Questions Bank
Calculus II
First Class

Part 4

INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE

CHAPTER 13

THE RIEMANN INTEGRAL

13.1. Background

Topics: summation notation, Riemann sums, Riemann integral, upper and lower Darboux sums, definite and indefinite integrals.

Here are two formulas which may prove helpful.

13.1.1. Proposition. *For every natural number n*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\,.$$

13.1.2. Proposition. *For every natural number n*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

13.1.3. Definition. Let $J = [a, b]$ be a fixed interval in the real line and $P = (x_0, x_1, \dots, x_n)$ be $n + 1$ points of J. Then P is a PARTITION of the interval J if:

(1) $x_0 = a$,
(2) $x_n = b$, and
(3) $x_{k-1} < x_k$ for $k = 1, 2, \dots, n$.

We denote the length of the k^{th} subinterval by Δx_k; that is, $\Delta x_k = x_k - x_{k-1}$. A partition $P = (x_0, x_1, \dots, x_n)$ is REGULAR if all the subintervals $[x_{k-1}, x_k]$ have the same length. In this case

$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n$$

and we write Δx for their common value.

13.1.4. Notation. Let f be a bounded function defined on the interval $[a, b]$ and $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a, b]$. Then we define

$$\mathrm{R}(P) := \sum_{k=1}^{n} f(x_k)\,\Delta x_k \tag{13.1}$$

$$\mathrm{L}(P) := \sum_{k=1}^{n} f(x_{k-1})\,\Delta x_k \tag{13.2}$$

$$\mathrm{M}(P) := \sum_{k=1}^{n} f\big((\tfrac{1}{2}(x_{k-1} + x_k)\big)\,\Delta x_k. \tag{13.3}$$

These are, respectively, the RIGHT, LEFT, and MIDPOINT SUMS OF f ASSOCIATED WITH THE PARTITION P. If P is a *regular* partition of $[a, b]$ consisting of n subintervals, then we may write R_n for $R(P)$, L_n for $L(P)$, and M_n for $M(P)$.

13.1.5. Definition. The AVERAGE VALUE of a function f over an interval $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f$.

13.2. Exercises

(1) $\displaystyle\sum_{k=1}^{5} k^2 =$ __________ .

(2) $\displaystyle\sum_{k=3}^{10} 4 =$ ________ .

(3) $\displaystyle\sum_{k=1}^{100} k(k-3) =$ __________________ .

(4) $\displaystyle\sum_{m=1}^{200} m^3 - \sum_{m=1}^{199} m^3 =$ ________________ .

(5) $\displaystyle\sum_{k=1}^{4} (-1)^k k^k =$ _____________ .

(6) Let $a_k = 2^k$ for each k. Then $\displaystyle\sum_{k=3}^{8} (a_k - a_{k-1}) =$ _____________ .

(7) $\displaystyle\sum_{k=3}^{50} \frac{1}{k^2-k} = \frac{12}{a}$ where $a =$ ________ . *Hint.* Find numbers p and q such that $\dfrac{1}{k^2-k} = \dfrac{p}{k-1} - \dfrac{q}{k}$.

(8) Express $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + \dfrac{1}{243}$ in summation notation. Answer: $\displaystyle\sum_{k=0}^{a} b^k$ where $a =$ ___________ and $b =$ ________ .

(9) $\displaystyle\sum_{k=1}^{4} (k-1)\,k\,(k+1) =$ ________ .

(10) $\displaystyle\sum_{k=0}^{5} 3^{k+4} = \sum_{j=a}^{b} 3^j$ where $a =$ ________ and $b =$ ________ .

(11) $\displaystyle\sum_{k=7}^{60} \frac{1}{3^{k-2}} = \sum_{j=-2}^{a} \frac{1}{3^{j+b}}$ where $a =$ ________ and $b =$ ________ .

(12) $\displaystyle\sum_{j=-3}^{70} \frac{1}{5^{j-7}} = \sum_{i=a}^{61} \frac{1}{5^{i+b}}$ where $a =$ ________ and $b =$ ________ .

(13) $\displaystyle\sum_{j=-4}^{18} \frac{1}{2^{j+3}} = \sum_{k=a}^{7} 2^{k-b}$ where $a =$ ________ and $b =$ ______ .

(14) $\displaystyle\sum_{k=4}^{60} \frac{1}{k^2-1} = \frac{a}{3660}$ where $a =$ ___________ . *Hint.* Write $\dfrac{1}{k^2-1}$ as the difference of two simpler fractions.

(15) $\displaystyle\sum_{k=2}^{34} \frac{1}{k^2+2k} = \frac{a}{2520}$ where $a =$ ________ . *Hint.* Write $\dfrac{1}{k^2+2k}$ as the difference of two simpler fractions.

(16) Let $f(x) = x^2$ on the interval $[0,4]$ and let $P = (0,1,2,4)$. Find the right,,left, and midpoint sums of f associated with the partition P.

Answer: $\mathrm{R}(P) =$ ________ ; $\mathrm{L}(P) =$ ______ ; and $\mathrm{M}(P) = \dfrac{a}{2}$ where $a =$ ________ .

(17) Let $f(x) = x^3 - x$ on the interval $[-2,3]$ and let $P = (-2,0,1,3)$. Find the right, left, and midpoint sums of f associated with the partition P.

Answer: $\mathrm{R}(P) =$ ________ ; $\mathrm{L}(P) =$ ________ ; and $\mathrm{M}(P) = \dfrac{a}{8}$ where $a =$ ________ .

(18) Let $f(x) = 3 - x$ on the interval $[0,2]$ and let P_n be the regular partition of $[0,2]$ into n subintervals. Then

(a) $\mathrm{R}_n = a + \dfrac{b}{n}$ where $a =$ _____ and $b =$ _____ .

(b) $\mathrm{L}_n = c + \dfrac{d}{n}$ where $c =$ _____ and $d =$ _____ .

(c) $\int_0^2 f =$ _____ .

(19) Let $f(x) = 2x - 3$ on the interval $[0,4]$ and let P_n be the regular partition of $[0,4]$ into n subintervals. Then

(a) $\mathrm{R}_n = a + \dfrac{b}{n}$ where $a =$ _____ and $b =$ _____ .

(b) $\mathrm{L}_n = c + \dfrac{d}{n}$ where $c =$ _____ and $d =$ _____ .

(c) $\int_0^4 f =$ _____ .

(20) Let $f(x) = x - 2$ on the interval $[1,7]$ and let P_n be the regular partition of $[1,7]$ into n subintervals. Then

(a) $\mathrm{R}_n = a + \dfrac{b}{n}$ where $a =$ _____ and $b =$ _____ .

(b) $\mathrm{L}_n = c + \dfrac{d}{n}$ where $c =$ _____ and $d =$ _____ .

(c) $\int_1^7 f =$ _____ .

(21) If $\displaystyle\int_1^e \ln x\,dx = 1$ and $\displaystyle\int_1^{e^2} \ln x\,dx = 1 + e^2$, then $\displaystyle\int_e^{e^2} \ln x\,dx =$ ________ .

(22) Suppose that $\displaystyle\int_{-10}^{17} f = 3$, $\displaystyle\int_{-7}^{8} f = 7$, $\displaystyle\int_{-3}^{1} f = -1$, $\displaystyle\int_{-3}^{8} f = 4$, $\displaystyle\int_{-1}^{2} f = 5$, $\displaystyle\int_{-1}^{17} f = 6$, and $\displaystyle\int_1^2 f = 1$. Then $\displaystyle\int_{-10}^{-7} f =$ ________ .

(23) For what value of x is $\displaystyle\int_4^{\sqrt{x}} f(t)\,dt$ sure to be 0? Answer: ________ .

(24) Suppose $\displaystyle\int_{-2}^{3} f(x)\,dx = 8$. Then $\displaystyle\int_3^{-2} f(\Xi)\,d\Xi =$ ________ .

(25) Find the value of the integral $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ by regarding it as the area under the graph of an appropriately chosen function and using an area formula from plane geometry.
Answer: ________ .

(26) Find the value of the integral $\int_{-2}^{2} (4-|x|)\,dx$ by regarding it as the area under the graph of an appropriately chosen function and using area formulas from plane geometry.
Answer: ________ .

(27) Let $a > 0$. Then $\int_0^a (\sqrt{a^2-x^2} - a + x)\,dx = \frac{1}{b}a^p(c-2)$ where $b =$ _____ , $p =$ _____ , and $c =$ _____ . *Hint.* Interpret the integral as an area.

(28) If the average value of a continuous function f over the interval $[0,2]$ is 3 and the average value of f over $[2,7]$ is 4, then the average value of f over $[0,7]$ is $\frac{a}{7}$ where $a =$ _____ .

(29) Let $f(x) = |2 - |x-3||$. Then $\int_0^8 f(x)\,dx =$ _____ .

(30) Let $f(x) = \begin{cases} 2+\sqrt{2x-x^2}, & \text{for } 0 \le x \le 2 \\ 4-x, & \text{for } x > 2 \end{cases}$. Then

$\int_0^2 f(x)\,dx = a + \frac{\pi}{b}$ where $a =$ _____ and $b =$ _____ ;

$\int_0^4 f(x)\,dx = c + \frac{\pi}{d}$ where $c =$ _____ and $d =$ _____ ; and

$\int_1^6 f(x)\,dx = p + \frac{\pi}{q}$ where $p =$ _____ and $q =$ _____ .

(31) Suppose $\int_0^3 f(x)\,dx = 4$, $\int_2^5 f(x)\,dx = 5$, and $\int_2^3 f(x)\,dx = -1$. Then $\int_0^2 f(x)\,dx =$ ______ , $\int_0^1 f(x+2)\,dx =$ ______ , $\int_0^2 \big(f(x)+2\big)\,dx =$ ______ , $\int_2^5 f(x-2)\,dx =$ ______ , $\int_0^5 f(x)\,dx =$ ______ , $\int_5^7 5f(x-2)\,dx =$ ______ , and $\int_{-2}^3 f(x+2)\,dx =$ ______ .

(32) $\int_{-1}^4 (|x| + |x-2|)\,dx =$ ________ .

(33) $\int_0^3 (|x-1| + |x-2|)\,dx =$ _____ .

13.3. Problems

(1) Prove proposition 13.1.1.

(2) Prove proposition 13.1.2.

(3) Show that $\displaystyle\sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}$ for each n. *Hint.* Let $s_n = \displaystyle\sum_{k=1}^{n} 2^{-k}$ and consider the quantity $s_n - \frac{1}{2}s_n$.

(4) Let $f(x) = x^3 + x$ for $0 \le x \le 2$. Approximate $\int_0^2 f(x)\,dx$ using the *midpoint sum*. That is, compute, *and simplify*, the Riemann sum M_n for arbitrary n. Take the limit as $n \to \infty$ of M_n to find the value of $\int_0^2 f(x)\,dx$. Determine the smallest number of subintervals that must be used so that the error in the approximation M_n is less than 10^{-5}.

(5) Without evaluating the integral show that

$$\frac{7}{4} \le \int_{1/4}^{2} \left(\frac{4}{3}x^3 - 4x^2 + 3x + 1 \right) dx \le 3.$$

(6) Let $f(x) = x^2 \sin\dfrac{1}{x}$ if $0 < x \le 1$ and $f(0) = 0$. Show that $\left| \displaystyle\int_0^1 f \right| \le \dfrac{1}{3}$.

(7) Suppose that $a < b$. Prove that $\displaystyle\int_a^b \big(f(x) - c\big)^2\,dx$ is smallest when c is the average value of f over the interval $[a, b]$.

(8) Show that if f is a continuous function on $[a, b]$, then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

Hint. Suppose that d is a positive number and we wish to prove that $|c| < d$. All we need to do is establish two things: that $c < d$ and that $-c < d$.

(9) Show that $1 \le \displaystyle\int_0^1 e^{x^2}\,dx \le \frac{e+1}{2}$. *Hint.* Examine the concavity properties of the curve $y = e^{x^2}$.

(10) Let $0 \le x \le 1$. Apply the *mean value theorem* to the function $f(x) = e^x$ over the interval $[0, x]$ to show that the curve $y = e^x$ lies between the lines $y = 1 + x$ and $y = 1 + 3x$ whenever x is between 0 and 1. Use this result to find useful upper and lower bounds for the value of $\int_0^1 e^x\,dx$ (that is, numbers m and M such that $m \le \int_0^1 e^x\,dx \le M$).

(11) Show that $\displaystyle\int_a^b \left(\int_c^d f(x)\,g(y)\,dy \right) dx = \left(\int_a^b f \right) \left(\int_c^d g \right)$.

(12) Without evaluating the integral show that

$$\frac{\pi}{3} \le \int_0^{\pi} \sin x\,dx \le \frac{5\pi}{6}.$$

(13) Consider the function $f(x) = x^2 + 1$ defined on the closed interval $[0, 2]$. For each natural number n let $P_n = (x_0, x_1, \dots, x_n)$ be a regular partition of the interval $[0, 2]$ into n subintervals. Denote the length of the k^{th} subinterval by Δx_k. (Thus for a regular partition $\Delta x_1 = \Delta x_2 = \dots = \Delta x_n$.)

Definition. Let P_n be a regular partition of $[0, 2]$ as above. For each k between 1 and n let a_k be the point in the k^{th} subinterval $[x_{k-1}, x_k]$ where f has its smallest value and b_k

be the point in $[x_{k-1}, x_k]$ where f has its largest value. Then let

$$L(n) = \sum_{k=1}^{n} f(a_k)\Delta x_k \qquad \text{and} \qquad U(n) = \sum_{k=1}^{n} f(b_k)\Delta x_k.$$

The number $L(n)$ is the LOWER SUM associated with the partition P and $U(n)$ is the UPPER SUM associated with P.

(a) Let $n = 1$. (That is, we do *not* subdivide $[0, 2]$.) Find P_1, Δx_1, a_1, b_1, $L(1)$, and $U(1)$. How good is $L(1)$ as an approximation to $\int_0^2 f$?
(b) Let $n = 2$. Find P_2. For $k = 1, 2$ find Δx_k, a_k, and b_k. Find $L(2)$ and $U(2)$. How good is $L(2)$ as an approximation to $\int_0^2 f$?
(c) Let $n = 3$. Find P_3. For $k = 1, 2, 3$ find Δx_k, a_k, and b_k. Find $L(3)$ and $U(3)$. How good is $L(3)$ as an approximation to $\int_0^2 f$?
(d) Let $n = 4$. Find P_4. For $k = 1, 2, 3, 4$ find Δx_k, a_k, and b_k. Find $L(4)$ and $U(4)$. How good is $L(4)$ as an approximation to $\int_0^2 f$?
(e) Let $n = 8$. Find P_8. For $k = 1, 2, \dots, 8$ find Δx_k, a_k, and b_k. Find $L(8)$ and $U(8)$. How good is $L(8)$ as an approximation to $\int_0^2 f$?
(f) Let $n = 20$. Find P_{20}. For $k = 1, 2, \dots, 20$ find Δx_k, a_k, and b_k. Find $L(20)$ and $U(20)$. How good is $L(20)$ as an approximation to $\int_0^2 f$?
(g) Now let n be an arbitrary natural number. (Note: "arbitrary" means "unspecified".) For $k = 1, 2, \dots, n$ find Δx_k, a_k, and b_k. Find $L(n)$ and $U(n)$. Explain carefully why $L(n) \le \int_0^2 f \le U(n)$. How good is $L(n)$ as an approximation to $\int_0^2 f$?
(h) Suppose we wish to approximate $\int_0^2 f$ by $L(n)$ for some n and have an error no greater than 10^{-5}. What is the smallest value of n that our previous calculations guarantee will do the job?
(i) Use the preceding to calculate $\int_0^2 f$ with an error of less than 10^{-5}.

13.4. Answers to Odd-Numbered Exercises

(1) 55

(3) $323,200$

(5) 232

(7) 25

(9) 90

(11) 51, 7

(13) -15, 6

(15) 979

(17) 48, -12, 93

(19) (a) 4, 16
(b) 4, -16
(c) 4

(21) e^2

(23) 16

(25) $\dfrac{9\pi}{2}$

(27) 4, 2, π

(29) 9

(31) 5, -1, 9, 4, 10, 30, 10

(33) 5

CHAPTER 14

THE FUNDAMENTAL THEOREM OF CALCULUS

14.1. Background

Topics: *Fundamental theorem of Calculus*, differentiation of indefinite integrals, evaluation of definite integrals using antiderivatives.

The next two results are versions of the most elementary form of the *fundamental theorem of calculus*. (For a much more sophisticated version see theorem 46.1.1.)

14.1.1. Theorem (Fundamental Theorem Of Calculus - Version I)**.** *Let a belong to an open interval J in the real line and $f\colon J \to \mathbb{R}$ be a continuous function. Define $F(x) = \int_a^x f$ for all $x \in J$. Then for each $x \in J$ the function F is differentiable at x and $DF(x) = f(x)$.*

14.1.2. Theorem (Fundamental Theorem of Calculus - Version II)**.** *Let a and b be points in an open interval $J \subseteq \mathbb{R}$ with $a < b$. If $f\colon J \to \mathbb{R}$ is continuous and g is an antiderivative of f on J, then*

$$\int_a^b f = g(b) - g(a)\,.$$

The next proposition is useful in problem 5. It says that the only circumstance in which a differentiable function F can fail to be continuously differentiable at a point a is when either the right- or left-hand limit of $F'(x)$ *fails to exist* at a.

14.1.3. Proposition. *Let F be a differentiable real valued function in some open interval containing the point a. If $l := \lim_{x\to a^-} F'(x)$ and $r := \lim_{x\to a^+} F'(x)$ both exist, then*

$$F'(a) = r = l = \lim_{x\to a} F'(x).$$

Proof. Suppose that F is differentiable on the interval $(a-\delta, a+\delta)$. For $x \in (a, a+\delta)$ the *mean value theorem* guarantees the existence of a point $c \in (a, x)$ such that

$$\frac{F(x) - F(a)}{x - a} = F'(c).$$

Taking the limit as x approaches a from the right we get $F'(a) = r$. A nearly identical argument yields $F'(a) = l$. This shows that F is continuously differentiable at a. □

14.2. Exercises

(1) Evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{(n+2k)^4}{n^5}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\dfrac{1}{c}\displaystyle\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{q}{r}$ where $q =$ ________ and $r =$ _____ .

(2) Evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{1}{n+3k}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\dfrac{1}{c}\displaystyle\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{u}{v}\ln u$ where $u =$ _____ and $v =$ _____ .

(3) Evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{n}{n^2+k^2}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\int_a^b f(x)\,dx$ where $a =$ _____ , $b =$ _____ , and $f(x) =$ ___________ . The value of the integral is _____________ .

(4) Evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{n}{(2n+7k)^2}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\dfrac{1}{c}\displaystyle\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ ______ . The value of the integral is $\dfrac{1}{r}$ where $r =$ ________ .

(5) Evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{(2n+5k)^2}{n^3}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\dfrac{1}{c}\displaystyle\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{r}{3}$ where $r =$ ________ .

(6) Evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{(2n+4k)^2}{n^3}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\dfrac{1}{c}\displaystyle\int_a^b x^p\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $p =$ _____ . The value of the integral is $\dfrac{r}{3}$ where $r =$ ________ .

(7) Let $J = \int_0^5 \sqrt{3x}\,dx$ and let P be the regular partition of $[0,5]$ into n subintervals. Find the left, right and midpoint approximations to J determined by P.

Answer: $L_n = \dfrac{5}{n}\sum\limits_{k=p}^{q}\sqrt{\dfrac{15k}{n}}$ where $p =$ _____ and $q =$ ______ .

$R_n = \dfrac{5}{n}\sum\limits_{k=r}^{s}\sqrt{\dfrac{15k}{n}}$ where $r =$ _____ and $s =$ _____ .

$M_n = \dfrac{5}{n}\sum\limits_{k=1}^{n}\sqrt{\dfrac{tk-u}{vn}}$ where $t =$ ______ , $u =$ ______ , and $v =$ _____ .

(8) Evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\ln(5k+n)-\ln n}{n}$ by expressing it as an integral and then using the *fundamental theorem of calculus* to evaluate the integral. The integral is $\displaystyle\frac{1}{c}\int_a^b f(x)\,dx$ where $a =$ _____ , $b =$ _____ , $c =$ _____ , and $f(x) =$ ____________ . The value of the integral is $\displaystyle\frac{r}{s}f(r)-1$ where $r =$ _____ and $s =$ _____ .

(9) Let $\displaystyle g(x)=\int_3^{\frac{1}{2}x}\frac{t^3+4t+4}{1+t^2}\,dt$. Then $Dg(2)=\dfrac{a}{4}$ where $a =$ _____ .

(10) Let $g(x) = (5+7\cos^2(2\pi x)-\sin(4\pi x))^{-1}$ and $\displaystyle f(x)=\int_{x^3}^{2}g(t)\,dt$. Then $Df(\frac{1}{2}) = -\dfrac{1}{a}$ where $a =$ ______ .

(11) Let $g(x) = (1+(x^4+7)^{1/3})^{-1/2}$ and $\displaystyle f(x)=\int_{x}^{x^3}g(t)\,dt$. Then $Df(1)=\dfrac{2}{\sqrt{a}}$ where $a =$ _____ .

(12) Let $\displaystyle f(x)=\int_{x^2}^{\sin\pi x}\frac{dt}{1+t^4}$. Then $Df(2) = a-\dfrac{4}{b}$ where $a =$ _____ and $b =$ _________ .

(13) Let $\displaystyle g(x)=\int_0^x\frac{u-1}{u-2}\,du$. Then
   (a) the domain of g is (_____ , _____);
   (b) g is increasing on (_____ , _____); and
   (c) g is concave down on (_____ , _____).

(14) Solve for x: $\displaystyle\int_0^x(2u-1)^2\,du=\frac{14}{3}$. Answer: $x =$ _____ .

(15) Solve for x: $\displaystyle\int_x^{x+2}u\,du=0$. Answer: $x =$ ______ .

(16) Find a number $x>0$ such that $\displaystyle\int_1^x(u-1)\,du=4$. Answer: $x=1+a\sqrt{a}$ where $a =$ _____ .

(17) Find $\displaystyle\int_3^6 f'(x)\,dx$ given that the graph of f includes the points $(0,4)$, $(3,5)$, $(6,-2)$, and $(8,-9)$. Answer: ______ .

(18) Let $\displaystyle g(x)=\int_0^x xf(t)\,dt$ where f is a continuous function. Then

$$Dg(x) = \underline{\hspace{10cm}}\ .$$

(19) Let $\displaystyle f(x)=\int_0^x\frac{1-t^2}{3+t^4}\,dt$. Then f is increasing on the interval (______ , ______) and is concave up on the intervals (______ , ______) and (______ , ______) .

(20) If $\displaystyle y=\int_0^s\sqrt{2+u^3}\,du$, then $\dfrac{dy}{ds} =$ _______________ .

(21) If $\displaystyle y=\int_2^{t^2}\cos\sqrt{x}\,dx$ and $t\ge 0$, then $\dfrac{dy}{dt} =$ ________________ .

(22) If $\displaystyle\int_{-2}^{x}f(t)\,dt=x^2\sin(\pi x)$ for every x, then $f(1/3)=\dfrac{\pi}{a}+\dfrac{1}{b}$ where $a =$ _____ and $b =$ _____ .

(23) Let $f(x) = \displaystyle\int_{x\ln x}^{x^3} \frac{dt}{3+\ln t}$ for $x \geq 1$. Then $Df(e) = \dfrac{1}{a}(b^2-1)$ where $a =$ ____ and $b =$ ____ .

(24) Let $f(x) = \displaystyle\int_{\ln(x+1)^4}^{\ln(x^2+1)} \frac{dt}{4+e^t}$ for $x > 0$. Then $Df(1) = \dfrac{1}{a}$ where $a =$ ________ .

(25) Let $f(x) = \displaystyle\int_{-2}^{\frac{1}{2}x^2e^{x-1}} \frac{t^2}{(4+\sin \pi t)^2}\,dt$. Then $Df(1) = \dfrac{3}{a}$ where $a =$ __________ .

(26) Let $f(x) = \displaystyle\int_0^{xe^{x^2}} \frac{dt}{5+(\ln t)^2}$. Then $Df(1) = \dfrac{a}{b}$ where $a =$ ____ and $b =$ ____ .

(27) Let $f(x) = \displaystyle\int_0^{e^{x^3}} \frac{dt}{6+(\ln t)^2}$. Then $Df(2) = \dfrac{6e^a}{b}$ where $a =$ ____ and $b =$ ______ .

(28) Let $f(x) = \displaystyle\int_{\ln x}^{\ln(x^2+3)} \frac{dt}{3+e^t}$ for $x \geq 1$. Then $Df(2) = -\dfrac{3}{a}$ where $a =$ ________ .

(29) Let $f(x) = (x^2+2x+2)^{-1}$ for all $x \in \mathbb{R}$. Then the interval on which the curve $y = \int_0^x f(t)\,dt$ is concave up is (______ , ______).

(30) $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+x^2}\,dx =$ ______ .

(31) $\displaystyle\lim_{\lambda\to 0^+} \int_\lambda^{2\lambda} e^{-x}x^{-1}\,dx =$ __________ . *Hint:* $\dfrac{e^{-x}}{x} = \dfrac{e^{-x}-1}{x} + \dfrac{1}{x}$.

(32) $\displaystyle\lim_{x\to 0} \frac{1}{x}\int_1^{1+5x} (4-\cos 2\pi t)^3\,dt =$ __________ .

(33) $\displaystyle\lim_{r\to 0} \frac{1}{r}\int_1^{e^{4r}} \sqrt{3+\frac{1}{x}}\,dx =$ ____ .

(34) $\displaystyle\lim_{u\to 0} \frac{1}{u}\int_2^{\ln(e^2+3u)} \sqrt{1+2t+5t^2}\,dt = ae^b$ where $a =$ ______ and $b =$ ______ .

(35) $\displaystyle\int_1^9 \frac{1}{x^{3/2}} = \frac{a}{3}$ where $a =$ ____ .

(36) $\displaystyle\int 12e^{4x}\,dx = ae^{bx} + c$ where $a =$ ____ , $b =$ ____ , and c is an arbitrary constant.

(37) $\displaystyle\int 40\cos 5x\,dx = a\sin bx + c$ where $a =$ ____ , $b =$ ____ , and c is an arbitrary constant.

(38) $\displaystyle\int_0^1 \frac{4}{\sqrt{4-x^2}}\,dx = \frac{a}{3}$ where $a =$ ________ .

(39) $\displaystyle\int_0^{\sqrt{3}} \frac{6}{9+x^2}\,dx =$ ____________ .

(40) $\displaystyle\int_0^{\pi/2} \cos x e^{\sin x}\,dx = a - b$ where $a =$ ______ and $b =$ ____ .

(41) $\displaystyle\int_0^{\pi} \sec^2 \tfrac{1}{4}x\,dx =$ ____ .

(42) If $a = 0$ and $b = \frac{1}{5}(e-1)$, then $\displaystyle\int_a^b \frac{15}{5x+1}\,dx =$ _____ .

(43) Let $f(x) = |x| + |\cos x|$ for all x. Then $\displaystyle\int_{-\pi/2}^{\pi} f = a + \frac{b}{8}\pi^p$ where $a =$ _____ , $b =$ _____ , and $p =$ _____ .

(44) $\displaystyle\int_{-1}^{2} \left|x^3 - x\right|\,dx = \frac{a}{4}$ where $a =$ ________ .

(45) $\displaystyle\int_0^{2\pi} (|\sin x| + \cos x)\,dx =$ _____ .

(46) $\displaystyle\int_0^{\pi/4} \sin^5 x\,\cos x\,dx = \frac{1}{a}$ where $a =$ ________ .

(47) $\displaystyle\int_1^3 (x^3 - 6x^2 + 2x - 7)\,dx =$ ___________ .

(48) $\displaystyle\int_0^4 (x^3 + 3\sqrt{x})\,dx =$ __________ .

(49) $\displaystyle\int_0^3 (5 - 2x^2)\,dx =$ ________ .

(50) $\displaystyle\int_1^5 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = a + \ln b$ where $a =$ ________ and $b =$ _____ .

(51) $\displaystyle\int_{\pi/6}^{\pi/2} \csc^2 x\,dx =$ ________ .

(52) $\displaystyle\int_0^5 \frac{dx}{25 + x^2} = \frac{\pi}{a}$ where $a =$ ________ .

(53) $\displaystyle\int_0^1 (x^3 + x)e^{x^4+2x^2}\,dx = \frac{1}{a}(e^p - 1)$ where $a =$ _____ and $p =$ _____ .

(54) Let $f(x) = \displaystyle\int_{3\pi}^{x} (7 + \cos(\sin t))\,dt$. Then $Df^{-1}(0) = \dfrac{1}{a}$ where $a =$ _____ .

(55) Let $f(x) = \displaystyle\int_{\pi/3}^{x^{1/3}} \arctan(2 + 2\sin t)\,dt$ for $x \geq 0$. Then $Df^{-1}(0) = \dfrac{4\pi}{a}$ where $a =$ _____ .
Hint. What is $\tan\frac{5\pi}{12}$?

(56) If $\log_2 x = \displaystyle\int_2^x \frac{1}{t}\,dt$, then $x = \exp\left(\dfrac{a^2}{a-1}\right)$ where $a =$ ________ .

14.3. Problems

(1) Estimate $\displaystyle\sum_{k=1}^{10^4} \sqrt{k}$ by interpreting it as a Riemann sum for an appropriate integral.

(2) Let $f(x) = x^3 + x$, let n be an arbitrary natural number, and let $P = (x_0, x_1, \dots x_n)$ be a regular partition of the interval $[0, 2]$ into n subintervals. (Note: "arbitrary" means "unspecified".) For each k between 1 and n let c_k be the midpoint of the k^{th} subinterval $[x_{k-1}, x_k]$.
  (a) Find the width Δx_k of each subinterval.
  (b) Find x_k for each $k = 0, \dots, n$.
  (c) Find c_k for each $k = 1, \dots, n$.
  (d) Find the corresponding Riemann midpoint sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$. Simplify the expression and put it in the form $a + b/n + c/n^2 + \dots$.
  (e) Find the limit of the Riemann sums in part (d) as $n \to \infty$.
  (f) Compute $\int_0^2 f$ using the *fundamental theorem of calculus*
  (g) What is the smallest number of subintervals we can use so that the Riemann sum found in (d) approximates the true value of the integral found in (f) with an error of less than 10^{-5}?

(3) Let $f(x) = -\frac{1}{2}x + \frac{3}{2}$ for $-1 \le x \le 3$. Partition the interval $[-1, 3]$ into n subintervals of equal length. Write down the corresponding right approximating sum R_n. Show how this expression can be simplified to the form $a + \frac{b}{n}$ for appropriate numbers a and b. Take the limit of this expression as n gets large to find the value of $\int_{-1}^{3} f(x)\,dx$. Check your answer in two different ways: using a geometrical argument and using the *fundamental theorem of calculus.*

(4) Let $f(x) = x^2 + 1$ for $0 \le x \le 3$. Partition the interval $[0, 3]$ into n subintervals of equal length. Write down the corresponding right approximating sum R_n. Show how this expression can be simplified to the form $a + \frac{b}{cn} + \frac{d}{cn^2}$ for appropriate numbers a, b, c, and d. Take the limit of this expression as n gets large to find the value of $\int_0^3 f(x)\,dx$. Check your answer using the *fundamental theorem of calculus.*

(5) Define functions f, g, and h as follows:

$$h(x) = \begin{cases} 1, & \text{for } 0 \le x \le 2 \\ x, & \text{for } 2 < x \le 4. \end{cases}$$

$$g(x) = \int_x^{x^2} h(t)\,dt \qquad \text{for } 0 \le x \le 2$$

$$f(x) = \int_0^x g(t)\,dt \qquad \text{for } 0 \le x \le 2.$$

  (a) For each of the functions h, g, and f answer the following questions:
    (i) Where is the function continuous? differentiable? twice differentiable?
    (ii) Where is the function positive? negative? increasing? decreasing? concave up? concave down?
    (iii) Where are the x-intercepts? maxima? minima? points of inflection?
  (b) Make a careful sketch of the graph of each of the functions.
  (c) What is the moral of this problem? That is, what do these examples suggest about the process of integration in general?

Hints for solution. When working with the first function h it is possible to get the "right answers" to questions (i)—(iii) but at the same time fail to give coherent reasons for the assertions made. This part of the problem is meant to encourage paying attention to the precise definitions of some of the terms. Indeed, the correct answers will vary from text to text. Some texts, for instance, distinguish between functions that are *increasing* and those that are *strictly increasing*. Other texts replace these terms by *nondecreasing* and *increasing*, respectively. Some texts define *concavity* only for functions which are twice differentiable; others define it in terms of the first derivative; still others define it geometrically.

This first part of the problem also provides an opportunity to review a few basic facts: differentiability implies continuity; continuity can be characterized (or defined) in terms of limits; and so on.

Unraveling the properties of the second function g is rather harder. Try not to be put off by the odd looking definition of g. The crucial insight here is that by carrying out the indicated integration it is possible to express g, at least piecewise, as a polynomial. From a polynomial expression it is a simple matter to extract the required information. Impatience at this stage is not a reliable friend. It is not a good idea to try to carry out the integration before you have thought through the problem and discovered the necessity of dividing the interval into two pieces. It may be helpful to compute the values of g at $x = 1.0, 1.1, 1.2, \dots, 1.9, 2.0$. Notice that about midway in these computations something odd happens. What is the precise point p where things change? Eventually one sees that g too is expressible as one polynomial on $[0, p]$ and as another polynomial on $(p\,, 2]$. Once g has been expressed piecewise by polynomials it is possible to proceed with questions (i)–(iii). To determine whether g is continuous at p, compute the right- and left-hand limits of g there.

The question of the differentiability of g is subtle and deserves some serious thought. It may be tempting to carry over the format of continuity argument to decide about the differentiability of g at p. Suppose we compute the right- and left-hand limits of the derivative of g at p and find that they are not equal. Can we then conclude that g is not differentiable at p? At first one is inclined to say *no*, that all we have shown is that the derivative of g is not continuous at p, which does not address the issue of the *existence* of $g'(p)$. Interestingly enough, it turns out that the only way in which a differentiable function F can fail to be continuously differentiable at a point a is for either the right- or left-hand limit of $F'(x)$ *to fail to exist* at a. The crucial result, which is a bit hard to find in beginning calculus texts, is proposition 14.1.3. Thus when we discover that a function F is differentiable at all points other than a, and that the limits $\lim_{x\to a^-} F'(x)$ and $\lim_{x\to a^+} F'(x)$ both exist but fail to be equal, there is only one possible explanation: F fails to be differentiable at a.

After finding a piecewise polynomial expression for g, another difficulty arises in determining whether g is concave up. It is easy to see that g is concave up on the intervals $(0, p)$ and $(p\,, 2)$. But this isn't enough to establish the property for the entire interval $(0, 2)$. In fact, according to the definition of concavity given in many texts g is *not* concave up. Why? Because, according to Finney and Thomas (see [2], page 237), for example, concavity is defined only for differentiable functions. A function is concave up on an interval only if its derivative is increasing on the interval. So if our function g fails to be differentiable at some point it can not be concave up. On the other hand, under any reasonable geometric definition of concavity g certainly *is* concave up on $(0, 2)$—although it is a bit hard to show. The solution to this dilemma is straightforward: pick a definition and stick to it.

Analysis of the last function f proceeds pretty much as for g. One new wrinkle is the difficulty in determining where f is positive. The point at which f changes sign is a root of a fifth degree polynomial. An approximation based either on the *intermediate value theorem* or *Newton's method* goes smoothly.

As with g, conclusions concerning the concavity of f may differ depending on the definitions used. This time both a geometrical definition and one based on the first derivative lead to one conclusion while a definition based on the second derivative leads to another.

Finally, for part (c) does it make any sense to regard integration as a "smoothing" operation? In what way?

(6) Show that if f is continuous, then

$$\int_0^x f(u)(x-u)\,du = \int_0^x \int_0^u f(t)\,dt\,du.$$

Hint. What can you say about functions F and G if you know that $F'(x) = G'(x)$ for all x and that $F(x_0) = G(x_0)$ at some point x_0?

(7) Let f be a continuous function and $a < b$. Show that $\displaystyle\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$.
Hint. Show that if F is an antiderivative of f, then the function $G\colon x \mapsto -F(-x)$ is an antiderivative of the function $g\colon x \mapsto f(-x)$.

(8) Let $a < b$, f be a continuous function defined on the interval $[a, b]$, and g be the function defined by $g(t) = \displaystyle\int_a^b (f(x) - t)^2\,dx$ for t in $\mathbb{R}$. Find the value for t at which g assumes a minimum. How do you know that this point is the location of a minimum (rather than a maximum)?

(9) Let λ be a positive constant. Define $F(x) = \displaystyle\int_x^{\lambda x} \frac{1}{t}\,dt$ for all $x > 0$. *Without mentioning logarithms* show that F is a constant function.

(10) Without computing the integrals give a simple *geometric* argument that shows that the sum of $\int_0^1 \sqrt{x}\,dx$ and $\int_0^1 x^2\,dx$ is 1. Then carry out the integrations.

14.4. Answers to Odd-Numbered Exercises

(1) 1, 3, 2, 4, 121, 5

(3) 0, 1, $\dfrac{1}{1+x^2}$, $\dfrac{\pi}{4}$

(5) 2, 7, 5, 2, 67

(7) 0, $n-1$, 1, n, 30, 15, 2

(9) 9

(11) 3

(13) (a) $-\infty$, 2
(b) $-\infty$, 1
(c) $-\infty$, 2

(15) -1

(17) -7

(19) -1, 1, $-\sqrt{3}$, 0, $\sqrt{3}$. ∞

(21) $2t\cos t$

(23) 2, e

(25) 200

(27) 8, 35

(29) $-\infty$, -1

(31) $\ln 2$

(33) 8

(35) 4

(37) 8, 5

(39) $\dfrac{\pi}{3}$

(41) 4

(43) 3, 5, 2

(45) 4

(47) -38

(49) -3

(51) $\sqrt{3}$

(53) 4, 3

(55) 5

CHAPTER 15

TECHNIQUES OF INTEGRATION

15.1. Background

Topics: antiderivatives, change of variables, trigonometric integrals, trigonometric substitutions, integration by parts, partial fractions, improper Riemann integrals.

15.2. Exercises

(1) $\displaystyle\int_0^1 \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{4}}}\,dx = \frac{a}{3}(1-\ln b)$ where $a =$ _____ and $b =$ _____.

(2) $\displaystyle\int_0^9 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = a + 4\ln b$ where $a =$ _____ and $b =$ _____.

(3) $\displaystyle\int_{\frac{1}{3\sqrt{3}}}^{3\sqrt{3}} \frac{1}{x^{\frac{4}{3}}+x^{\frac{2}{3}}}\,dx =$ ________.

(4) $\displaystyle\int_0^{1/2} \frac{\arctan 2x}{1+4x^2}\,dx = \frac{\pi^2}{a}$ where $a =$ _____.

(5) $\displaystyle\int_0^{\sqrt{2}} x\,10^{1+x^2}\,dx = \frac{a}{\ln 10}$ where $a =$ __________ .

(6) $\displaystyle\int_1^{16} \frac{x-1}{x+\sqrt{x}}\,dx =$ _______ .

(7) $\displaystyle\int_1^9 \frac{dx}{(x+1)\sqrt{x}+2x} = \frac{1}{a}$ where $a =$ _____ .

(8) $\displaystyle\int_{27/8}^8 \frac{2\,dx}{x^{5/3}-3x^{4/3}+3x-x^{2/3}} =$ _____ .

(9) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \frac{a}{5} - \frac{b}{10\sqrt{2}}$ where $a =$ _____ and $b =$ _______ .

(10) $\displaystyle\int_0^{\pi/8} \tan 2x \sec^2 2x\,dx = \frac{1}{a}$ where $a =$ _____ .

(11) $\displaystyle\int_1^{4/3} \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx = \frac{1}{a}$ where $a =$ ________ .

(12) $\displaystyle\int_1^4 \frac{4x-1}{2x+\sqrt{x}}\,dx =$ _____ .

(13) $\displaystyle\int_0^{1/\sqrt{2}} x\sin^3(\pi x^2)\cos(\pi x^2)\,dx = \frac{1}{a}$ where $a =$ __________ .

(14) $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{(5+\tan x)^2}\,dx = \frac{1}{a}$ where $a =$ ________ .

(15) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{a}{3}$ where $a =$ _____ .

(16) $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{4x^2+9}} = \frac{1}{a}$ where $a =$ _____ .

(17) $\displaystyle\int_1^2 \frac{2x^2\,dx}{(x^3+1)^2} = \frac{a}{27}$ where $a =$ _____ .

(18) $\displaystyle\int_1^{\sqrt{10}} x\sqrt{x^2-1}\,dx =$ _____ .

(19) $\displaystyle\int_4^9 \frac{x-9}{3\sqrt{x}+x}\,dx =$ ________ .

(20) $\displaystyle\int \frac{r^5\,dr}{\sqrt{4-r^6}} = -\frac{1}{a}\sqrt{4-r^6} + c$ where $a =$ _____ and c is an arbitrary constant.

(21) $\displaystyle\int_0^{\pi/4} \frac{\tan^3 x \sec^2 x}{\left(1+\tan^4 x\right)^3}\,dx = \frac{3}{a}$ where $a =$ ________ .

(22) $\displaystyle\int_0^{\sqrt{3}} \frac{x\,dx}{x^2-4} = -\ln a$ where $a =$ _____ .

(23) $\displaystyle\int_1^3 \frac{dx}{x^{1/2}+x^{3/2}} = \frac{\pi}{a}$ where $a =$ _____ .

(24) $\displaystyle\int_e^{e^e} \frac{1}{x\ln x(1+(\ln\ln x)^2)}\,dx = \frac{\pi}{a}$ where $a =$ _____ .

(25) $\displaystyle\int_{1/8}^{\frac{1}{2\sqrt{2}}} \frac{dx}{\sqrt{x^{4/3}-x^2}} = \frac{a}{4}$ where $a =$ _____ .

(26) $\displaystyle\int_{-5}^{\sqrt{3}-5} \frac{dx}{\sqrt{-x^2-10x-21}} = \frac{\pi}{a}$ where $a =$ _____ .

(27) $\displaystyle\int_{-3/2}^{0} \frac{dx}{4x^2+12x+18} = \frac{\pi}{a}$ where $a =$ ________ .

(28) $\displaystyle\int_{\arctan e}^{\arctan e^3} \frac{\csc 2x}{\ln(\tan x)}\,dx = \frac{1}{2}\ln a$ where $a =$ _____ .

(29) $\displaystyle\int_0^1 \frac{(\arctan x)^2}{1+x^2}\,dx = \frac{\pi^p}{a}$ where $p =$ _____ and $a =$ ___________ .

(30) $\displaystyle\int_0^{\ln 3} \frac{e^{x/2}}{1+e^x}\,dx = \frac{\pi}{a}$ where $a =$ _____ .

(31) $\displaystyle\int_0^{1/2} \frac{3\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{\pi^p}{a}$ where $p =$ _____ and $a =$ ________ .

(32) $\displaystyle\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} = \frac{\pi}{a}$ where $a =$ _____ .

(33) $\displaystyle\int_0^1 (x+2)e^{x^2+4x}\,dx = \frac{1}{a}\left(e^p-1\right)$ where $a =$ _____ and $p =$ _____ .

(34) $\displaystyle\int_{\sqrt{\ln\frac{\pi}{2}}}^{\sqrt{\ln \pi}} xe^{x^2}\cos(3e^{x^2})\,dx = \frac{1}{a}$ where $a =$ _____ .

(35) $\displaystyle\int_{\ln 2}^{\ln 8} \frac{1-e^x}{1+e^x}\,dx = a\ln\frac{a}{b}$ where $a =$ _____ and $b =$ _____ .

(36) $\displaystyle\int_0^1 x\,5^{-x^2}\,dx = \frac{a}{b\ln b}$ where $a =$ _____ and $b =$ _____ .

(37) $\displaystyle\int_e^{e^4} \frac{\log_7 x}{x}\,dx = \frac{a}{2\ln 7}$ where $a =$ ________ .

(38) $\displaystyle\int_{e^2}^{e^3} x^{-1}(\log_3 x)^2\,dx = \frac{a}{b(\ln b)^2}$ where $a =$ ________ and $b =$ _____ .

(39) $\displaystyle\int_0^{\pi/2} \sin^7 u\,du = \frac{16}{a}$ where $a =$ ________ .

(40) $\displaystyle\int_0^{\pi/3} \sec^5 x \tan x\,dx = \frac{a}{5}$ where $a =$ ________ .

(41) $\displaystyle\int_0^{\pi/4} \sec^3 x\,dx = a + \tfrac{1}{2}\ln b$ where $a =$ ________ and $b =$ ___________ .

(42) $\displaystyle\int_0^{\pi/3} \tan^3 x \sec^3 x\,dx = \frac{a}{15}$ where $a =$ ________ .

(43) $\displaystyle\int_0^{\pi/2} \sin^4 x\,dx = \frac{3\pi}{a}$ where $a =$ ________ .

(44) $\displaystyle\int_0^{\pi/2} \cos^3 x \sin^5 x\,dx = \frac{1}{a}$ where $a =$ ________ .

(45) $\displaystyle\int_0^{\pi/2} \sin^4 x\, \cos^5 x\,dx = \frac{8}{a}$ where $a =$ _________ .

(46) $\displaystyle\int_0^{\pi/3} \tan^3 x\,dx = a - \ln b$ where $a =$ ________ and $b =$ _____ .

(47) $\displaystyle\int_0^{\pi} \sin^6 x\,dx = \frac{a\pi}{16}$ where $a =$ _____ .

(48) $\displaystyle\int_0^{\pi/3} \sec^6 x\,dx = \frac{a}{5}\sqrt{3}$ where $a =$ ________ .

(49) $\displaystyle\int_0^{3/4} \frac{dx}{\sqrt{9-4x^2}} =$ _____________ .

(50) $\displaystyle\int_{1/\sqrt{2}}^{1} \frac{dx}{x\sqrt{4x^2-1}} = \frac{\pi}{a}$ where $a =$ ________ .

(51) $\displaystyle\int \frac{dx}{\sqrt{8-4x-4x^2}} = a\ \arcsin\big(\tfrac{1}{3}f(x)\big) + c$ where $a =$ ______ , $f(x) =$ __________ , and c is an arbitrary constant.

(52) $\displaystyle\int \frac{dx}{x^2+2x+5} = a\ \arctan(a\,f(x)) + c$ where $a =$ ______ , $f(x) =$ __________ , and c is an arbitrary constant.

(53) $\displaystyle\int_1^2 \frac{dx}{x(1+x^4)} = \frac{1}{4}\ln\frac{32}{a}$ where $a =$ ________ .

(54) $\displaystyle\int_1^{\sqrt{13}} \frac{dx}{x^2\sqrt{3+x^2}} = a\left(1 - \frac{b}{\sqrt{13}}\right)$ where $a =$ ________ and $b =$ _____ .

(55) $\displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx = a\sqrt{3} - b$ where $a =$ _____ and $b =$ _____ .

(56) $\displaystyle\int_{3\sqrt{2}}^{6} \frac{dx}{\sqrt{x^2-9}} = \ln(a+\sqrt{b}) - \ln(1+\sqrt{2})$ where $a =$ _____ and $b =$ _____ .

(57) $\displaystyle\int_0^3 \frac{dw}{\sqrt{9+w^2}} = \ln(a+\sqrt{b})$ where $a =$ _____ and $b =$ _____ .

(58) $\displaystyle\int \frac{x^2}{\left(4-x^2\right)^{3/2}}\,dx = \frac{x}{f(x)} - g(x/2) + c$ where $f(x) =$ ____________ , $g(x) =$ ____________ , and c is an arbitrary constant.

(59) $\displaystyle\int_0^{\sqrt{3}} \arctan x\,dx = \frac{a}{\sqrt{3}} - \ln b$ where $a =$ _____ and $b =$ _____ .

(60) $\displaystyle\int_0^{\sqrt{3}} x\arctan x\,dx = \frac{a\pi}{b} - \frac{1}{a}\sqrt{b}$ where $a =$ _____ and $b =$ _____ .

(61) $\displaystyle\int_1^{2} x^3 \ln x\,dx = a\ln 2 - \frac{b}{16}$ where $a =$ _____ and $b =$ ________ .

(62) $\displaystyle\int_1^{e} x^2 \ln x\,dx = \frac{1}{a}(be^p + 1)$ where $a =$ _____ , $b =$ _____ , and $p =$ _____ .

(63) $\displaystyle\int x^2 e^x\,dx = p(x)e^x + c$ where $p(x) =$ _______________ and c is an arbitrary constant.

(64) $\displaystyle\int_0^{1} \arctan x\,dx = \frac{\pi}{a} - \frac{1}{b}\ln b$ where $a =$ _____ and $b =$ _____ .

(65) $\displaystyle\int_0^{1} \operatorname{arccot} x\,dx = \frac{a}{4} + \frac{1}{b}\ln b$ where $a =$ _____ and $b =$ _____ .

(66) $\displaystyle\int_0^{\pi/6} x\sin x\,dx = \frac{a - \sqrt{b}\,\pi}{12}$ where $a =$ _____ and $b =$ _____ .

(67) $\displaystyle\int x^2 \cos x\,dx = f(x)\sin x + g(x)\cos x + c$ where $f(x) =$ ___________ , $g(x) =$ ________ , and c is an arbitrary constant.

(68) Expand $\dfrac{x^2+2x-2}{x^3(x-1)}$ by partial fractions.

Answer: $\dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x^3} + \dfrac{d}{x-1}$ where $a =$ ______ , $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(69) Expand $\dfrac{x^3+x^2+7}{x^2+x-2}$ by partial fractions.

Answer: $f(x) + \dfrac{a}{x-1} + \dfrac{b}{x+2}$ where $f(x) =$ _____ , $a =$ _____ and $b =$ ______ .

(70) $\displaystyle\int_1^{2} \frac{2}{w^3+2w}\,dw = \frac{1}{a}\ln a$ where $a =$ _____ .

(71) $\displaystyle\int_{-3}^{0} \frac{-2w^3+w^2+2w+13}{w^2+2w+3}\,dw = a + \ln b$ where $a =$ ________ and $b =$ _____ .

(72) $\displaystyle\int \frac{3x^2+x+6}{x^4+3x^2+2}\,dx = -a\ln(x^2+2) + a\ln(g(x)) + 3\arctan x + c$ where $a =$ ________ , $g(x) =$ ____________ , and c is an arbitrary constant.

(73) $\displaystyle\int \frac{1-4x-3x^2-3x^3}{x^4+x^3+x^2}\,dx = \frac{a}{x} - 5\ln x + \ln(g(x)) + c$ where $a =$ ______ , $g(x) =$ __________ , and c is an arbitrary constant.

(74) $\displaystyle\int \frac{4x^3-2x^2+x}{x^4-x^3-x+1}\,dx = f(x) + 2g(x) + \ln(x^2+x+1) + c$ where $f(x) =$ ____________ , $g(x) =$ ____________ , and c is an arbitrary constant.

(75) $\displaystyle\int \frac{x^2+3}{x^3+x}\,dx = a\ln x - \ln(g(x)) + c$ where $a =$ _____ , $g(x) =$ __________ , and c is an arbitrary constant.

(76) $\displaystyle\int \frac{2x^2-3x+9}{x^3-3x^2+7x-5}\,dx = a\ln(x-1) + \frac{1}{a}\arctan\left(\frac{1}{a}\,g(x)\right) + c$ where $a =$ ______ , $g(x) =$ ____________ , and c is an arbitrary constant.

(77) $\displaystyle\int \frac{2x^3+x^2+2x-1}{x^4-1}\,dx = \ln(x-1) + \ln(x+1) + f(x) + c$ where $f(x) =$ _____________ and c is an arbitrary constant.

(78) $\displaystyle\int \frac{x^5-2x^4+x^3-3x^2+2x-5}{x^3-2x^2+x-2}\,dx = g(x) + a\ln(x-2) + b\arctan x + c$ where $a =$ ______ , $b =$ _____ , $g(x) =$ _______________ , and c is an arbitrary constant.

(79) $\displaystyle\int \frac{x^7+9x^5+2x^3+4x^2+9}{x^4+9x^2}\,dx = f(x) - \frac{1}{x} + \ln(g(x)) + \arctan(h(x)) + c$ where $f(x) =$ __________ , $g(x) =$ __________ , $h(x) =$ ________ , and c is an arbitrary constant.

(80) $\displaystyle\int_0^{\pi/2} \frac{dx}{8+4\sin x+7\cos x} = \ln\left(\frac{a}{9}\right)$ where $a =$ ______ . *Hint.* Try substituting $u = \tan\frac{x}{2}$.

(81) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx = \frac{\pi}{a}$ where $a =$ _____ . *Hint.* Try substituting $u = \tan\frac{x}{2}$.

(82) $\displaystyle\int \frac{2x^2+9x+9}{(x-1)(x^2+4x+5)}\,dx = a\ln(x-1) + f(x) + c$ where $a =$ _____ , $f(x) =$ _______________ , and c is an arbitrary constant.

(83) $\displaystyle\int \frac{2x^4+x^3+4x^2+2}{x^5+2x^3+x}\,dx = a\,f(x) + \frac{1}{a}\arctan x - \frac{1}{a}g(x) + c$ where $a =$ ______ , $f(x) =$ __________ , $g(x) =$ _______________ , and c is an arbitrary constant.

(84) $\displaystyle\int \frac{5u^2+11u-4}{u^3+u^2-2u}\,du = a\ln|u| + b\ln|u-1| - \ln|u+2| + c$ where $a =$ _____ , $b =$ _____ , and c is an arbitrary constant.

(85) $\displaystyle\int_0^{\pi/2} (\cot x - x\csc^2 x)\,dx =$ ______ .

(86) $\displaystyle\int_0^{e} x^2\ln x\,dx = ae^p$ where $a =$ ______ and $p =$ _____ .

(87) $\displaystyle\int_2^4 \frac{x\,dx}{\sqrt{|9-x^2|}} = \sqrt{5} + \sqrt{a}$ where $a =$ _____ .

(88) $\displaystyle\int_0^\infty e^{-x}\sin x\,dx = \frac{1}{a}$ where $a =$ _____ .

(89) If we choose $k =$ _____ , then the improper integral $\displaystyle\int_0^\infty \left(\frac{k}{3x+1} - \frac{2x}{x^2+1}\right)dx$ converges. In this case the value of the integral is $2\ln a$ where $a =$ _____ .

(90) $\displaystyle\int_2^4 \frac{1}{\sqrt{2x-4}}\,dx =$ _____ .

(91) $\displaystyle\int_{e^2}^\infty \frac{dx}{x(\ln x)^2} = \frac{1}{a}$ where $a =$ _____ .

(92) Does the improper integral $\int_0^8 x^{-1/3}\,dx$ converge? Answer: ________ . If it converges, its value is _____ .

(93) Does the improper integral $\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{2x-1}}\,dx$ converge? Answer: ________ . If it converges, its value is ________ .

(94) Does the improper integral $\int_{-1}^1 \frac{1}{x^2}\,dx$ converge? Answer: ________ . If it converges, its value is _____ .

(95) Does the improper integral $\int_{\frac{2}{3}}^1 \frac{1}{3x-2}\,dx$ converges? Answer: ________ . If it converges, its value is ________ .

(96) Does the improper integral $\int_0^\infty \frac{1}{1+9x^2}\,dx$ converge? Answer: ________ . If it converges, its value is ________ .

(97) Does the improper integral $\int_0^\infty x^4 e^{-x^5}\,dx$ converge? Answer: ________ . If it converges, its value is ________ .

(98) $\int_{-1}^1 \frac{dx}{\sqrt{|x|}} =$ _____ .

(99) $\int_{3^{1/4}}^\infty \frac{x\,dx}{1+x^4} = \frac{\pi}{a}$ where $a =$ ______ .

(100) $\int_0^\infty \frac{x\,dx}{(1+x^2)^4} = \frac{1}{a}$ where $a =$ _____ .

(101) $\int_0^{\sqrt{3}} \frac{x}{\sqrt{9-x^4}}\,dx = \frac{\pi}{a}$ where $a =$ _____ .

(102) $\int_{\sqrt{3}}^\infty \frac{1}{1+x^2}\,dx = \frac{a}{6}$ where $a =$ _____ .

(103) $\int_1^\infty xe^{-x}\,dx = \frac{2}{a}$ where $a =$ _____ .

(104) $\int_0^\infty x^{12}e^{-x}\,dx = n!$ where $n =$ ______ .

(105) $\int_{1/2}^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{a}{2}$ where $a =$ ______ .

(106) $\lim_{\lambda\to 0^+} \frac{1}{\ln\lambda}\int_\lambda^a \frac{\cos x}{x}\,dx =$ ______ . *Hint.* Problem 10 may help.

(107) Let $f(x) = \int_{3\pi}^x (7+\cos(\sin t))\,dt$. Then $Df^{-1}(0) = \frac{1}{a}$ where $a =$ _____ .

(108) Let $f(x) = \int_{\pi/3}^{x^{1/3}} \arctan(2+2\sin t)\,dt$ for $x \geq 0$. Then $Df^{-1}(0) = \frac{4\pi}{a}$ where $a =$ _____ . *Hint.* What is $\tan\frac{5\pi}{12}$?

(109) Let $f(x) = \int_1^x \frac{3t^2+t+1}{5t^4+t^2+2}\,dt$. Then $Df^{-1}(0) = \frac{a}{5}$ where $a =$ _____ .

(110) Let $f(x) = \displaystyle\int_0^x t^3\sqrt{t^4+9}\,dt$ for $x \geq 0$. Then $Df^{-1}\left(\frac{49}{3}\right) = \dfrac{1}{a}$ where $a =$ ________ .

(111) $\displaystyle\lim_{\lambda\to 0^+}\int_\lambda^{2\lambda} \frac{e^{-x}}{x}\,dx = \ln a$ where $a =$ _____ . *Hint:* $\dfrac{e^{-x}}{x} = \dfrac{e^{-x}-1}{x} + \dfrac{1}{x}$.