(x) = \infty, \quad \lim_{x\to+\infty} g(x) = \infty, \quad \text{and} \quad \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L.$$

8. Prove that if f' and g' exist and

$$\lim_{x\to-\infty} f(x) = \infty, \quad \lim_{x\to-\infty} g(x) = \infty, \quad \text{and} \quad \lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L.$$

9. Suppose that f' and f'' exist in an open interval (a, b) containing c. Then prove that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

10. Suppose that f' is continuous in an open interval (a, b) containing c. Then prove that

$$\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} = f'(c).$$

11. Suppose that $f(x)$ and $g(x)$ are two polynomials such that

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \ a_0 \neq 0,$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m, \ b_0 \neq 0.$$

Then prove that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } m > n \\ +\infty \text{ or } -\infty & \text{if } m < n \\ a_0/b_0 & \text{if } m = n \end{cases}$$

12. Suppose that f and g are differentiable functions, except possibly at c, and

$$\lim_{x\to c} f(x) = 0, \quad \lim_{x\to c} g(x) = 0 \quad \text{and} \quad \lim_{x\to c} g(x)\ln(f(x)) = L.$$

Then prove that

$$\lim_{x\to c} (f(x))^{g(x)} = e^L.$$

13. Suppose that f and g are differentiable functions, except possibly at c, and

$$\lim_{x\to c} f(x) = +\infty, \quad \lim_{x\to c} g(x) = 0 \quad \text{and} \quad \lim_{x\to c} g(x)\ln(f(x)) = L.$$

Then prove that

$$\lim_{x\to c} (f(x))^{g(x)} = e^L.$$

14. Suppose that f and g are differentiable functions, except possibly at c, and

$$\lim_{x\to c} f(x) = 1, \quad \lim_{x\to c} g(x) = +\infty \quad \text{and} \quad \lim_{x\to c} g(x)\ln(f(x)) = L.$$

Then prove that

$$\lim_{x\to c} (f(x))^{g(x)} = e^L.$$

15. Suppose that f and g are differentiable functions, except possibly at c, and

$$\lim_{x\to c} f(x) = 0, \quad \lim_{x\to c} g(x) = +\infty \quad \text{and} \quad \lim_{x\to c} \frac{f(x)}{(1/g(x))} = L.$$

Then prove that

$$\lim_{x\to c} f(x)g(x) = L.$$

16. Prove that $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$.

17. Prove that $\lim_{x\to 0} (1-x)^{\frac{1}{x}} = \frac{1}{e}$.

18. Prove that $\lim_{x\to +\infty} \frac{x^n}{e^x} = 0$ for each natural number n.

19. Prove that $\lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = 0$.

20. Prove that $\lim_{x\to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x = 1$.

In problems 21–50 evaluate each of the limits.

21. $\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$

22. $\lim_{x\to 0} \frac{1-\cos x^2}{x^2}$

23. $\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$

24. $\lim_{x\to 0} \frac{\tan(mx)}{\tan(nx)}$

25. $\lim_{x\to 0} \frac{e^{3x}-1}{x}$

26. $\lim_{x\to 0} (1+2x)^{3/x}$

27. $\lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h}$

28. $\lim_{h\to 0} \frac{e^{x+h}-e^x}{h}$

29. $\lim_{x\to 0} (1+mx)^{n/x}$

30. $\lim_{x\to \infty} \frac{\ln(100+x)}{x}$

31. $\lim_{x\to 0} (1+\sin mx)^{n/x}$

32. $\lim_{x\to 0^+} (\sin x)^x$

33. $\lim_{x\to 0^+} (x)^{\sin x}$

34. $\lim_{x\to\infty} \dfrac{x^4-2x^3+10}{3x^4+2x^3-7x+1}$

35. $\lim_{x\to 0^+} \tan(2x)\ln(x)$

36. $\lim_{x\to +\infty} x\sin\left(\dfrac{2\pi}{x}\right)$

37. $\lim_{x\to 0} (x+e^x)^{2/x}$

38. $\lim_{x\to\infty} \left(\dfrac{3+2x}{4+2x}\right)^x$

39. $\lim_{x\to 0} (1+\sin mx)^{n/x}$

40. $\lim_{x\to 0^+} (x)^{\sin(3x)}$

41. $\lim_{x\to 0^+} (e^{3x}-1)^{2/\ln x}$

42. $\lim_{x\to 0} \left(\dfrac{1}{x^2}-\dfrac{\cos 4x}{x^2}\right)$

43. $\lim_{x\to 0^+} \dfrac{\cot(ax)}{\cot(bx)}$

44. $\lim_{x\to +\infty} \dfrac{\ln x}{x}$

45. $\lim_{x\to 0^+} \dfrac{x}{\ln x}$

46. $\lim_{x\to 0^+} \left(\dfrac{1}{x}-\dfrac{2}{\ln x}\right)$

47. $\lim_{x\to +\infty} \dfrac{2x+3\sin x}{4x+2\sin x}$

48. $\lim_{x\to +\infty} x(b^{1/x}-1), b>0,\ b\neq 1$

49. $\lim_{h\to 0} \left(\dfrac{b^{x+h}-b^x}{h}\right), b>0, b\neq 1$

50. $\lim_{h\to 0} \dfrac{\log_b(x+h)-\log_b x}{h}, b>0, b\neq 1$

51. $\lim_{x\to 0} \dfrac{(e^x-1)\sin x}{\cos x-\cos^2 x}$

52. $\lim_{x\to +\infty} x\ln\left(\dfrac{x+1}{x-1}\right)$

53. $\lim_{x\to 0^+} \dfrac{\sin 5x}{1-\cos 4x}$

54. $\lim_{x\to 1} \dfrac{2x-3x^6+x^7}{(1-x)^3}$

55. $\lim_{x\to +\infty} e^x\ln\left(\dfrac{e^x+1}{e^x}\right)$

56. $\lim_{x\to 0} \dfrac{\tan x-\sin x}{x^3}$

57. $\lim\limits_{x\to 0} \dfrac{x^3 \sin 2x}{(1-\cos x)^2}$

58. $\lim\limits_{x\to 0} \dfrac{5^x - 3^x}{x^2}$

59. $\lim\limits_{x\to 0} \dfrac{1}{x} \ln\left(\dfrac{1+x}{1-x}\right)$

60. $\lim\limits_{x\to 0} \dfrac{\arctan x - x}{x^3}$

61. $\lim\limits_{x\to 0} \dfrac{\sin(\pi \cos x)}{x \sin x}$

62. $\lim\limits_{x\to +\infty} \dfrac{\ln(1+xe^{2x})}{x^2}$

63. $\lim\limits_{x\to +\infty} \dfrac{(\ln x)^n}{x}, n = 1, 2, \cdots$

64. $\lim\limits_{x\to +\infty} \dfrac{1}{\sqrt{x}} \ln\left(\dfrac{x+e^{2x}}{x}\right)$

65. $\lim\limits_{x\to +\infty} \dfrac{\ln x}{(1+x^3)^{1/2}}$

66. $\lim\limits_{x\to 0^+} \dfrac{\ln(\tan 3x)}{\ln(\tan 4x)}$

67. $\lim\limits_{x\to 0^+} (1-3^{-x})^{-2x}$

68. $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$

69. $\lim\limits_{x\to +\infty} (e^{-x} + e^{-2x})^{1/x}$

70. $\lim\limits_{x\to +\infty} \left(\cos\left(\dfrac{3}{x}\right)\right)^{x^2}$

71. $\lim\limits_{x\to 0^+} \left(\ln\left(\dfrac{1}{x}\right)\right)^{x}$

72. $\lim\limits_{x\to +\infty} \left(1+\dfrac{1}{2x}\right)^{x^2}$

73. $\lim\limits_{x\to +\infty} \left(1+\dfrac{1}{2x}\right)^{3x+\ln x}$

74. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin 2x}\right)$

75. $\lim\limits_{x\to +\infty} x\left(\sqrt{x^2+b^2} - x\right)$

76. $\lim\limits_{x\to 0} \left(\dfrac{1}{x \sin x} - \dfrac{1}{x^2}\right)$

77. $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{5}{x^2+x-6}\right)$

78. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \ln\left(\dfrac{1}{x}\right)\right)$

79. $\lim\limits_{x\to 0} \left(\cot x - \dfrac{1}{x}\right)$

80. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan^2 x}\right)$

81. $\lim_{x\to 0}\left(\frac{e^{-x}}{x}-\frac{1}{e^x-1}\right)$

82. $\lim_{x\to\infty}\frac{x-\sin x}{x}$

83. $\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x}$

84. $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

85. $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}$

86. $\lim_{x\to+\infty}\frac{\ln(\ln x)}{\ln(x-\ln x)}$

87. $\lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{x\ln x}\right)$

88. $\lim_{x\to+\infty}\frac{1}{x}\int_1^x\frac{\ln t}{1+t}\,dt$

89. $\lim_{x\to+\infty}(\ln(1+e^x)-x)$

90. $\lim_{x\to+\infty}\frac{1}{x^2}\left(\int_0^x\sin^2 x\;dx\right)$

91. Suppose that f is defined and differentiable in an open interval (a,b). Suppose that $a<c<b$ and $f''(c)$ exists. Prove that

$$f''(c)=\lim_{x\to c}\frac{f(x)-f(c)-(x-c)f'(c)}{((x-c)^2/2!)}.$$

92. Suppose that f is defined and $f', f'', \cdots, f^{(n-1)}$ exist in an open interval (a,b). Also, suppose that $a<c<b$ and $f^{(n)}(c)$ exists

(a) Prove that

$$f^{(n)}(c)=\lim_{x\to c}\frac{f(x)-f(c)-(x-c)f'(c)-\cdots-\frac{(x-c)^{n-1}}{(n-1)!}f^{n-1}(c)}{\frac{(x-c)^n}{n!}}.$$

(b) Show that there is a function $E_n(x)$ defined on (a,b), except possibly at c, such that

$$\begin{aligned}f(x)&=f(c)+(x-c)f'(c)+\cdots+\frac{(x-c)^{n-1}}{(n-1)!}\,f^{(n-1)}(x)\\&\quad+\frac{(x-c)^n}{n!}\,f^{(n)}(c)+E_n(x)\,\frac{(x-c)^nE_n(x)}{n!}\end{aligned}$$

and $\lim_{n\to c} E_n(x) = 0$. Find $E_2(x)$ if $c = 0$ and

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$$

(c) If $f'(c) = \cdots = f^{(n-1)}(c) = 0$, n is even, and f has a relative minimum at $x = c$, then show that $f^{(n)}(c) \geq 0$. What can be said if f has a relative maximum at c? What are the sufficient conditions for a relative maximum or minimum at c when $f'(c) = \cdots = f^{(n-1)}(c) = 0$? What can be said if n is odd and $f'(c) = \cdots = f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$.

93. Suppose that f and g are defined, have derivatives of order $1, 2, \cdots, n-1$ in an open interval $(a, b), a < c < b, f^{(n)}(c)$ and $g^{(n)}(c)$ exist and $g^{(n)}(c) \neq 0$. Prove that if f and g, as well as their first $n-1$ derivatives are 0, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

Evaluate the following limits:

94. $\lim_{x\to 0} \left(\dfrac{x^2 \sin \frac{1}{x}}{x}\right)$

95. $\lim_{x\to 0} \dfrac{\cos\left(\frac{\pi}{2} \cos x\right)}{\sin^2 x}$

96. $\lim_{x\to 1} x^{\left(\frac{1}{1-x}\right)}$

97. $\lim_{x\to 0^+} x(\ln(x))^n, n = 1, 2, 3, \cdots$

98. $\lim_{x\to 1^+} \dfrac{x^x - x}{1 - x + \ln x}$

99. $\lim_{x\to +\infty} \dfrac{x^{3/2} \ln x}{(1 + x^4)^{1/2}}$

100. $\lim_{x\to +\infty} x^n \ln\left(\dfrac{1 + e^x}{e^x}\right), \ n = 1, 2, \cdots$

101. $\lim_{x\to 0} \dfrac{x \int_0^x e^{-t^2} dx}{1 - e^{-x^2}}$

7.4 Improper Integrals

1. Suppose that f is continuous on $(-\infty, \infty)$ and $g'(x) = f(x)$. Then define each of the following improper integrals: