Questions Bank
Axiomatic System
Second Class

Chapter 1

Exercise 1.1. Consider the following axiom set.

Postulate 1. There are at least two buildings on campus.

Postulate 2. There is exactly one sidewalk between any two buildings.

Postulate 3. Not all the buildings have the same sidewalk between them.

a. What are the primitive terms in this axiom set?

b. Deduce the following theorems:

Theorem 1. There are at least three buildings on campus.

Theorem 2. There are at least two sidewalks on campus.

c. Show by the use of models that it is possible to have

exactly two sidewalks and three buildings;

at least two sidewalks and four buildings; and, exactly three sidewalks and three buildings.

d. Is the system complete? Explain.

e. Find two isomorphic models.

f. Demonstrate the independence of the axioms.

Exercise 1.2. Consider the following axiom set.

A1. Every hive is a collection of bees.

A2. Any two distinct hives have one and only one bee in common.

A3. Every bee belongs to two and only two hives.

A4. There are exactly four hives.

a. What are the undefined terms in this axiom set?

b. Deduce the following theorems:

T1. There are exactly six bees.

T2. There are exactly three bees in each hive.

T3. For each bee there is exactly one other bee not in the same hive with it.

c. Find two isomorphic models.

d. Demonstrate the independence of the axioms.

Exercise 1.3. Consider the following axiom set.

P1. Every herd is a collection of cows.

P2. There exist at least two cows.

P3. For any two cows, there exists one and only one herd containing both cows.

P4. For any herd, there exists a cow not in the herd.

P5. For any herd and any cow not in the herd, there exists one and only one other herd containing

the cow and not containing any cow that is in the given herd.

a. What are the primitive terms in this axiom set?

b. Deduce the following theorems:

T1. Every cow is contained in at least two herds.

T2. There exist at least four distinct cows.

T3. There exist at least six distinct herds.

c. Find two isomorphic models.

d. Demonstrate the independence of the axioms.

Chapter 2

Exercise 2: Prove the Followings

Proposition I.1. To construct an equilateral triangle.

Proposition I.2 To place a straight line equal to a given straight line with one end at a given point.

Proposition I.3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Proposition I.4. (SAS) If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals to the triangle, and the remaining angles equal the remaining angles respectively.

Proposition I.5. In isosceles triangles, the angles at the base equal one another; and if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition I.6. If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Proposition I.7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

Proposition I.8 (SSS) If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proposition I.9. To bisect a given rectilinear angle.

Proposition I.10. To bisect a given finite straight line.

Proposition I.11. To draw a straight line at right angles to a given straight line from a given point on it.

Proposition I.12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Proposition I.13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition I.14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition I.15. If two straight lines cut one another, then they make the vertical angles equal to one another.

Proposition I.16. (Exterior Angle Theorem) In any triangle, if any one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles.

Proposition I.17. In any triangle, two angles taken together in any manner are less than two right angles.

Proposition I.18. In any triangle, the angle opposite the greater side is greater.

Proposition I.19. In any triangle, the side opposite the greater angle is greater.

Proposition I.20. In any triangle, the sum of any two sides is greater than the remaining one.

Proposition I.21. If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangles, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Proposition I.22. To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition I.23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition I.24. If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Proposition I.25. If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have one of the angles contained by the equal straight lines greater than the other.

Proposition I.26. (ASA or AAS) If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angles equals the remaining angle.

Proposition I.27. If a straight line falling on two straight lines make the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition I.28. If a straight line falling on two straight lines make the exterior angles equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angle, then the straight lines are parallel to one another.

Exercise 2.1: If two sides of a triangle are equal, the line which bisects the angle between the equal sides bisects the third side.

Exercise 2.2: If two sides of a triangle are equal, the line joining the corner (or vertex) between the equal sides and the mid-point of the third side bisects the angle between the equal sides.

Exercise 2.3: line PM is perpendicular to line AB at point M and PM bisects AB at M Prove that PA =PB.

Exercise 2.4: If two angles of a triangle are equal, the sides opposite these angles are equal.

Exercise 2.5: If a quadrilateral has three right angles, its fourth angle is a right angle also.

Exercise 2.6: If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

Exercise 2.7: A radius perpendicular to a chord of a circle bisects the chord.

Exercise 2.8: The sum of the angles of a triangle is two right angles.

Exercise 2.9: Equal chords of a circle are equally distant from the center of the circle.

Exercise 2.10: The opposite angles of a parallelogram are equal.

Chapter 3

Theorem 3.10 (Stewart) *If* $\dfrac{BP}{PC} = \dfrac{m}{n}$*, then* $nAB^2 + mAC^2 = (m+n)AP^2 + \dfrac{mn}{m+n}BC^2$

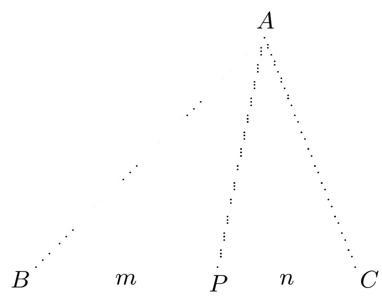

Figure 3.10: Stewart's theorem

Theorem 3.11 (Pappus' theorem) *Let P be the midpoint of the side BC of a triangle ABC. Then*

$$AB^2 + AC^2 = 2(AP^2 + BP^2).$$

. The 3 medians AD, BE and CF of $\triangle ABC$ are concurrent. Their common point, denoted by G, is called the *centroid* of $\triangle ABC$.

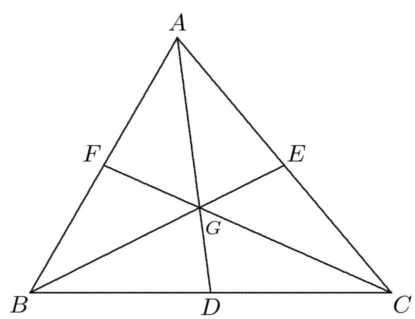

Figure 3.12: Medians

We have

(1) $(AGF) = (BGF) = (BGD) = (CGD) = (CGE) = (AGE)$.

(2) $AG : GD = BG : GE = CG : GF = 2 : 1$.

(3) (Apollonius'theorem)

$AD^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2$,

$BE^2 = \frac{1}{2}(c^2 + a^2) - \frac{1}{4}b^2$,

$CF^2 = \frac{1}{2}(a^2 + b^2) - \frac{1}{4}c^2$.

. The internal bisectors of the 3 angles of $\triangle ABC$ are concurrent. Their common point, denoted by I, is called the *incentre* of $\triangle ABC$. It is equidistant to the sides of the triangle. Let r denote the distance from I to each side. The circle centred at I with radius r is called the *incircle* of $\triangle ABC$, and r is called the *inradius*.

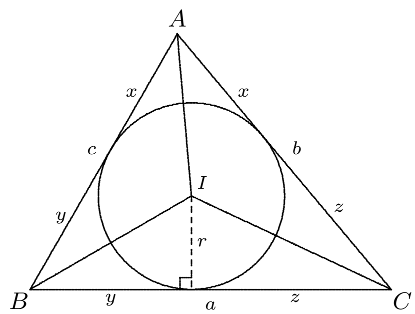

Figure 3.13: Angle bisectors

Let $s = \frac{1}{2}(a+b+c)$ be the *semi-perimeter*. We have

(1) $x = s-a$, $y = s-b$ and $z = s-c$.

(2) $(ABC) = sr$.

(3) $abc = 4srR$.

Exercise 3.1 Prove that $\sin A = (2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)^{\frac{1}{2}}/(2bc)$.

Theorem 3.12 *The orthocentre of an acute-angled triangle is the incentre of its orthic triangle.*

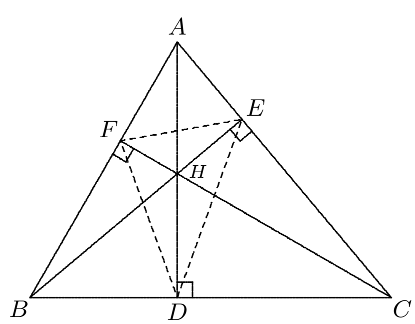

Figure 3.14: Altitudes

Exercise 3.2 In an acute-angled $\triangle ABC$ $AB < AC$, BD and CE are the altitudes. Prove that
(i) $BD < CE$
(ii) $AD < AE$
(iii) $AB^2 + CE^2 < AC^2 + BD^2$
(iv) $AB + CE < AC + BD$
(v) Is it true that $AB^n + CE^n < AC^n + BD^n$ for all positive integer n?

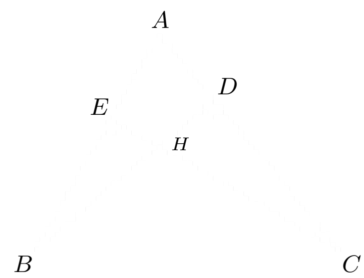

Figure 3.16: $AB^2 + CE^2 < AC^2 + BD^2$

Exercise 3.3 Prove Heron's formula that for a triangle ABC, we have

$$(ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

Exercise 3.4 Prove that if I is the incentre of the triangle ABC, then $AI^2 = bc(s-a)/s$.

Exercise 3.5 Prove that for any triangle ABC,

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc} \text{ and } \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

(5) $\triangle ABC$ is the orthic triangle of $\triangle I_a I_b I_c$.

Exercise 3.6 Prove that $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.

Exercise 3.7 Prove the identity

$$abc = s(s-b)(s-c) + s(s-c)(s-a) + s(s-a)(s-b) - (s-a)(s-b)(s-c),$$

where $2s = a + b + c$.

Exercise 3.8 Prove that $4R = r_a + r_b + r_c - r$

Exercise 3.9 Suppose the Euler line passes through a vertex of the triangle. Show that the triangle is either right-angled or isosceles or both.

Chapter 4

Theorem 4.6 *If P is a point not on the arc CA of the circumcircle of the triangle ABC, then*

$$AC \quad PB + BC \cdot PA > AB \cdot PC.$$

Exercise 4.2 In a parallelogram $ABCD$, a circle passing through A meets AB, AD and AC at P Q and R respectively. Prove that $AP \quad AB + AQ \cdot AD = AR \cdot AC$. See figure 4.11.

Exercise 4.3 In a trapezium $ABCD$, AB is parallel to DC and E is the midpoint of BC. Prove that $2(AED) = (ABCD)$.

Exercise 4.4 Suppose the quadrilateral $ABCD$ has an inscribed circle. Show that $AB + CD = BC + DA$.

Exercise 4.5 Suppose the cyclic quadrilateral $ABCD$ has an inscribed circle. Show that $(ABCD) = \sqrt{abcd}$.

Exercise 4.6 Let $ABCD$ be a convex quadrilateral. Prove that its area K is given by

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right).$$

Exercise 4.7 Let $ABCDE$ be the pentagon whose vertices are intersections of the extensions of non-neighboring sides of a pentagon $HIJKL$. Prove that the neighboring pairs of the circumcircles of the triangles ALH, BHI, CIJ, DJK, EKL intersect at 5 concyclic points P, Q, R, S, T.

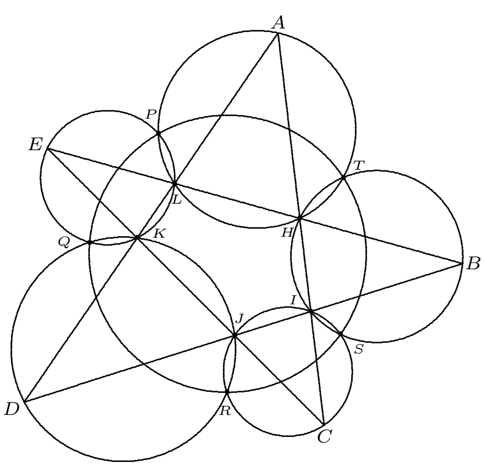

Figure 4.13: Miquel's 5-circle theorem

[Hint: Note that J, S, B, E are concyclic since $\angle EBS = \angle HBS = \angle CIS = \angle CJS$. Similarly, J, Q, E, B are concyclic. Thus J, S, B, E, Q are concyclic. Now try to show P, T, S, Q are concyclic by showing that $\angle QPT + \angle QST = 180°$.]

Corollary 4.9 *The point P lies on the circumcircle of $\triangle ABC$ if and only if the area of the pedal triangle is zero if and only if A_1, B_1, C_1 are collinear.*

Exercise 4.8 Show that the third pedal triangle is similar to the original triangle.

Exercise 4.9 Let P be a point on the circumcircle of the triangle ABC. Prove that its Simson line with respect to the triangle ABC bisects PH, where H is the orthocentre of the triangle ABC

Exercise 4.10 Let P and P' be diametrically opposite points on the circumcircle of the triangle ABC. Prove that the Simson lines of P and P' meet at right angle on the nine-point circle of the triangle.

Exercise 4.11 Prove Brahmagupta-Mahavira formula: Let $ABCD$ be a cyclic quadrilateral with $AB = b, BC = c, CD = d, DA = a$ and $AC = m, BD = n$. Then

$$\frac{m}{n} = \frac{ab + cd}{ad + bc}.$$

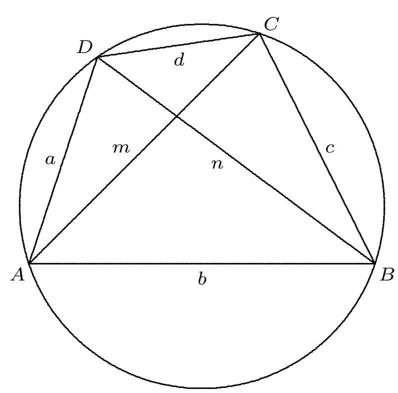

Figure 4.17: Brahmagupta-Mahavira formula

[Hint: Interchange the sides with lengths a and b, also a and d. Apply Ptolemy's theorem.]