Axiomatic System

Mathematics Department

Second year – First Course

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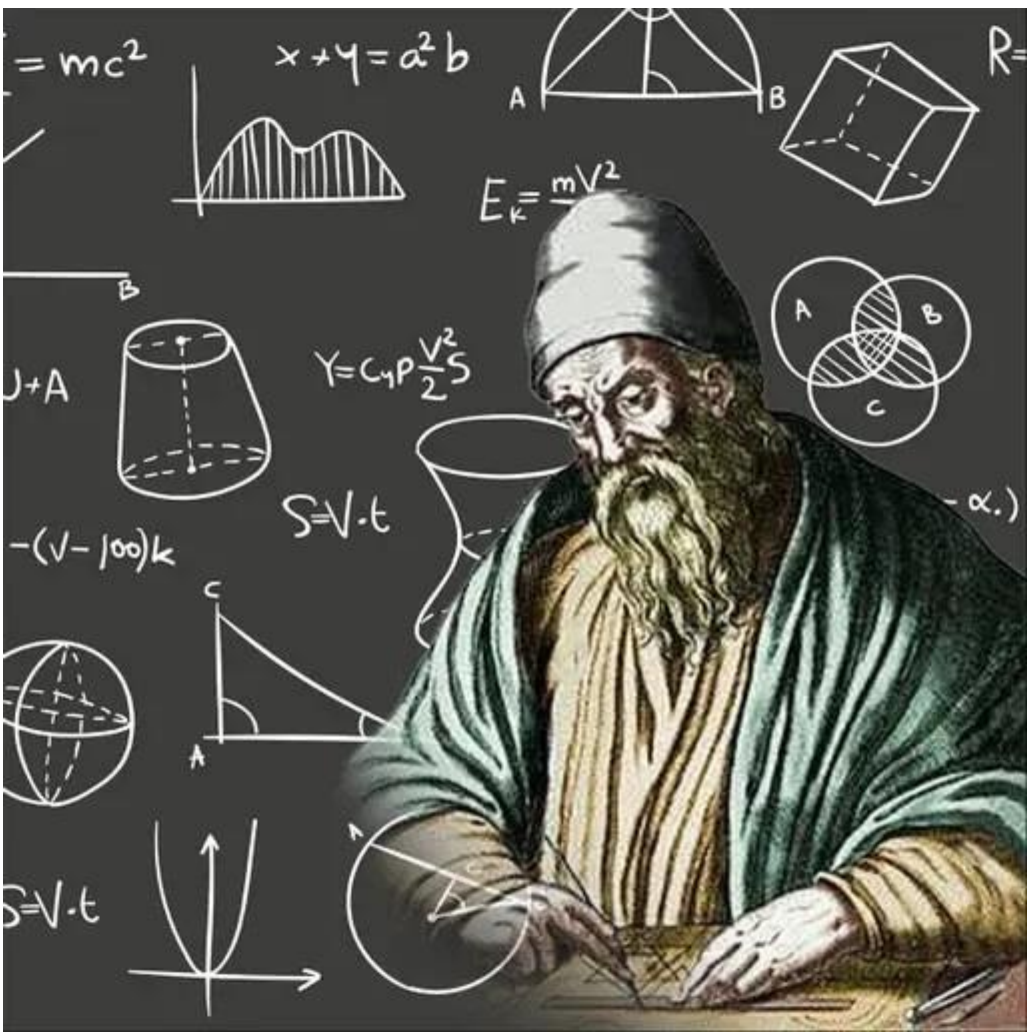

Chapter One Introduction to Axiomatic System

1.1 History of geometry

- The word ‘geometry’ comes from the Greek words ‘geo’, meaning earth, and ‘metria’, meaning measure.
- Along with arithmetic, geometry was one of the two fields of pre-modern mathematics.
- Ancient Egyptians used geometry principles as far back as 3000 BC, using equations to approximate the area of circles among other formulas.
- Babylonians measured the circumference of a circle as approximately 3 times the diameter, which is fairly close to today’s measurement which uses the value of Pi (around 3.14).
- A Greek mathematician named Euclid who lived around the year 300 BC is often referred to as the ‘Father of Geometry’ for his amazing geometry works that included the influential ‘Elements’, which remained the main textbook for teaching mathematics until around the early 20th century.

- Greeks constructed aesthetically pleasing buildings and artworks based on the golden ratio of approximately 1.618.
- Greek philosopher and mathematician Pythagoras lived around the year 500 BC and is known for his Pythagorean theorem relating to the three sides of a right angle triangle: $a^2 + b^2 = c^2$
- Archimedes of Syracuse lived around the year 250 BC and played a large role in the history of geometry including a method for determining the volume of objects with irregular shapes.
- The compass and straight edge were powerful tools in the advancement of geometry, allowing the construction of various lengths, angles and geometric shapes.
- Modern day geometry has made developments in a number of areas, including those that make use of the raw computing power of today’s computers.

1.2 Major branches of geometry

Euclidean geometry

In several ancient cultures there developed a form of geometry suited to the relationships between lengths, areas, and volumes of physical objects. This geometry was codified in Euclid’s *Elements* about 300 BCE on the basis of 10 axioms, or postulates, from which several hundred theorems were proved by deductive logic. The *Elements* epitomized the axiomatic-deductive method for many centuries.

Analytic geometry

Analytic geometry was initiated by the French mathematician René Descartes (1596–1650), who introduced rectangular coordinates to locate points and to enable lines and curves to be represented with algebraic equations. Algebraic geometry is a modern extension of the subject to multidimensional and non-Euclidean spaces.

Projective geometry

Projective geometry originated with the French mathematician Girard Desargues (1591–1661) to deal with those properties of geometric figures that are not altered by projecting their image, or “shadow,” onto another surface.

Differential geometry

The German mathematician Carl Friedrich Gauss (1777–1855), in connection with practical problems of surveying and geodesy, initiated the field of differential geometry. Using differential calculus, he characterized the intrinsic properties of curves and surfaces. For instance, he showed that the intrinsic curvature of a cylinder is the same as that of a plane, as can be seen by cutting a cylinder along its axis and flattening, but not the same as that of a sphere, which cannot be flattened without distortion.

Non-Euclidean geometries

Beginning in the 19th century, various mathematicians substituted alternatives to Euclid’s parallel postulate, which, in its modern form, reads, “given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.” They hoped to show that the alternatives were logically impossible. Instead, they discovered that consistent non-Euclidean geometries exist.

Topology

Topology, the youngest and most sophisticated branch of geometry, focuses on the properties of geometric objects that remain unchanged upon continuous deformation shrinking, stretching, and folding, but not tearing. The continuous development of topology dates from 1911, when the Dutch mathematician L.E.J. Brouwer (1881–1966) introduced methods generally applicable to the topic.

2.1 Introduction to Axiomatic Systems

A way of arriving at a scientific theory in which certain primitive assumptions, the so-called axioms, are postulated as the basis of the theory, while the remaining propositions of the theory are obtained as logical consequences of these axioms.

In mathematics, the axiomatic method originated in the works of the ancient Greeks on geometry. The most brilliant example of the application of the axiomatic method which remained unique up to the 19th century was the geometric system known as Euclid's Elements (ca. 300 B.C.). At the time the problem of the description of the logical tools employed to derive the consequences of an axiom had not yet been posed, but the Euclidean system was a very clear attempt to obtain all the basic statements of geometry by pure derivation based on a relatively small number of postulates (axioms) whose truth was considered to be self-evident.

2.2 Elements of an Axiomatic Systems

- Undefined terms (Primitives Notions)
  We need undefined terms for any axiomatic system to build the axioms upon them. Which are basic worlds or things for the system.

- Postulates (Axioms)
  An axiom is a list of statements dealing with undefined terms and definitions that are chosen to remain unproved.

- Defined terms
  Definitions of the system are all other technical terms of the system are ultimately defined by means of the undefined terms.

- Propositions (Theorem)
  A theorem is any statement that can be proven using logical deduction from the axioms.

2.3 Axiomatic System

An axiomatic System is a list of axioms and theorems that dealing with undefined terms.

Example: - Three-Point Geometry

Undefined terms (points, Lines)

Axioms for the Three Point Geometry:

1. There exist exactly 3 points in this geometry.
2. Two distinct points are on exactly one line.
3. Not all the points of the geometry are on the same line.
4. Two distinct lines are on at least one point.

Theorem 1: Two distinct lines are on exactly one point.

To prove this, note that by axiom 4 we need only show that two distinct lines are on at most one point.

Assume, to the contrary, that distinct lines l and m, meet at points P and Q.
This contradicts axiom 2, which says that the points P and Q lie on exactly one line.
Thus, our assumption is false, and two distinct lines are on at most one point.
Proving the theorem.

Theorem 2: The three-point geometry has exactly three lines.

Let the line determined by two of the points, say A and B, be denoted by m (Axiom 2).
We know that the third point, C, is not on m by Axiom 3.
AC is thus a line different from m, and BC is also a line different from m.
These two lines cannot be equal to each other since that would imply that the three points are on the same line.
So there are at least 3 lines.
If there was a fourth line, it would have to meet each of the other lines at a point by Theorem 1.1.
As those three lines do not pass through a common point, the fourth line must have at least two points on it contradicting Axiom 2.

2.3 Model

A model for an axiomatic system is a way to define the undefined terms so that the axioms are true. Sometimes it is easy to find a model for an axiomatic system, and sometimes it is more difficult.

Example: -
Here are some examples of axiomatic systems.

Committees
Undefined terms: committee, member
Axiom 1: Each committee is a set of three members.
Axiom 2: Each member is on exactly two committees.
Axiom 3: No two members may be together on more than one committee.
Axiom 4: There is at least one committee.

Monoid
Undefined terms: element, product of two elements
Axiom 1: Given two elements x and y, the product of x and y, denoted $x * y$, is a uniquely defined element.

Axiom 2: Given elements x, y, and z, the equation $x * (y * z) = (x * y) * z$ is always true.
Axiom 3: There is an element e, called the identity, such that $e * x = x * e = x$ and for all elements .

Here are some examples of models for the "monoid" system.

- the elements are real numbers, and the product of two elements is the product of those two numbers
- the elements are 22 matrices, and the product is the product of those two matrices

Here is a model for the Committees system (but certainly not the only one):

Members	Alan, Beth, Chris, Dave, Elena, Fred
Committees	{Alan, Beth, Chris} {Alan, Dave, Elena} {Beth, Dave, Fred} {Chris, Elena, Fred}

We have defined the undefined terms, and now we have to check that the axioms are actually satisfied. It is easy to see that Axioms 1 and 4 are satisfied.

Axiom 2 says "Each member is on exactly two committees." To check this axiom, we look at each member, and list the number of committees they are on. If that number is 2 for every member, then the axiom is true.

Member	*Committees*	*Number = 2?*
Alan	{**Alan**, Beth, Chris}, {**Alan**, Dave, Elena}	yes
Beth	{Alan, **Beth**, Chris}, {**Beth**, Dave, Fred}	yes
Chris	{Alan, Beth, **Chris**}, {**Chris**, Elena, Fred}	yes
Dave	{Alan, **Dave**, Elena}, {Beth, **Dave**, Fred}	yes
Elena	{Alan, Dave, **Elena**}, {Chris, **Elena**, Fred}	yes
Fred	{Beth, Dave, **Fred**}, {Chris, Elena, **Fred**}	yes

Axiom 3 says "No two members may be together on more than one committee." For this axiom, we have to look at all 15 pairs of members and make sure that none of the pairs is on more than one committee. So it is acceptable to have the pair of members be on zero committees or one committee, but not two or more.

Pair of Members	*Committee(s)*	*Number ≤ 1?*
Alan & Beth	{**Alan**, **Beth**, Chris}	yes
Alan & Chris	{**Alan**, Beth, **Chris**}	yes
Alan & Dave	{**Alan**, **Dave**, Elena}	yes
Alan & Elena	{**Alan**, Dave, **Elena**}	yes
Alan & Fred	*none*	yes
Beth & Chris	{Alan, **Beth**, **Chris**}	yes
Beth & Dave	{**Beth**, **Dave**, Fred}	yes
Beth & Elena	*none*	yes
Beth & Fred	{**Beth**, Dave, **Fred**}	yes
Chris & Dave	*none*	yes
Chris & Elena	{**Chris**, **Elena**, Fred}	yes
Chris & Fred	{**Chris**, Elena, **Fred**}	yes
Dave & Elena	{Alan, **Dave**, **Elena**}	yes
Dave & Fred	{Beth, **Dave**, **Fred**}	yes
Elena & Fred	{Chris, **Elena**, **Fred**}	yes

3.1 Property of Axiomatic System

Independent
An axiom is called independent if it cannot be proven from the other axioms.

Example:
Consider Axiom 1 from the Committee system. Let's omit it and see what kind of model we can come up with.

Members	Adam, Brian, Carla, Dana
Committees	{Adam, Brian} {Brian, Carla, Dana} {Adam, Carla} {Dana}

Notice that we found a model where Axiom 1 is not true; we have committees that do not have exactly three members. Since all of the other axioms are true in this model, then so is any statement that we could prove using those axioms. But since Axiom 1 is not true, it follows that Axiom 1 is not provable from the other axioms. Thus Axiom 1 is independent.

Consistency

If there is a model for an axiomatic system, then the system is called consistent. Otherwise, the system is inconsistent. In order to prove that a system is consistent, all we need to do is come up with a model: a definition of the undefined terms where the axioms are all true. In order to prove that a system is inconsistent, we have to somehow prove that no such model exists (this is much harder!).

Example: The following axiomatic system is not consistent

Undefined Terms: boys, girls

A1. There are exactly 2 boys.

A2. There are exactly 3 girls.

A3. Each boy likes exactly 2 girls.

A4. No two boys like the same girl.

Completeness

An axiomatic system is complete if every true statement can be proven from the axioms.

Example:

Twin Primes Conjecture: There are an infinite number of pairs of primes whose difference is 2.

Some examples of “twin” primes are 3 and 5, 5 and 7, 11 and 13, 101 and 103, etc. Computers have found very large pairs of twin primes, but so far no one has been able to prove this theorem. It is possible that a proof will never be found.

Categorical

An axiomatic system is categorical if (informally put) all systems obtained by giving specific interpretations to the undefined terms of the abstract systems all essentially the same.

Example: -

Undefined terms (points, Lines)

Axioms for the Four Point Geometry:

1. Each line is a set of four point
2. Each point is contained by precisely two line
3. Two distinct lines intersect at exactly one point

Is non categorical

3.2 Four Line Geometry

The Axioms for the Four Line Geometry:

Ax1. There exist exactly 4 lines.

Ax2. Any two distinct lines have exactly one point on both of them.

Ax3. Each point is on exactly two lines.

Theorem 1: The four-line geometry has exactly six points.

There are exactly 6 pairs of lines (4 choose 2), and every pair meets at a point. Since each point lies on only two lines by Ax3, these six pairs of lines give 6 distinct points. To prove the statement, we need to show that there are no more points than these 6. However, by axiom 3, each point is on two lines of the geometry and every such point has been accounted for there are no other points

Theorem 2: Each line of the four-line geometry has exactly 3 points on it.

Proof:

Consider any line. The three other lines must each have a point in common with the given line (Axiom 2).
These three points are distinct, otherwise Axiom 3 is violated.
There can be no other points on the line since if there was, there would have to be another line on the point by Axiom 3 and we can't have that without violating Axiom 1

3.3 Fano's Geometry

Ax1. There exists at least one line.

Ax2. Every line of the geometry has exactly 3 points on it.

Ax3. Not all points of the geometry are on the same line.

Ax4. For two distinct points, there exists exactly one line on both of them.

Ax5. Each two lines have at least one point on both of them.

Theorem 1: Each two lines have exactly one point in common.

Proof:

Assume that two distinct lines $l \neq m$ have two distinct points in common P and Q. (C! Ax4) since these two points would then be on two distinct lines.

Theorem 2: Fano's geometry consists of exactly seven points and seven lines.

Proof:

First, we have to show that there are at least 7 points and seven lines (by drawing)

Assume that there is an 8th point.

By axiom 4 it must be on a line with point 1.

By axiom 5 this line must meet the line containing points 3,4 and 7.

But the line cannot meet at one of these points (C! Ax4)

So the point of intersection would have to be a fourth point on the line 347(C!

Ax2).

Thus there are exactly seven points and seven lines.

4.1 Euclid's Axioms of Geometry

Let the following be postulated

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.
(Euclid's Parallel Postulate)

4.2 Hilbert's Axioms of Geometry

Undefined Terms: *point, line, incidence, betweenness, and congruence.*

Incidence Geometry

Axiom I-1: *For every point P and for every point Q not equal to P there exists a unique line ℓ that passes through P and Q.*

Axiom I-2: *For every line ℓ there exist at least two distinct points incident with ℓ .*

Axiom I-3: *There exist three distinct points with the property that no line is incident with all three of them.*

Betweenness Axioms

Axiom B-1: If $A * B * C$, *then A, B, and C are three distinct points all lying on the same line and* $C * B * A$.

Axiom B-2: *Given any two distinct points B and D, there exists points A, C, and E lying on $\overleftrightarrow{BD}$ such that* $A * B * D$, $B * C * D$, and $B * D * E$.

Axiom B-3: *If A, B, and C are three distinct point lying on the same line, then one and only one of the points is between the other two.*

Axiom B-4: (Plane Separation Axiom) *For every line ℓ and for any three points A, B, and C not lying on ℓ :*

(i) If A and B are on the same side of ℓ and B and C are on the same side of ℓ , then A and C are on the same side of ℓ .

(ii) If A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ , then A and C are on the same side of ℓ .

Congruence Theorems

Axiom C-1: *If A and B are distinct points and if A' is any point, then for each ray* r *emanating from A' there is a* unique *point B' on* r *such that $B \neq A$ and $AB \cong A'B'$.*

Axiom C-2: *If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.*

Axiom C-3: *If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.*

Axiom C-4: *Given any $\angle BAC$ and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a* unique *ray $A'C'$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\angle BAC \cong \angle B'A'C'$.*

Axiom C-5: *If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$. Moreover, every angle is congruent to itself.*

Axiom C-6: (**SAS**) *If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.*

Axioms of Continuity

Archimedes' Axiom: *If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E.*

Dedekind's Axiom: *Suppose that the set of all points on a line ℓ is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and* vice versa. *Then there is a unique point, O, lying on ℓ such that $P_1 * O * P_2$ if and only if $P_1 \in \Sigma_1$ and $P_2 \in \Sigma_2$ and $O \neq P_1, P_2$.*

Elementary Continuity Principle: *If one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.*

Circular Continuity Principle: *If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.*

Axiom of Parallelism

Euclidean Parallel Postulate: *Through a given external point there is at most one-line parallel to a given line.*

4.3 Birkhoff's Axioms of Geometry

B1. There exist nonempty subsets of the plane called lines, with the property that each two points belong to exactly one line.

B2. Corresponding to any two points A and B in the plane there exists a unique real number $d(AB) = d(BA)$, the distance from A to B, which is 0 if and only if $A = B$.

B3. (*Birkhoff Ruler Axiom*) If k is a line and $\mathbb{R}$ denotes the set of real numbers, there exists a one-to-one correspondence $(X \to x)$ between the points X in k and the numbers $x \in \mathbb{R}$ such that $d(A, B) = |a - b|$ where $A \to a$ and $B \to b$.

B4. For each line k there are exactly two nonempty convex sets R' and R'' satisfying

a) $R' \cup k \cup R''$ is the entire plane,

b) $R' \cap R'' = \emptyset$, $R' \cap k = \emptyset$, and $R'' \cap k = \emptyset$,

c) if $X \in R$ and $Y \in R''$ then $\overline{XY} \cap k \neq \emptyset$.

B5. For each angle $\angle ABC$ there exists a unique real number x with $0 \leq x \leq 180$ which is the (degree) measure of the angle $x = \angle ABC°$.

B6. If ray $\overrightarrow{BD}$ lies in $\angle ABC$, then $\angle ABD° + \angle DBC° = \angle ABC°$.

B7. If $\overrightarrow{AB}$ is a ray in the edge, k, of an open half plane $H(k; P)$ then there exist a one-to-one correspondence between the open rays in $H(k; P)$ emanating from A and the set of real numbers between 0 and 180 so that if $\overrightarrow{AX} \to x$ then $\angle BAX° = x$.

B8. (SAS) If a correspondence of two triangles, or a triangle with itself, is such that two sides and the angle between them are respectively congruent to the corresponding two sides and the angle between them, the correspondence is a congruence of triangles.

B9. (Euclidean Parallel Postulate) Through a given external point there is at most one-line parallel to a given line.

Exercise 1.1. Consider the following axiom set.

Postulate 1. There are at least two buildings on campus.

Postulate 2. There is exactly one sidewalk between any two buildings.

Postulate 3. Not all the buildings have the same sidewalk between them.

a. What are the primitive terms in this axiom set?

b. Deduce the following theorems:

Theorem 1. There are at least three buildings on campus.

Theorem 2. There are at least two sidewalks on campus.

c. Show by the use of models that it is possible to have

exactly two sidewalks and three buildings;

at least two sidewalks and four buildings; and, exactly three sidewalks and three buildings.

d. Is the system complete? Explain.

e. Find two isomorphic models.

f. Demonstrate the independence of the axioms.

Exercise 1.2. Consider the following axiom set.

A1. Every hive is a collection of bees.

A2. Any two distinct hives have one and only one bee in common.

A3. Every bee belongs to two and only two hives.

A4. There are exactly four hives.

a. What are the undefined terms in this axiom set?

b. Deduce the following theorems:

T1. There are exactly six bees.

T2. There are exactly three bees in each hive.

T3. For each bee there is exactly one other bee not in the same hive with it.

c. Find two isomorphic models.

d. Demonstrate the independence of the axioms.

Exercise 1.3. Consider the following axiom set.

P1. Every herd is a collection of cows.

P2. There exist at least two cows.

P3. For any two cows, there exists one and only one herd containing both cows.

P4. For any herd, there exists a cow not in the herd.

P5. For any herd and any cow not in the herd, there exists one and only one other herd containing

the cow and not containing any cow that is in the given herd.

a. What are the primitive terms in this axiom set?

b. Deduce the following theorems:

T1. Every cow is contained in at least two herds.

T2. There exist at least four distinct cows.

T3. There exist at least six distinct herds.

c. Find two isomorphic models.

d. Demonstrate the independence of the axioms.

Chapter Two

Basic Results in Book I of the Elements

2.1 The first 28 propositions

A plane geometry is "*neutral*" if it does not include a parallel postulate or its logical consequences. The first 28 propositions of Book I of Euclid's *Elements* are results in a neutral geometry that are proved based on the first 4 axioms and the common notions.

Proposition I.1. To construct an equilateral triangle.

Proposition I.2. To place a straight line equal to a given straight line with one end at a given point.

Proposition I.3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Proposition I.4. (SAS) If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals to the triangle, and the remaining angles equal the remaining angles respectively.

Proposition I.5. In isosceles triangles, the angles at the base equal one another; and if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition I.6. If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Proposition I.7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

Proposition I.8. (SSS) If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proposition I.9. To bisect a given rectilinear angle.

Proposition I.10. To bisect a given finite straight line.

Proposition I.11. To draw a straight line at right angles to a given straight line from a given point on it.

Proposition I.12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Proposition I.13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition I.14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition I.15. If two straight lines cut one another, then they make the vertical angles equal to one another.

Proposition I.16. (Exterior Angle Theorem) In any triangle, if any one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles.

Proposition I.17. In any triangle, two angles taken together in any manner are less than two right angles.

Proposition I.18. In any triangle, the angle opposite the greater side is greater.

Proposition I.19. In any triangle, the side opposite the greater angle is greater.

Proposition I.20. In any triangle, the sum of any two sides is greater than the remaining one.

Proposition I.21. If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangles, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Proposition I.22. To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition I.23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition I.24. If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Proposition I.25. If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have one of the angles contained by the equal straight lines greater than the other.

Proposition I.26. (ASA or AAS) If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angles equals the remaining angle.

Proposition I.27. If a straight line falling on two straight lines make the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition I.28. If a straight line falling on two straight lines make the exterior angles equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angle, then the straight lines are parallel to one another.

Proposition I.1, I.2, and I.3 are basically proved by construction using straightedge and compass. Proposition I.4 (SAS) is deduced by means of the uniqueness of straight line segment joining two points. Apparently Euclid places it early in his list so that he can make use it in proving later results. Before we proceed, let's state the definition of "*congruent triangle*."

Definition 2.1 *Two triangles are "congruent" if and only of there is some "way" to match vertices of one to the other such that corresponding sides are equal in length and corresponding angles are equal in size.*

If $\triangle ABC$ is congruent to $\triangle XYZ$, we shall use the notation $\triangle ABC \cong \triangle XYZ$. Thus $\triangle ABC \cong \triangle XYZ$ if and only if $AB = XY$, $AC = XZ$, $BC = YZ$ and $\angle BAC = \angle YXZ, \angle CBA = \angle ZYX, \angle ACB = \angle XZY$.

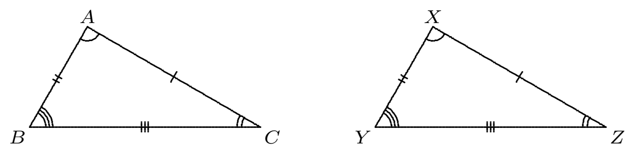

Figure 2.1: Congruent triangles

Let's state and prove proposition I.5 and I.6 in modern language

Proposition I.5. In $\triangle ABC$, if $AB = AC$, then $\angle ABC = \angle ACB$, same for the exterior angles at B and C.

Proof. Let the angle bisector of $\angle A$ meet BC at D. Then by (SAS), $\triangle BAD \cong \triangle CAD$. Thus $\angle ABC = \angle ACB$. (Alternatively, take D to be the midpoint of BC and use (SSS) to conclude that $\triangle BAD \cong \triangle CAD$.)

2.2 Pasch's axiom

There is a hidden assumption that the bisector actually intersects the third side of the triangle. This seems intuitively obvious to us, as we see that any triangle has an "*inside*" and an "*outside.*" That is "*the triangle separates the plane into two regions*" which is a simple version of the Jordan curve theorem! In fact, Euclid assumes this separation property without proof and does not include it as one of his axioms. Pasch (1843-1930) was the first to notice this hidden assumption of Euclid. Later he formulates this property specifically; and it is now known as "*Pasch's axiom*".

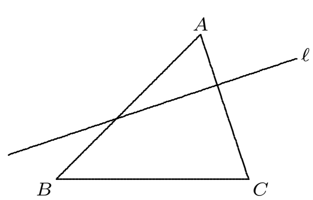

Figure 2.2: Pasch's axiom

Pasch's axiom *Let ℓ be a line passing through the side AB of a triangle ABC. Then ℓ must pass through a either a point on AC or on BC.*
Proposition I.6. In $\triangle ABC$, if $\angle ABC = \angle ACB$, then $AB = AC$.
Proof. Suppose $AB \neq AC$. Then one of them is greater. Let $AB > AC$. Mark off a point D on AB such that $DB = AC$. Also $CB = BC$ and $\angle ACB = \angle DBC$. Thus triangles ACB is congruent to triangle DBC, the less equal to the greater, which is absurd. Therefore $AB = AC$.

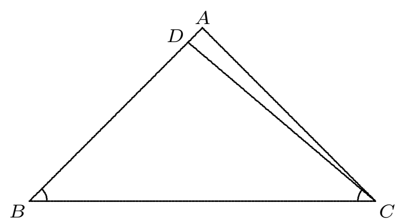

Figure 2.3: Proposition 6

Similarly, it is not true that $AB < AC$. Consequently, $AB = AC$.

Propositions I.7 and I.8 are the (SSS) congruent criterion. Proposition I.7 is self-evident by construction and proposition I.8 follows from I.7. Propositions I.9 to I.15 follow from definitions and construction. Propositions I.16 and I.17 are discussed in chapter 1. The proofs use crucially axiom 1 and 2.

Proposition I.18. In the triangle ABC, if $AB > AC$, then $\angle C > \angle B$.
Proof. Mark off a point D on AB such that $AD = AC$.

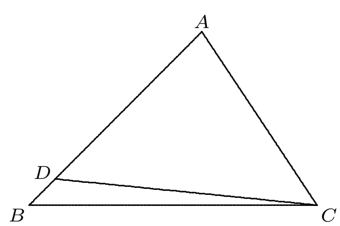

Figure 2.4: Proposition 18

By proposition I.5, $\angle ADC = \angle ACD$. Thus $\angle C > \angle ACD = \angle ADC > \angle B$ by the exterior angle theorem (proposition I.16).

Proposition I.19. In the triangle ABC, if $\angle B > \angle C$, then $AC > AB$.
Proof. If $AB = AC$, then by proposition I.5 we have $\angle B = \angle C$. If $AB > AC$, then by proposition 18 we have $\angle C > \angle B$. Thus both cases lead to a contradiction. Hence, we must have $AC > AB$.

Proposition I.20. (Triangle Inequality) For any triangle ABC, $AB + BC > AC$.
Proof. Exercise.

Proposition I.21. Let D be a point inside the triangle ABC. Then $AB + AC > DB + DC$ and $\angle BDC > \angle BAC$.

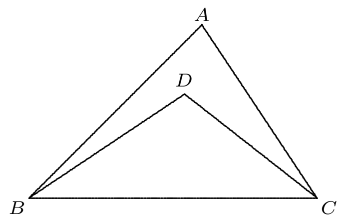

Figure 2.5: Proposition 21

Proof. This follows from the triangle inequality (proposition I.20) and the exterior angle theorem (proposition I.16).

Also Proposition I.22 follows from the triangle inequality (proposition I.20). Proposition 23 is on copying an angle by means of a straightedge and a compass. It can be justified using (SSS) condition.

Proposition I.24. For the triangles ABC and PQR with $AB = PQ$ and $AC = PR$, if $\angle A > \angle P$ then $BC > QR$.

Proof. Stack the triangle PQR onto ABC so that PQ matches with AB. Since $\angle A > \angle P$, the ray AR is within $\angle BAC$. Join BR and CR. Suppose R is outside the triangle ABC.

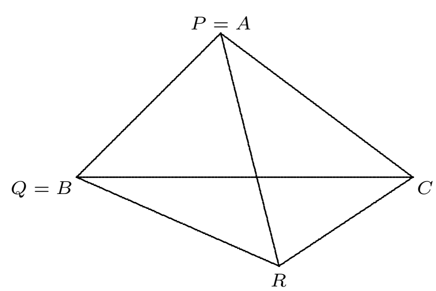

Figure 2.6: Proposition 24

As $AC = AR$ (or PR), $\angle ARC = \angle ACR$. Thus $\angle BRC > \angle ARC = \angle ACR > \angle BCR$. Therefore, $BC > QR$. We leave it as an exercise for the case where R is inside ABC.

Proposition I.25. For the triangles ABC and PQR with $AB = PQ$ and $AC = PR$, if $BC > QR$, then $\angle A > \angle P$.

Proof. If $\angle A = \angle P$, then by (SAS) the two triangles are congruent. But $BC \neq QR$, we have a contradiction. If $\angle A < \angle P$, then by proposition I.24, $BC < QR$, which also contradicts the given condition. Thus we must have $\angle A > \angle P$.

Proposition I.26. (ASA) For the triangles ABC and PQR, if $\angle B = \angle Q, \angle C = \angle R$ and $BC = QR$ then $\triangle ABC \cong \triangle PQR$.

Proof. Suppose $AB > PQ$. Mark off a point D on AB such that $BD = QP$. Then by (SAS), $DBC \cong PQR$ so that $\angle BCD = \angle QRP = \angle R$. But then $\angle BCD < \angle C = \angle R$, a contradiction.

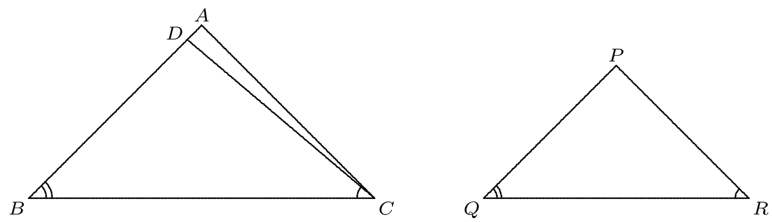

Figure 2.7: Proposition 26

Similarly we get a contradiction if $AB < PQ$. Thus $AB = PQ$. Then by (SAS), $\triangle ABC \cong \triangle PQR$. The (AAS) case is left as an exercise.

Exercise 2.1: If two sides of a triangle are equal, the line which bisects the angle between the equal sides bisects the third side.

Exercise 2.2: If two sides of a triangle are equal, the line joining the corner (or vertex) between the equal sides and the mid-point of the third side bisects the angle between the equal sides.

Exercise 2.3: line PM is perpendicular to line AB at point M and PM bisects AB at M Prove that PA =PB.

Exercise 2.4: If two angles of a triangle are equal, the sides opposite these angles are equal.

Exercise 2.5: If a quadrilateral has three right angles, its fourth angle is a right angle also.

Exercise 2.6: If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

Exercise 2.7: A radius perpendicular to a chord of a circle bisects the chord.

Exercise 2.8: The sum of the angles of a triangle is two right angles.

Exercise 2.9: Equal chords of a circle are equally distant from the center of the circle.

Exercise 2.10: The opposite angles of a parallelogram are equal.

Chapter Three

Triangles

In this chapter, we prove some basic properties of triangles in Euclidean geometry.

3.1 Basic properties of triangles

Theorem 3.1 (Congruent Triangles) *Given two triangles* ABC *and* $A'B'C'$,

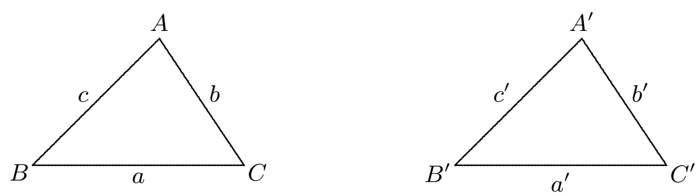

Figure 3.1: Congruent Triangles

the following statements are equivalent.
(a) $\triangle ABC$ *is congruent to* $\triangle A'B'C'$. *(*$\triangle ABC \cong \triangle A'B'C'$*)*
(b) $a = a', b = b', c = c'$. *(SSS)*
(c) $b = b'$, $\angle A = \angle A'$, $c = c'$. *(SAS)*
(d) $\angle A = \angle A'$, $b = b'$, $\angle C = \angle C'$. *(ASA)*
(e) $\angle A = \angle A'$, $\angle B = \angle B'$, $a = a'$. *(AAS)*

Theorem 3.2 *Given two triangles* ABC *and* $A'B'C'$ *where* $\angle C = \angle C' = 90°$,

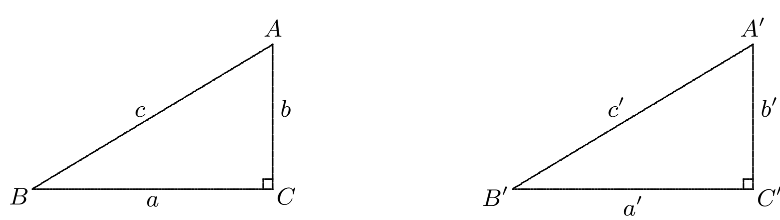

Figure 3.2: Congruent right Triangles

the following statements are equivalent.

(a) $\triangle ABC \cong \triangle A'B'C'$.
(b) $\angle C = \angle C' = 90°$, $a = a', c = c'$. *(RHS)*
(b) $\angle C = \angle C' = 90°$, $b = b', c = c'$. *(RHS)*

Theorem 3.3 (Similar triangles) *Given two triangles* ABC *and* $A'B'C'$,

Figure 3.3: Similar Triangles

the following are equivalent.

(a) $\triangle ABC$ *is similar to* $\triangle A'B'C'$. $(\triangle ABC \sim \triangle A'B'C')$
(b) $\angle A = \angle A'$ *and* $\angle B = \angle B'$.
(c) $\angle A = \angle A'$ *and* $b : b' = c : c'$.
(d) $a : a' = b : b' = c : c'$.

Theorem 3.4 (The midpoint theorem) *Let* D *and* E *be points on the sides* AB *and* AC *of the triangle* ABC *respectively. Then* $AD = DB$ *and* $AE = EC$ *if and only if* DE *is parallel to* BC *and* $DE = \frac{1}{2}BC$.

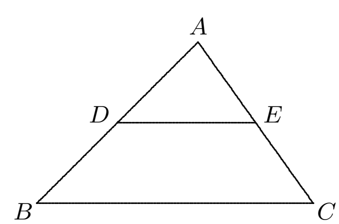

Figure 3.4: The midpoint theorem

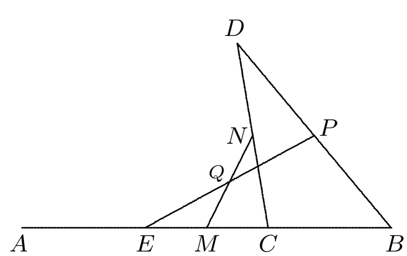

Figure 3.5: The midpoint of AC is E

Example 3.1 In figure 3.5, M, N, and P are respectively the mid-points of the line segments AB, CD and BD. Let Q be the mid-point of MN and let PQ be extended to meet AB at E. Show that $AE = EC$.

Solution. Join NP. Because N is the mid-point of CD and P is the midpoint of BD, we have NP is parallel to AB. Since $NQ = MQ$, we see that $\triangle NPQ$ is congruent to $\triangle MEQ$. Thus $EM = NP = \frac{1}{2}BC$. Therefore, $2EM = BC = MB - MC = AM - MC = AC - 2MC = AC - 2(EC - EM) = AC - 2EC + 2EM$. Thus $AC = 2EC$ and E is the mid-point of AC.

Definition 3.1 *For any polygonal figure $A_1A_2 \cdots A_n$, the area bounded by its sides is denoted by $(A_1A_2 \cdots A_n)$.*

For example if ABC is a triangle, then (ABC) denotes the area of $\triangle ABC$; and if $ABCD$ is a quadrilateral, then $(ABCD)$ denotes its area, etc.

Theorem 3.5 (Varignon) *The figure formed when the midpoints of the sides of a quadrilateral are joined is a parallelogram, and its area is half that of the quadrilateral.*

Proof. Let P, Q, R, S be the midpoints of the sides AB, BC, CD, DA of a quadrilateral respectively. The fact that $PQRS$ is a parallelogram follows from the midpoint theorem. Even $ABCD$ is a "*cross-quadrilateral*", the result still holds.

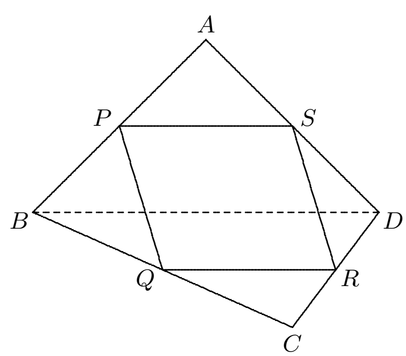

Figure 3.6: Varignon's theorem

As for the area, we have

$$\begin{aligned}(PQRS) &= (ABCD) - (PBQ) - (RDS) - (QCR) - (SAP)\\ &= (ABCD) - \tfrac{1}{4}(ABC) - \tfrac{1}{4}(CDA) - \tfrac{1}{4}(BCD) - \tfrac{1}{4}(DAB)\\ &= (ABCD) - \tfrac{1}{4}(ABCD) - \tfrac{1}{4}(ABCD)\\ &= \tfrac{1}{2}(ABCD).\end{aligned}$$

If "*sign area*" is used, the result still holds.

Theorem 3.6 (Steiner-Lehmus) *Let BD be the bisector of $\angle B$ and let CE be the bisector of $\angle C$. The following statements are equivalent:*
(a) $AB = AC$
(b) $\angle B = \angle C$
(c) $BD = CE$

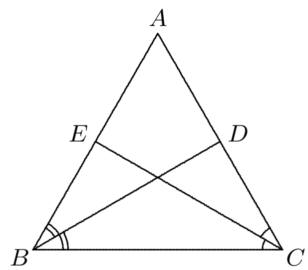

Figure 3.7: Steiner-Lehmus Theorem

The result on (c) implies (a) is called the Steiner-Lehmus Theorem. The proof relies on two lemmas.

Lemma 3.7 *If two chords of a circle subtend different acute angles at points on the circle, the smaller angle belongs to the shorter chord.*

Proof. Two equal chords subtend equal angles at the center and equal angles (half as big) at suitable points on the circumference. Of two unequal chords, the shorter, being farther from the center, subtends a smaller angle there and consequently a smaller acute angle at the circumference.

Lemma 3.8 *If a triangle has two different angles, the smaller angle has the longer internal angle bisector.*

Proof. Let ABC be the triangle with $\angle B > \angle C$. Let's take $\beta = \frac{1}{2}\angle B$ and $\gamma = \frac{1}{2}\angle C$. Thus $\beta > \gamma$. Let BE and CF be the internal angle bisectors at angles B and C respectively. Since $\angle EBF = \beta > \gamma$, we can mark off a point M on CF such that $\angle EBM = \gamma$. Then B, C, E, M lie on a circle.

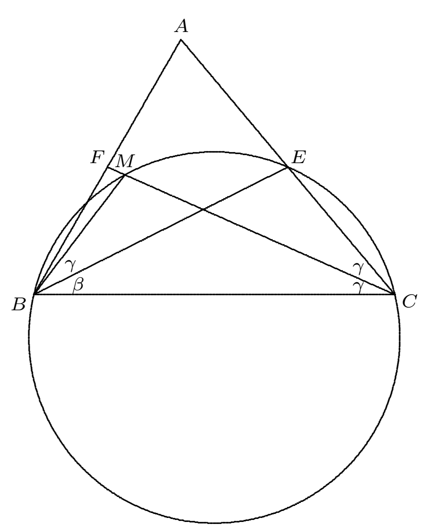

Figure 3.8: The smaller angle has the longer internal angle bisector

Note that $\beta+\gamma<\beta+\gamma+\frac{1}{2}\angle A=90^\circ$. Also $\angle C=2\gamma<\beta+\gamma=\angle CBM$. Hence $CF>CM>BE$. To prove the theorem, we prove by contradiction. Suppose $AC>AB$. Then $\angle B>\angle C$. By lemma 2, $CF>BE$, a contradiction. Can you produce a constructive proof of this result?

Theorem 3.9 (The angle bisector theorem) *If AD is the (internal or external) angle bisector of $\angle A$ in a triangle ABC, then $AB:AC=BD:DC$.*

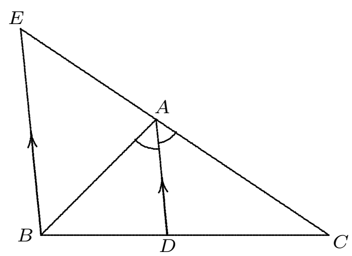

Figure 3.9: Angle bisectors

Proof. The theorem can be proved by applying sine law to $\triangle ABD$ and $\triangle ACD$. An alternate proof is as follow. Construct a line through B parallel to AD meeting the extension of CA at E. Then $\angle ABE=\angle BAD=\angle DAC=\angle AEB$. Thus $AE=AB$. Since $\triangle CAD$ is similar to $\triangle CEB$,we have $AB/AC=AE/AC=BD/DC$. The proof for the external angle bisector is similar.

Theorem 3.10 (Stewart) *If $\dfrac{BP}{PC}=\dfrac{m}{n}$, then $nAB^2+mAC^2=(m+n)AP^2+\dfrac{mn}{m+n}BC^2$.*

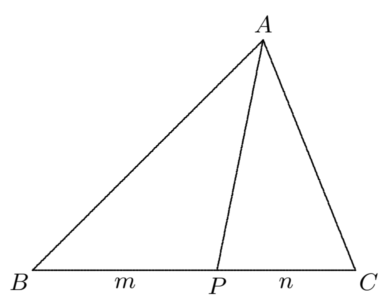

Figure 3.10: Stewart's theorem

Proof. Apply cosine law to the triangles ABP and APC for the two complementary angles at P.

Theorem 3.11 (Pappus' theorem) *Let P be the midpoint of the side BC of a triangle ABC. Then*

$$AB^2+AC^2=2(AP^2+BP^2).$$

3.2 Special points of a triangle

1. Perpendicular bisectors. The three perpendicular bisectors to the sides of a triangle ABC meet at a common point O, called the *circumcentre* of the triangle. The point O is equidistant to the three vertices of the triangle. Thus the circle centred at O with radius OA passes through the three vertices of the triangle. This circle is called the *circumcircle* of the triangle and the radius R is called the *circumradius* of the triangle.

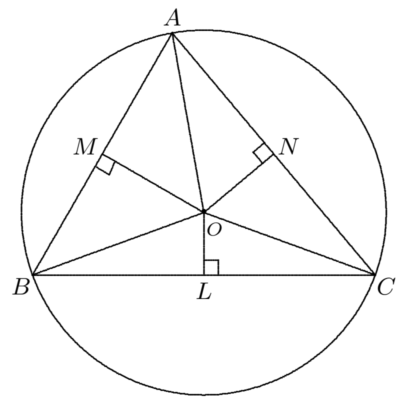

Figure 3.11: Perpendicular bisectors

For any triangle ABC with circumradius R, we have the *sine rule*: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

2. Medians. The 3 medians AD, BE and CF of $\triangle ABC$ are concurrent. Their common point, denoted by G, is called the *centroid* of $\triangle ABC$.

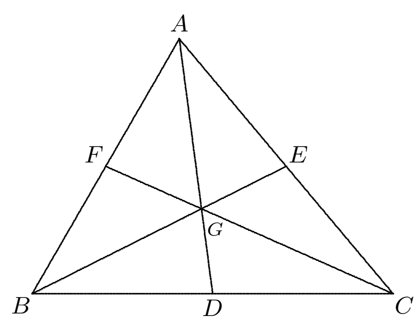

Figure 3.12: Medians

We have

(1) $(AGF) = (BGF) = (BGD) = (CGD) = (CGE) = (AGE)$.

(2) $AG : GD = BG : GE = CG : GF = 2 : 1$.

(3) (Apollonius'theorem)

$AD^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2,$

$BE^2 = \frac{1}{2}(c^2 + a^2) - \frac{1}{4}b^2,$

$CF^2 = \frac{1}{2}(a^2 + b^2) - \frac{1}{4}c^2.$

3. Angle bisectors. The internal bisectors of the 3 angles of $\triangle ABC$ are concurrent. Their common point, denoted by I, is called the *incentre* of $\triangle ABC$. It is equidistant to the sides of the triangle. Let r denote the distance from I to each side. The circle centred at I with radius r is called the *incircle* of $\triangle ABC$, and r is called the *inradius*.

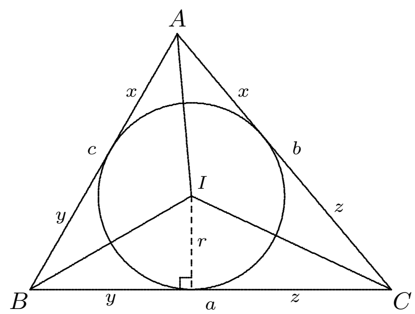

Figure 3.13: Angle bisectors

Let $s = \frac{1}{2}(a+b+c)$ be the *semi-perimeter*. We have

(1) $x = s-a$, $y = s-b$ and $z = s-c$.

(2) $(ABC) = sr$.

(3) $abc = 4srR$.

To prove (3), we have $4srR = 4(ABC)R = 2(ab\sin C)R = abc$.

Exercise 3.1 Prove that $\sin A = (2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)^{\frac{1}{2}}/(2bc)$.

4. Altitudes. The 3 altitudes AD, BE and CF of $\triangle ABC$ are concurrent. The point of concurrence, denoted by H, is called the *orthocentre* of $\triangle ABC$. The triangle DEF is called the *orthic triangle* of $\triangle ABC$. We have the following result.

Theorem 3.12 *The orthocentre of an acute-angled triangle is the incentre of its orthic triangle.*

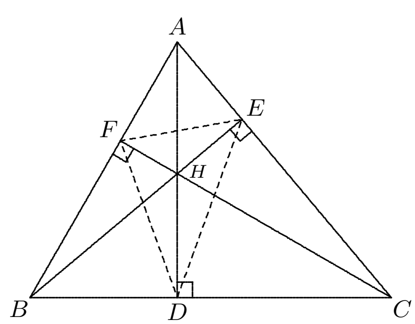

Figure 3.14: Altitudes

Example 3.2 Show that the three altitudes of a triangle are concurrent.

Solution. Draw lines PQ, QR, RP through C, A, B and parallel to AB, BC, CA respectively. Then PQR forms a triangle whose perpendicular bisectors are the altitudes of the triangle ABC.

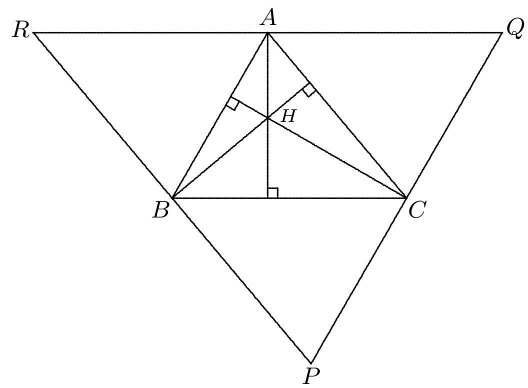

Figure 3.15: The three altitudes of a triangle are concurrent

Exercise 3.2 In an acute-angled $\triangle ABC$, $AB < AC$, BD and CE are the altitudes. Prove that
(i) $BD < CE$
(ii) $AD < AE$
(iii) $AB^2 + CE^2 < AC^2 + BD^2$
(iv) $AB + CE < AC + BD$.
(v) Is it true that $AB^n + CE^n < AC^n + BD^n$ for all positive integer n?

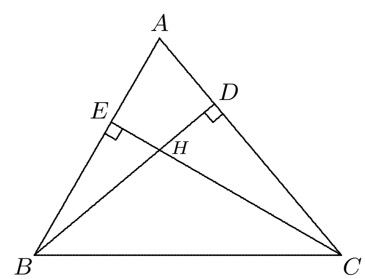

Figure 3.16: $AB^2 + CE^2 < AC^2 + BD^2$

Exercise 3.3 Prove Heron's formula that for a triangle ABC, we have

$$(ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

Exercise 3.4 Prove that if I is the incentre of the triangle ABC, then $AI^2 = bc(s-a)/s$.

Exercise 3.5 Prove that for any triangle ABC,

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc} \text{ and } \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}.$$

5. External bisectors. The external bisectors of any two angles of $\triangle ABC$ are concurrent with the internal bisector of the third angle.

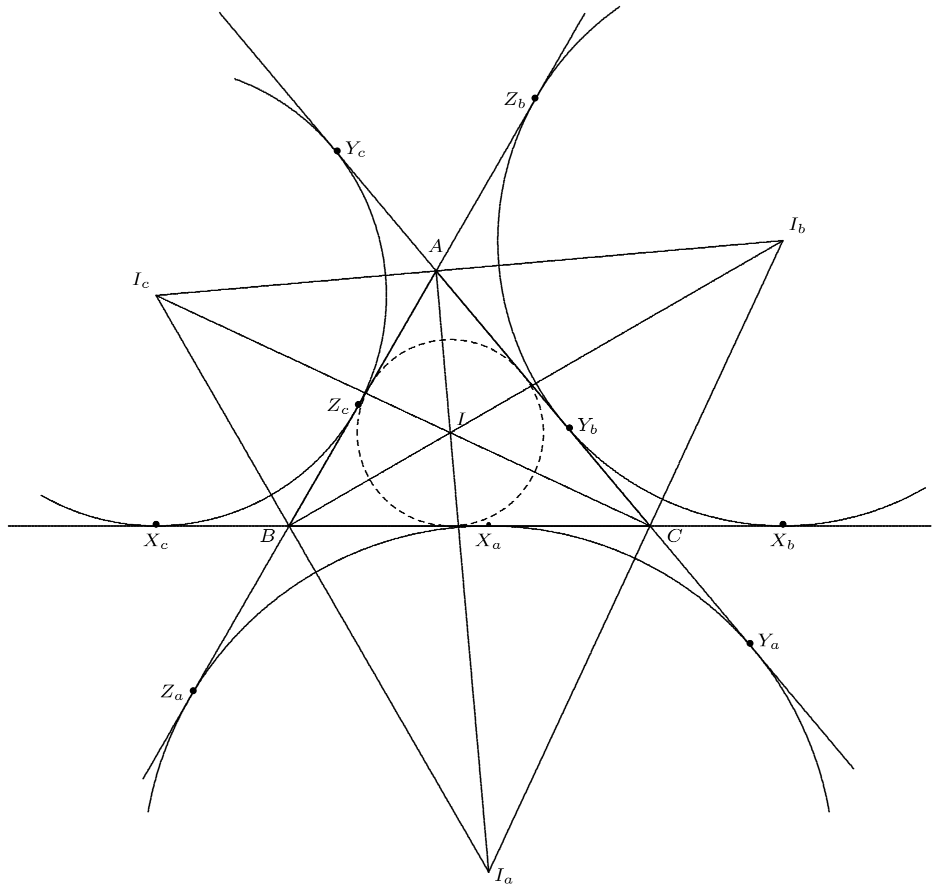

Figure 3.17: External angle bisectors

We call the circles centred at I_a, I_b, I_c with radii r_a, r_b, r_c respectively the *excircles* of the $\triangle ABC$, their centres I_a, I_b, I_c, the *excentres* and their radii r_a, r_b, r_c the *exradii*. Note that

(1) $AY_a = AZ_a = BZ_b = BX_b = CX_c = CY_c = s.$

$[2AY_a = AY_a + AZ_a = AB + BZ_a + AC + CY_a = AB + BX_a + X_aC + AC = AB + BC + AC = 2s.]$

(2) $BX_c = BZ_c = CX_b = CY_b = s - a$. $[BX_c = CX_c - BC = s - a.]$

$CY_a = CX_a = AY_c = AZ_c = s - b.$

$AZ_b = AY_b = BZ_a = BX_a = s - c.$

(3) $(ABC) = (s-a)r_a = (s-b)r_b = (s-c)r_c.$

$[(ABC) = \frac{1}{2}I_aZ_a \cdot AB + \frac{1}{2}I_aY_a \cdot AC - \frac{1}{2}I_aX_a \cdot BC = \frac{1}{2}r_a(c+b-a) = r_a(s-a).]$

(4) $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}.$

(5) $\triangle ABC$ is the orthic triangle of $\triangle I_a I_b I_c$.

Exercise 3.6 Prove that $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.

Exercise 3.7 Prove the identity

$$abc = s(s-b)(s-c) + s(s-c)(s-a) + s(s-a)(s-b) - (s-a)(s-b)(s-c),$$

where $2s = a + b + c$.

Exercise 3.8 Prove that $4R = r_a + r_b + r_c - r$

3.3 The nine-point circle

Theorem 3.13 *Let L be the foot of the perpendicular from O to BC. Then $AH = 2OL$.*

Proof. As $\triangle AEB$ is similar to $\triangle OLB$ with $AB : OB = c : R = 2\sin C$, we have $AE : OL = 2\sin C$. On the other hand, $\angle AHE = \angle C$ so that $AE : AH = AD : AC = \sin C$. Consequently, $AH = 2OL$. Alternatively, extend CO meeting the circumcircle of $\triangle ABC$ at the point P. Then $APBH$ is a parallelogram. Thus $AH = PB = 2OL$.

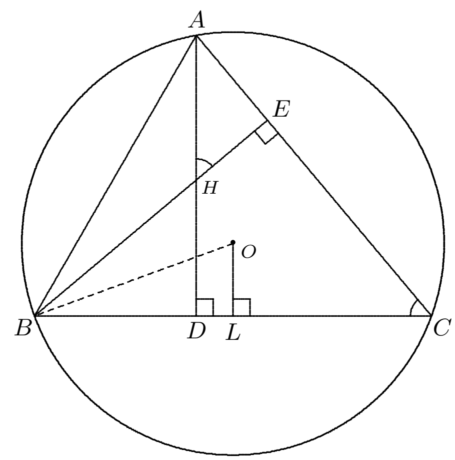

Figure 3.18

Theorem 3.14 *The circumcentre O, centroid G and orthocentre H of $\triangle ABC$ are collinear. The centroid G divides the segment OH into the ratio* $1 : 2$.

The line on which O, G, H lie is called the *Euler line* of $\triangle ABC$.

Proof. Since AH and OL are parallel, $\angle HAG = \angle OLG$. Also $AH = 2LO$ and $AG = 2LG$. Thus $\triangle HAG$ is similar to $\triangle OLG$ so that $\angle AGH = \angle LGO$. Therefore O, G, H are collinear.

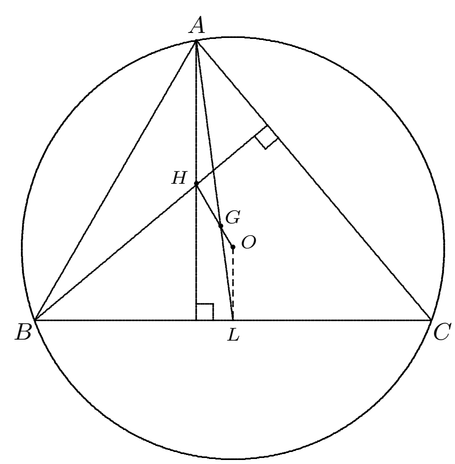

Figure 3.19

Let N be the midpoint of OH, where O is the circumcentre and H is the orthocentre of $\triangle ABC$. Using the fact that $OG : GH = 1 : 2$, we have $NG : GO = 1 : 2$. Since $GL : GA = 1 : 2$

and $\angle NGL = \angle OGA$, we see that $\triangle NGL$ is similar to $\triangle OGA$. Thus NL is parallel to OA and $NL : OA = 1 : 2$. If we take H_1 to be the midpoint of AH, then L, N, H_1 are collinear, NH_1 is parallel to OA and $NH_1 = \frac{1}{2}OA$. Since N is the midpoint of OH, we also have $ND = NL$. Consequently, $ND = NL = NH_1 = \frac{1}{2}OA = \frac{1}{2}R$.
Alternatively, if we take H_1=midpoint of AH, then $\triangle NHH_1$ is congruent to $\triangle NOL$ because $HH_1 = \frac{1}{2}AH = OL, NH = NO, \angle H_1HN = \angle LON$. Then L, N, H_1 are collinear. Thus $NH_1 = NL = ND = \frac{1}{2}OA$. [Here G is not involved in the proof.]

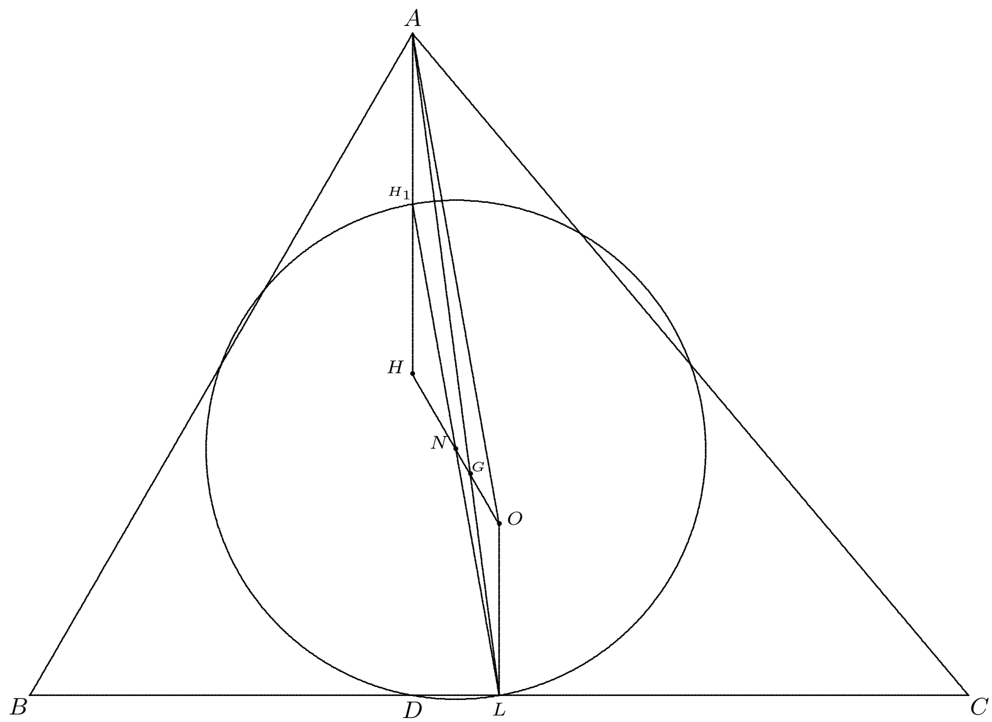

Figure 3.20: The Nine-point Circle

Theorem 3.15 (The Nine-point Circle) *The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices to the orthocentre, all lie on the same circle of radius $\frac{1}{2}R$ with centre at the midpoint of the Euler line. This circle is known as the nine-point circle or the Euler circle of the triangle.*

Exercise 3.9 Suppose the Euler line passes through a vertex of the triangle. Show that the triangle is either right-angled or isosceles or both.

Chapter Four

Quadrilaterals

Quadrilaterals are 4-sided polygons. Among them those whose vertices lie on a circle are called cyclic quadrilaterals. Cyclic quadrilaterals are the simplest objects like triangles in plane geometry and they possess remarkable properties. In this chapter, we shall explore some basic properties of quadrilaterals in Euclidean geometry.

4.1 Basic properties

1. For a quadrilateral $ABCD$, the following statements are equivalent:
(i) $ABCD$ is a parallelogram.
(ii) $AB \parallel DC$ and $AD \parallel BC$.
(iii) $AB = DC$ and $AD = BC$.
(iv) $AB \parallel DC$ and $AB = DC$.
(v) AC and BD bisect each other.

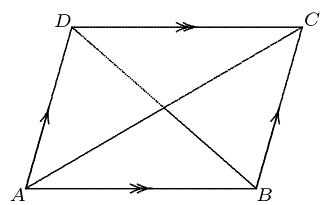

Figure 4.1

2. For a parallelogram $ABCD$, the following statements are equivalent:
(i) $ABCD$ is a rectangle.
(ii) $\angle A = 90°$.
(iii) $AC = BD$.

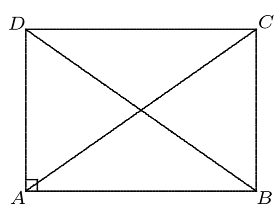

Figure 4.2

3. For a parallelogram $ABCD$, the following statements are equivalent:
(i) $ABCD$ is a rhombus
(ii) $AB = BC$.
(iii) $AC \perp BD$.
(iv) AC bisects $\angle A$.

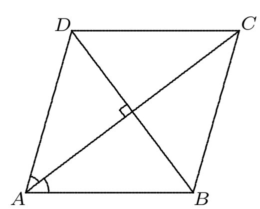

Figure 4.3

Example 4.1 In the figure, E, F are the midpoints of AB and BC respectively. Suppose DE and DF intersect AC at M and N respectively such that $AM = MN = NC$. Prove that $ABCD$ is a parallelogram.

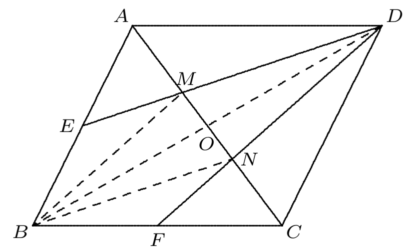

Figure 4.4

Solution. Join BM and BN. Let BD intersect AC at O. As $AE = EB, AM = MN$, we have EM is parallel to BN. Similarly, BM is parallel to FN. Therefore, $BMDN$ is a parallelogram. From this, we have $OB = OD$ and $OM = ON$. Since $AM = NC$, we also have $OA = OC$. Now the diagonals of $ABCD$ bisect each other. This means that $ABCD$ is a parallelogram.

Theorem 4.1 *The segments joining the midpoints of pairs of opposite sides of a quadrilateral and the segment joining the midpoints of the diagonals are concurrent and bisect one another.*

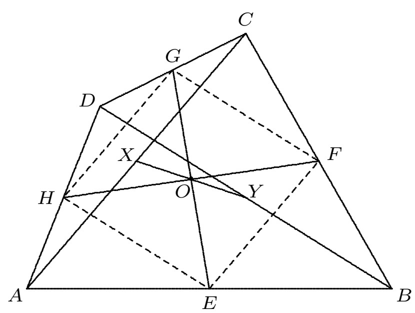

Figure 4.5: XY passes through O

Proof. Consider a quadrilateral $ABCD$ with midpoints E, F, G, H of its sides as shown in the figure. By Varignon's theorem, $EFGH$ is a parallelogram. Thus the diagonals EG and FH of this parallelogram bisect each other. Now consider the quadrilateral (a crossed-quadrilateral in the figure) $ABDC$. By Varignon's theorem, the midpoints E, Y, G, X of its sides form a parallelogram. Thus EG and XY bisect each other. Consequently, EG, FH and XY are concurrent at their common midpoint O.

Definition 4.1 *A quadrilateral $ABCD$ is called a cyclic quadrilateral if its 4 vertices lie on a common circle. In this case the 4 points A, B, C, D are said to be concyclic.*

Regarding cyclic quadrilaterals, we have the following characterizations.

Theorem 4.2 *Let $ABCD$ be a convex quadrilateral. The following statements are equivalent.*
(a) $ABCD$ is a cyclic quadrilateral.
(b) $\angle BAC = \angle BDC$.
(c) $\angle A + \angle C = 180°$.
(d) $\angle ABE = \angle D$.

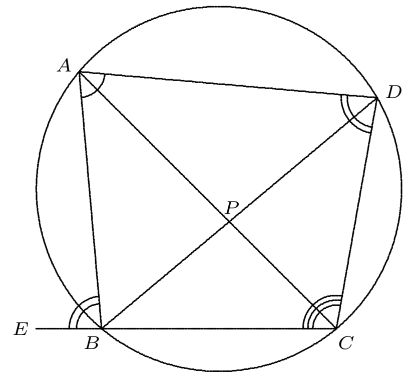

Figure 4.6: A cyclic quadrilateral

Proof. That (a) implies (b) follows from the property of circles, namely the angle subtended by a chord at any point on the circumference and on one side of the chord is a constant. To prove (b) implies (c), observe that $\triangle APB$ is similar to $\triangle DPC$. This in turn implies that $\triangle APD$ is similar to $\triangle BPC$. Thus $\angle BAC = \angle BDC$, $\angle ABD = \angle ACD$, $\angle CAD = \angle CBD$ and $\angle ADB = \angle ACB$. Therefore, $\angle A + \angle C = \angle BAC + \angle CAD + \angle ACB + \angle ACD = \frac{1}{2}(\angle A + \angle B + \angle C + \angle D) = 180°$. That (c) is equivalent to (d) is obvious. The part that (d) implies (a) is left as an exercise.

Exercise 4.1 Suppose the diagonals of a cyclic quadrilateral $ABCD$ intersect at a point P. Prove that $AP \cdot PC = BP \cdot PD$.

Theorem 4.3 *If a cyclic quadrilateral has perpendicular diagonals intersecting at P, then the line through P perpendicular to any side bisects the opposite side.*

Proof. Let XH be the line through P perpendicular to BC. We wish to prove X is the midpoint of AD.

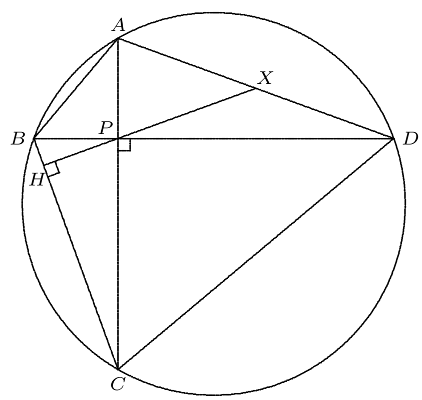

Figure 4.7: A cyclic quadrilateral with perpendicular diagonals

We have $\angle DPX = \angle BPH = \angle PCH = \angle ACB = \angle ADB = \angle XDP$. Thus the triangle XPD is isosceles. Similarly, the triangle XAP is isosceles. Consequently $XA = XP = XD$.

4.2 Ptolemy's theorem

Theorem 4.4 (The Simson line) *The feet of the perpendiculars from any point P on the circumcircle of a triangle ABC to the sides of the triangle are collinear.*

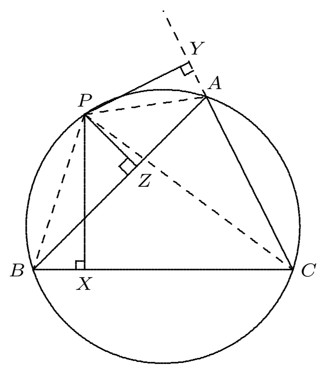

Figure 4.8: The Simson line

Proof. Referring to figure 4.8, we see that $PZAY, PXCY$ and $PACB$ are cyclic quadrilaterals. Therefore, $\angle PYZ = \angle PAZ = \angle PCX = \angle PYX$. This shows that Y, Z, X are collinear.

(Note that the converse of the statement in this theorem is also true. That is, if the feet of the perpendiculars from a point P to the sides of the triangle ABC are collinear, then P lies on the circumcircle of $\triangle ABC$.) The line containing the feet is known as the *Simson line*.

Theorem 4.5 (Ptolemy) *For any cyclic quadrilateral, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.*

Proof. Let $PBCA$ be a cyclic quadrilateral and let X, Y, Z be the feet of the perpendiculars from P onto the sides BC, AC, AB respectively. By previous theorem, X, Y, Z lie on the Simson line.

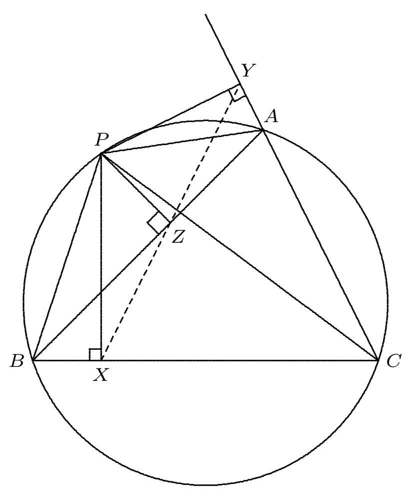

Figure 4.9: Ptolemy's theorem

The quadrilateral $AYPZ$ is cyclic. Since $\angle PYA = 90°$, the circle passing through A, Y, P, Z has diameter PA. Thus

$$\frac{YZ}{PA} = \sin\angle YAZ = \sin\angle BAC = \frac{a}{2R}.$$

That is $YZ = aPA/(2R)$. Similarly, by considering the cyclic quadrilaterals $PZXB$ and $PXCY$, we have $XZ = bPB/(2R)$ and $XY = cPC/(2R)$. As X, Y, Z lie on the Simson line, we have $XZ + ZY = XY$ so that $bPB/(2R) + aPA/(2R) = cPC/(2R)$. Canceling the common factor $2R$, we get $bPB + aPA = cPC$. That is

$$AC \cdot PB + BC \cdot PA = AB \cdot PC.$$

Ptolemy's Theorem can be strengthened by observing that if P is any point not on the circumcircle of $\triangle ABC$, then the equality $XZ + ZY = XY$ has to be replaced by the inequality $XZ + ZY > XY$ so that $AC \cdot PB + BC \cdot PA > AB \cdot PC$.

Theorem 4.6 *If P is a point not on the arc CA of the circumcircle of the triangle ABC, then*

$$AC \cdot PB + BC \cdot PA > AB \cdot PC.$$

Example 4.2 Let P be a point of the minor arc CD of the circumcircle of a square $ABCD$. Prove that

$$PA(PA + PC) = PB(PB + PD).$$

Solution. Refer to figure 4.10. Let $AB = a$. Applying Ptolemy's theorem to the cyclic quadrilaterals $PDAB$ and $PABC$, we have $PD \cdot BA + PB \cdot DA = PA \cdot DB$, and $PA \cdot BC + PC \cdot AB = PB \cdot AC$. That is $a(PD + PB) = \sqrt{2}a \cdot PA$ and $a(PA + PC) = \sqrt{2}a \cdot PB$. Canceling a common factor of a for both equations, we get $PD + PB = \sqrt{2}PA$ and $PA + PC = \sqrt{2}PB$. Thus $PA(PA + PC) = \sqrt{2}PA \cdot PB = PB(PB + PD)$.

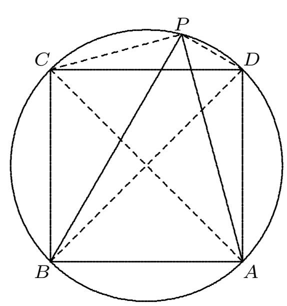

Figure 4.10

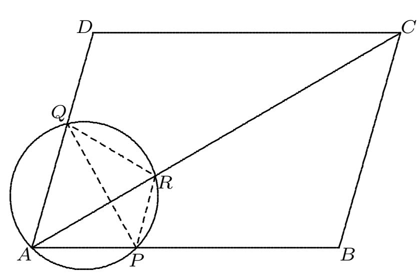

Figure 4.11

Exercise 4.2 In a parallelogram $ABCD$, a circle passing through A meets AB, AD and AC at P, Q and R respectively. Prove that $AP \cdot AB + AQ \cdot AD = AR \cdot AC$. See figure 4.11.

4.3 Area of a quadrilateral

Theorem 4.7 (Brahmagupta's Formula) *If a cyclic quadrilateral has sides a, b, c, d and semi-perimeter s, then its area K is given by*

$$K^2 = (s-a)(s-b)(s-c)(s-d).$$

Proof. Let $ABCD$ be a cyclic quadrilateral. Let the length of BD be n. First note that $\angle A + \angle C = 180^\circ$ so that $\cos A = -\cos C$ and $\sin A = \sin C$. Thus by Cosine law,

$$a^2 + b^2 - 2ab\cos A = n^2 = c^2 + d^2 - 2cd\cos C,$$

giving

$$2(ab+cd)\cos A = a^2 + b^2 - c^2 - d^2. \tag{4.1}$$

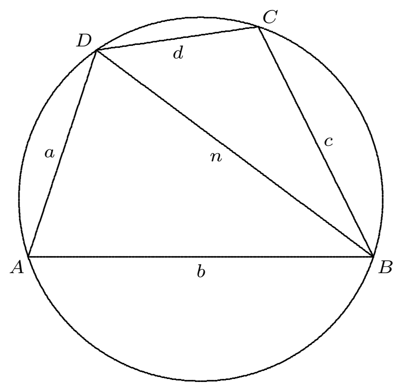

Figure 4.12: Brahmagupta's Formula

Since

$$K = \frac{1}{2}ab\sin A + \frac{1}{2}cd\sin C = \frac{1}{2}(ab+cd)\sin A,$$

we also have

$$2(ab+cd)\sin A = 4K. \tag{4.2}$$

Adding the squares of (4.1) and (4.2), we obtain

$$4(ab+cd)^2 = (a^2+b^2-c^2-d^2)^2 + 16K^2,$$

giving

$$16K^2 = (2ab+2cd)^2 - (a^2+b^2-c^2-d^2)^2.$$

Thus $$\begin{aligned} 16K^2 &= (2ab+2cd)^2 - (a^2+b^2-c^2-d^2)^2 \\ &= (2ab+2cd+a^2+b^2-c^2-d^2)(2ab+2cd-a^2-b^2+c^2+d^2) \end{aligned}$$

$$= ((a+b)^2 - (c-d)^2)((c+d)^2 - (a-b)^2)$$
$$= (a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$$
$$= (2s-2d)(2s-2c)(2s-2b)(2s-2a).$$

Therefore, $K^2 = (s-a)(s-b)(s-c)(s-d)$.
Setting $d = 0$, we obtain Heron's formula for the area of a triangle:

$$(ABC)^2 = s(s-a)(s-b)(s-c).$$

Exercise 4.3 In a trapezium $ABCD$, AB is parallel to DC and E is the midpoint of BC. Prove that $2(AED) = (ABCD)$.

Exercise 4.4 Suppose the quadrilateral $ABCD$ has an inscribed circle. Show that $AB + CD = BC + DA$.

Exercise 4.5 Suppose the cyclic quadrilateral $ABCD$ has an inscribed circle. Show that $(ABCD) = \sqrt{abcd}$.

Exercise 4.6 Let $ABCD$ be a convex quadrilateral. Prove that its area K is given by

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right).$$

Exercise 4.7 Let $ABCDE$ be the pentagon whose vertices are intersections of the extensions of non-neighboring sides of a pentagon $HIJKL$. Prove that the neighboring pairs of the circumcircles of the triangles ALH, BHI, CIJ, DJK, EKL intersect at 5 concyclic points P, Q, R, S, T.

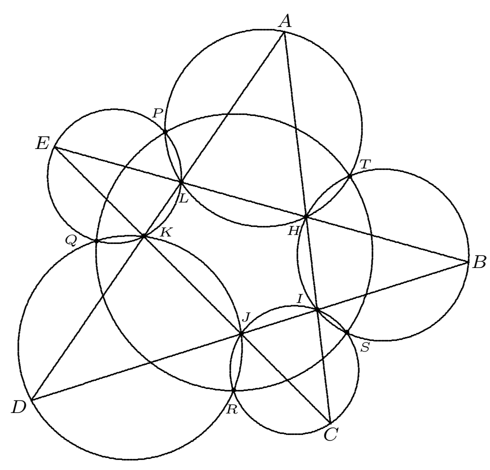

Figure 4.13: Miquel's 5-circle theorem

[Hint: Note that J, S, B, E are concyclic since $\angle EBS = \angle HBS = \angle CIS = \angle CJS$. Similarly, J, Q, E, B are concyclic. Thus J, S, B, E, Q are concyclic. Now try to show P, T, S, Q are concyclic by showing that $\angle QPT + \angle QST = 180°$.]

Remark 4.1 This is Miquel's 5-circle theorem first proved by Miquel in 1838. This problem was proposed by president Jiang Zemin of PRC to the students of Háo Jiāng Secondary School in Macau during his visit to the school in 20 December 2000.

4.4 Pedal triangles

Definition 4.2 *For any point P on the plane of a triangle ABC, the foot of the perpendiculars from P onto the sides of the triangle ABC form a triangle $A_1B_1C_1$ called the pedal triangle of the point P with respect to the triangle ABC.*

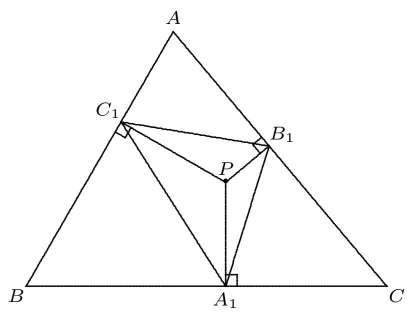

Figure 4.14: Pedal triangle

Theorem 4.8 *Let $A_1B_1C_1$ be the pedal triangle of the point P with respect to the triangle ABC. Then*

$$(A_1B_1C_1) = \frac{R^2 - OP^2}{4R^2}(ABC),$$

where O is the circumcentre and R is the circumradius of the triangle ABC.

Proof. Extend BP meeting the circumcircle of $\triangle ABC$ at B_2. Join B_2C. As in the figure, $\angle A_1 = \alpha + \beta = \angle B_2CP$.

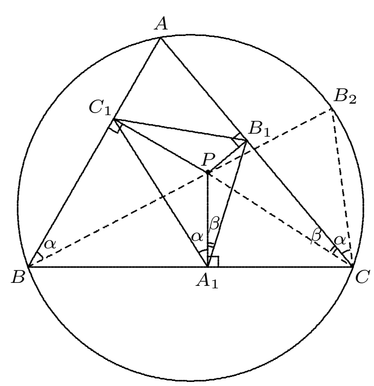

Figure 4.15: Area of the pedal triangle

Thus

$$(A_1B_1C_1) = \frac{1}{2}A_1B_1 \cdot A_1C_1 \cdot \sin A_1 = \frac{1}{2}(PC\sin C)(PB\sin B)\sin\angle B_2CP.$$

Also,

$$\frac{\sin\angle B_2CP}{\sin A} = \frac{\sin\angle B_2CP}{\sin\angle BB_2C} = \frac{PB_2}{PC}.$$

Thus,

$$\begin{aligned}(A_1B_1C_1) &= \frac{1}{2}PB_2 \cdot PB\sin A\sin B\sin C\\ &= \frac{1}{2}(R^2 - OP^2)\sin A\sin B\sin C\\ &= \frac{R^2 - OP^2}{4R^2}(ABC).\end{aligned}$$

The above result is a generalization of Simson's theorem.

Corollary 4.9 *The point P lies on the circumcircle of $\triangle ABC$ if and only if the area of the pedal triangle is zero if and only if A_1, B_1, C_1 are collinear.*

Exercise 4.8 Show that the third pedal triangle is similar to the original triangle.

Exercise 4.9 Let P be a point on the circumcircle of the triangle ABC. Prove that its Simson line with respect to the triangle ABC bisects PH, where H is the orthocentre of the triangle ABC.

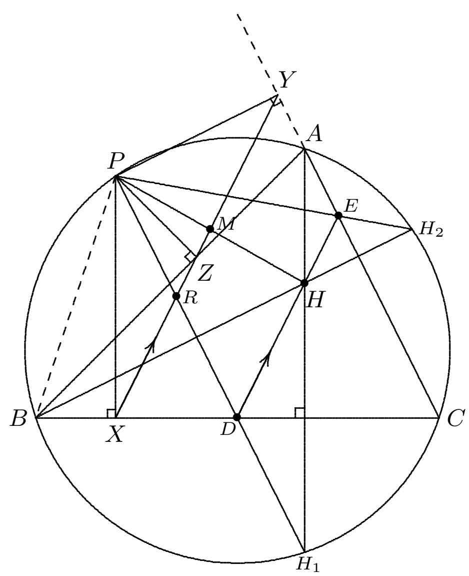

Figure 4.16: The Simson line bisects PH.

[Hint: Let X, Y and Z be the feet of perpendiculars from P onto the sides BC, CA and AB respectively. It is well-known that X, Y and Z are collinear. The line on which they lie is called the Simson line. Extend AH, BH and CH meeting the circumcircle of the triangle ABC at H_1, H_2 and H_3 respectively. Let PH_1 intersect BC at D, PH_2 intersect CA at E and PH_3 intersect AB at F. Join PB.]

Exercise 4.10 Let P and P' be diametrically opposite points on the circumcircle of the triangle ABC. Prove that the Simson lines of P and P' meet at right angle on the nine-point circle of the triangle.

Exercise 4.11 Prove Brahmagupta-Mahavira formula: Let $ABCD$ be a cyclic quadrilateral with $AB = b, BC = c, CD = d, DA = a$ and $AC = m, BD = n$. Then

$$\frac{m}{n} = \frac{ab + cd}{ad + bc}.$$

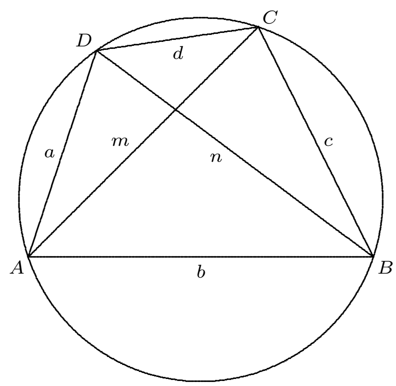

Figure 4.17: Brahmagupta-Mahavira formula

[Hint: Interchange the sides with lengths a and b, also a and d. Apply Ptolemy's theorem.]