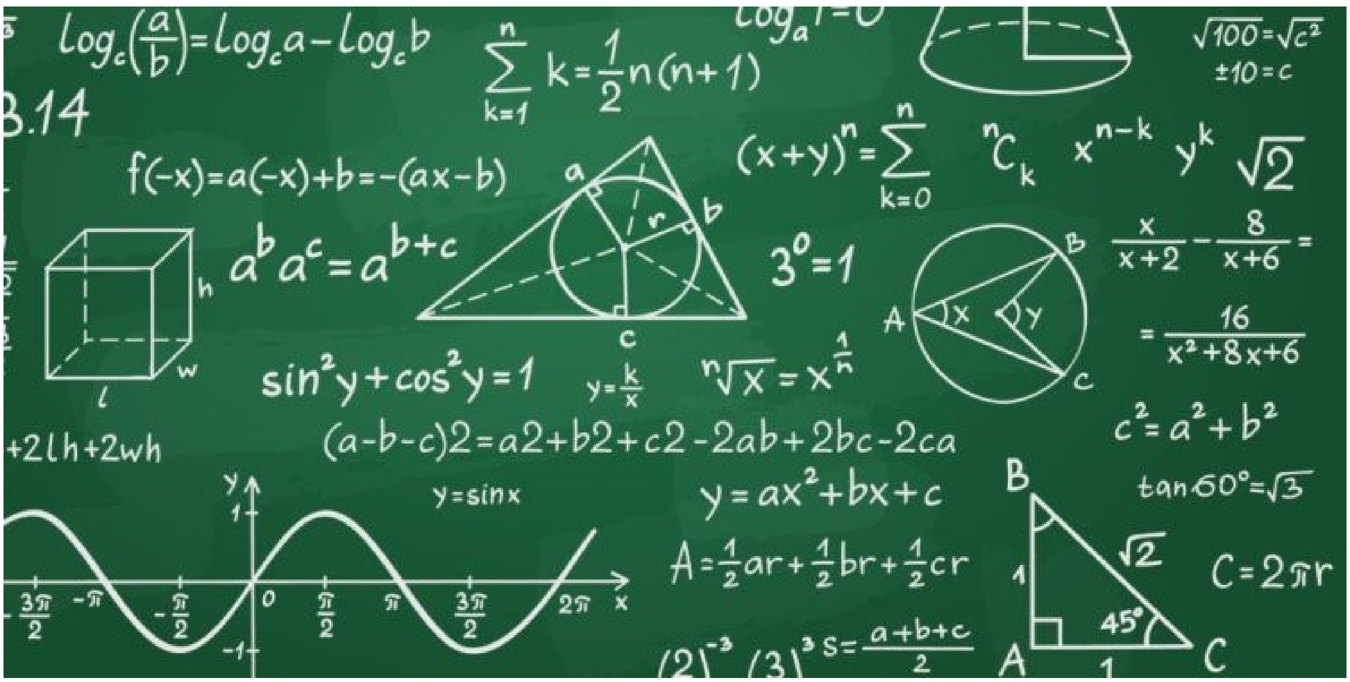
Geometry

Mathematics Department

Second Class – Second Semester

2023 - 2024



Chapter 5

Concurrence

When several lines meet at a common point, they are said to be *concurrent*. The concurrence of lines occurs very often in many geometric configurations. The point of concurrence usually plays a significant and special role in the geometry of the figure. In this chapter, we will introduce several of these points and the classical Ceva's theorem which gives a necessary and sufficient condition for three cevians of a triangle to be concurrent. We will illustrate with many applications that stem out from Ceva's theorem.

5.1 Ceva's theorem

Definition 5.1 *The line segment joining a vertex of $\triangle ABC$ to any given point on the opposite side (or extended) is called a cevian.*

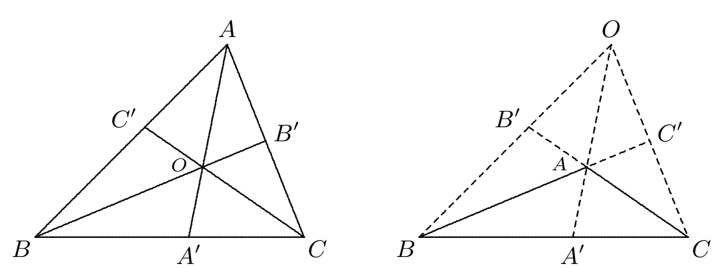


Figure 5.1: Three cevians meet a point

Theorem 5.1 (Ceva) *Three cevians AA', BB', CC' of $\triangle ABC$ are concurrent if and only if*

$$\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}\cdot\frac{AC'}{C'B}=1.$$

[*Here directed segments are used.*]

Proof. First suppose the 3 cevians AA', BB', CC' are concurrent. Draw a line through A parallel to BC meeting the extension of BB' and CC' at D and E respectively. See Figure 5.2. Then

$$\frac{CB'}{B'A}=\frac{BC}{AD},\quad \frac{AC'}{C'B}=\frac{EA}{BC}.$$

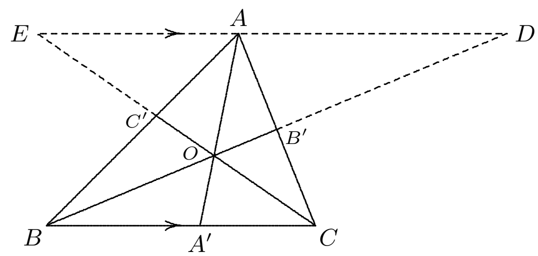


Figure 5.2: Ceva's Theorem

Since $\frac{BA'}{AD} = \frac{A'O}{OA} = \frac{A'C}{EA}$, we have $\frac{BA'}{A'C} = \frac{AD}{EA}$. Thus

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = \frac{AD}{EA} \cdot \frac{BC}{AD} \cdot \frac{EA}{BC} = 1.$$

To prove the converse, suppose

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1. \tag{5.1}$$

Let's consider the case where A', B', C' lie in the interior of BC, CA, AB, respectively. The case that two of them are outside is similar. Let BB' and CC' meet at a point O. Then connect AO meeting BC at a point A''. It suffices to prove $A' = A''$. By the forward implication of Ceva's theorem ,we have

$$\frac{BA''}{A''C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1. \tag{5.2}$$

Comparing equations (5.1) and (5.2), we have $\frac{BA'}{A'C} = \frac{BA''}{A''C}$. Thus $A' = A''$.
There is an alternate proof using area. As

$$\frac{BA'}{A'C} = \frac{(ABA')}{(AA'C)} = \frac{(OBA')}{(OA'C)} = \frac{(ABO)}{(ACO)},\ \frac{CB'}{B'A} = \frac{(BCO)}{(BAO)},\ \frac{AC'}{C'B} = \frac{(CAO)}{(CBO)}$$

we have

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1.$$

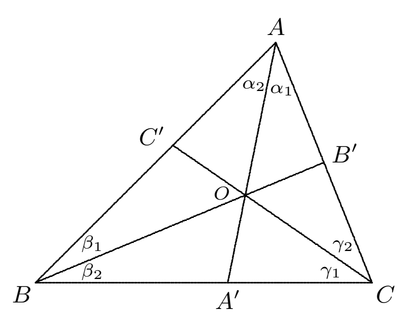


Figure 5.3: Trigonometric version of Ceva's Theorem

There is a trigonometric version of Ceva's theorem in terms of the sines of the angles that the cevians make with the sides of the triangles at the vertices. Refer to Figure 5.3. Let $\angle CAA' = \alpha_1$, $\angle A'AB = \alpha_2$, $\angle ABB' = \beta_1$, $\angle B'BC = \beta_2$, $\angle BCC' = \gamma_1$ and $\angle C'CA = \gamma_2$.

Then $\sin\alpha_1 = A'C\cdot\dfrac{\sin C}{AA'}$, $\sin\alpha_2 = BA'\cdot\dfrac{\sin B}{AA'}$, so that $\dfrac{\sin\alpha_2}{\sin\alpha_1} = \dfrac{BA'}{A'C}\dfrac{\sin B}{\sin C}$. Similarly,

$$\frac{\sin\beta_2}{\sin\beta_1} = \frac{CB'}{B'A}\frac{\sin C}{\sin A} \text{ and } \frac{\sin\gamma_2}{\sin\gamma_1} = \frac{AC'}{C'B}\frac{\sin A}{\sin B}.$$

Therefore, by Ceva's theorem, AA', $BB'CC'$ are concurrent if and only if

$$\frac{\sin\alpha_2}{\sin\alpha_1}\cdot\frac{\sin\beta_2}{\sin\beta_1}\cdot\frac{\sin\gamma_2}{\sin\gamma_1} = 1.$$

Example 5.1 We can use the trigonometric version of Ceva's theorem to deduce that the three altitudes of a triangle are concurrent.

5.2 Common points of concurrence

The common points of concurrence that arise from a triangle consist of the following.

1. The 3 medians of $\triangle ABC$ are concurrent. Their common point, denoted by G, is called the **centroid** of $\triangle ABC$.
2. The 3 altitudes of $\triangle ABC$ are concurrent. Their common point, denoted by H, is called the **orthocentre** of $\triangle ABC$.
3. The internal bisectors of the 3 angles of $\triangle ABC$ are concurrent. Their common point, denoted by I, is called the **incentre** of $\triangle ABC$.
4. The internal bisector of $\angle A$ and the external bisectors of the other two angles of $\triangle ABC$ are concurrent. Their common point, denoted by I_a, is called the **excentre** of $\triangle ABC$. Similarly, there are excentres I_b and I_c.
5. The three perpendicular bisectors of a triangle $\triangle ABC$ are concurrent. Their common point, denoted by O is called the **circumcentre** of $\triangle ABC$.
6. The cevians where the feet are the tangency points of the incircle (or excircle) of a triangle are concurrent. This common point is called the **Gergonne** point. Thus there are 4 Gergonne points for a triangle.

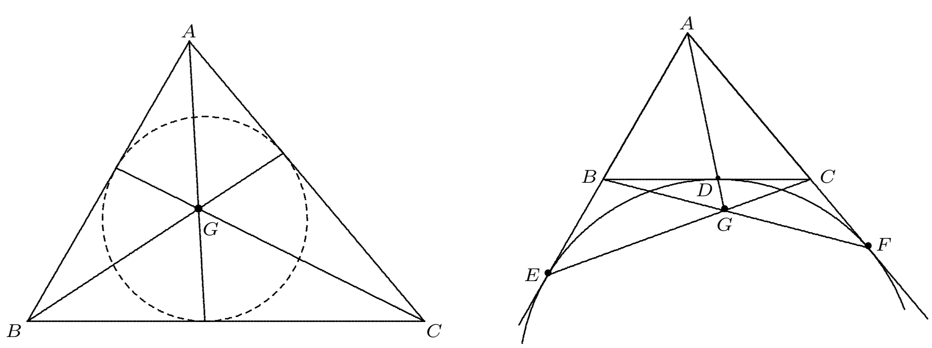


Figure 5.4: Gergonne point

Example 5.2 In $\triangle ABC$, D, E and F are the feet of the altitudes from A, B and C onto the sides BC, CA and AB respectively. Prove that the perpendiculars from A onto EF, from B onto DF and from C onto EF are concurrent.

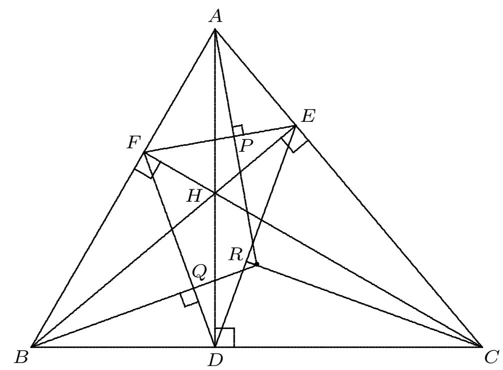


Figure 5.5: A point of concurrence

Solution. We shall use the trigonometric version of Ceva's theorem. First $\sin\angle FAP = \cos\angle AFP = \cos C$. Similarly, $\sin\angle PAE = \cos B$, $\sin\angle ECR = \cos B$, $\sin\angle RCD = \cos A$, $\sin\angle DBQ = \cos A$ and $\sin\angle QBF = \cos C$. Thus

$$\frac{\sin\angle FAP}{\sin\angle PAE}\cdot\frac{\sin\angle ECR}{\sin\angle RCD}\cdot\frac{\sin\angle DBQ}{\sin\angle QBF} = 1,$$

and by Ceva's theorem, AP, BQ and CR are concurrent. In fact the point of concurrence is the circumcentre of the triangle ABC.

Example 5.3 In an acute-angled triangle ABC, N is a point on the altitude AM. The line CN, BN meet AB and AC respectively at F and E. Prove that $\angle EMN = \angle FMN$.

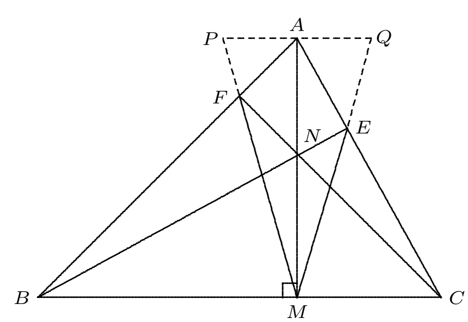


Figure 5.6: $\angle EMN = \angle FMN$

Solution. Construct a line through A parallel to BC meeting the extensions of MF and ME at P and Q respectively. Thus $\angle MAP = 90°$. As $\triangle PAF$ is similar to $\triangle MBF$ and $\triangle QAE$ is similar to $\triangle MCE$, we have

$$PA = \frac{AF}{FB} \cdot BM, \quad AQ = \frac{EA}{EC} \cdot MC.$$

Thus $\frac{PA}{AQ} = \frac{AF}{FB} \cdot \frac{BM}{MC} \cdot \frac{CE}{EA} = 1$, by Ceva's Theorem. Therefore, $PA = AQ$. It follows that $\angle EMN = \angle FMN$.

Example 5.4 On the plane, there are 3 mutually and externally disjoint circles Γ_1, Γ_2 and Γ_3 centred at X_1, X_2 and X_3 respectively. The two internal common tangents of Γ_2 and Γ_3, (Γ_3 and Γ_1, Γ_1 and Γ_2) meet at P, (Q, R respectively). Prove that X_1P, X_2Q and X_3R are concurrent.

Solution. Let the radii of Γ_1, Γ_2 and Γ_3 be r_1, r_2 and r_3 respectively.

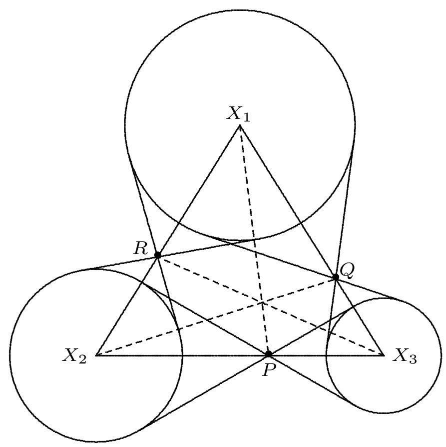


Figure 5.7: X_1P, X_2Q and X_3R are concurrent

Then $X_1R : RX_2 := r_1 : r_2$, $X_2P : PX_3 := r_2 : r_3$ and $X_3Q : QX_1 := r_3 : r_1$. Thus

$$\frac{X_1R}{RX_2} \cdot \frac{X_2P}{PX_3} \cdot \frac{X_3Q}{QX_1} = 1.$$

By Ceva's Theorem, X_1P, X_2Q and X_3Z are concurrent.

Example 5.5 Prove that the 3 cevians of a triangle ABC such that each of them bisects the perimeter of the triangle ABC are concurrent.

Solution. Let $BC = a, AC = b, AB = c$ and $s = \frac{1}{2}(a + b + c)$. Let A', B', C' be the points on BC, AC, AB such that AA', BB', CC' each bisects the perimeter of $\triangle ABC$. Then $BA' + A'C = a$ and $c + BA' = b + A'C$. Thus $BA' = s - c$ and $A'C = s - b$. Similarly, $CB' = s - a, B'A = s - c, AC' = s - b$ and $C'B = s - a$. Thus

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1,$$

so that by Ceva's Theorem, AA', BB', CC' are concurrent. The point of concurrence is called the **Nagel point** of $\triangle ABC$. It is also the point of concurrence of the cevians that join the vertices of the triangle to the points of tangency of the excircles on the opposite sides.

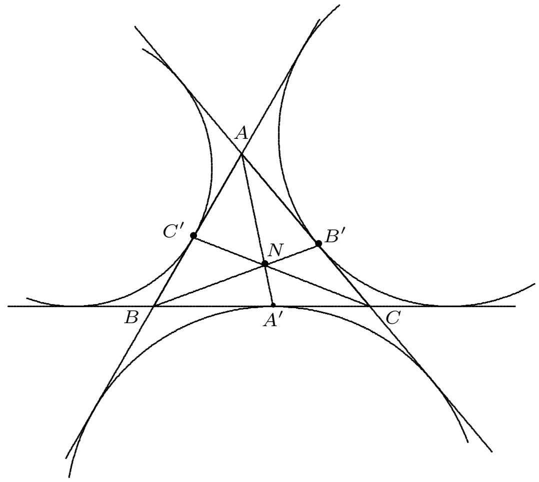


Figure 5.8: Nagel point

Remark 5.1 If D, E, F are the points of tangency of the incircle to the sides BC, CA and AB, and DX, EY, FZ are the diameters of the incircle respectively, then AX, BY, CZ concurs at the Nagel point. In fact we can prove that the extension of AX, BY and CZ meet BC, CA and AB at A', B' and C', respectively. To see this, we show that the point A' on BC which is the point of tangency of the excircle with the side BC together with the points X and A are collinear. This is because a *homothety* mapping the incircle to this excircle must map the highest point X of the incircle to the highest point A' of the excircle.

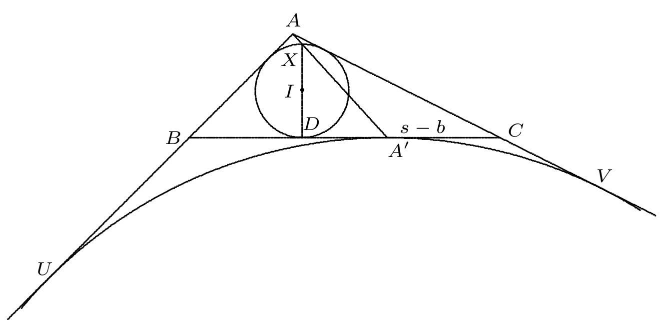


Figure 5.9: The incircle and excircle

Exercise 5.1 Let $ABCD$ be a trapezium with AB parallel to CD. Let M and N be the midpoints of AB and CD respectively. Prove that MN, AC and BD are concurrent.

Exercise 5.2 Suppose a circle cuts the sides of a triangle $A_1A_2A_3$ at the points $X_1, Y_1, X_2, Y_2, X_3, Y_3$. Show that if A_1X_1, A_2X_2, A_3X_3 are concurrent, then A_1Y_1, A_2Y_2, A_3Y_3 are concurrent.

[Hint: Observe that $X_1A_2 \cdot Y_1A_2 = X_3A_2 \cdot Y_3A_2$.]

Exercise 5.3 Let P be a point inside the triangle ABC. The bisector of $\angle BPC$, $\angle CPA$, and $\angle APB$ meet BC, CA and AB at X, Y and Z, respectively. Prove that AX, BY, CZ are concurrent.

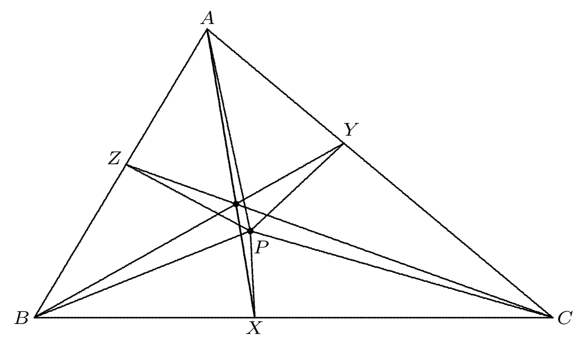


Figure 5.10: AX, BY, CZ are concurrent

Exercise 5.4 Let Γ be a circle with center I, the incentre of triangle ABC. Let D, E, F be points of intersection of Γ with the lines from I that are perpendicular to the sides BC, CA, AB respectively. Prove that AD, BE, CF are concurrent.

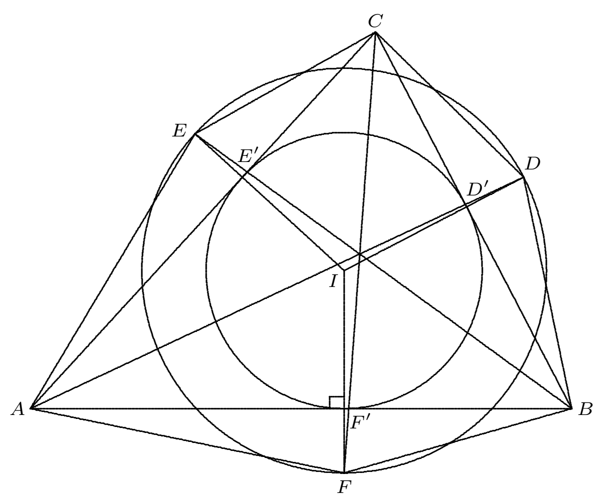


Figure 5.11: A generalization of the Gergonne point

[Hint: Let the intersection of AD, BE, CF with BC, CA, AB be D', E', F' respectively. It is easy to establish that $\angle FAF' = \angle EAE'$, $FBF' = \angle DBD'$, $\angle DCD' = \angle ECE'$. Also $AE = AF$, $BF = BD$, $CD = CE$. The ratio $AF'/F'B$ equals to the ratio of the altitudes from A and B on CF of the triangles AFC and BFC and hence equals to the ratio of their areas. Now apply Ceva's theorem.]

Exercise 5.5 Let A_1, B_1 and C_1 be points in the interiors of the sides BC, CA and AB of a triangle ABC respectively. Prove that the perpendiculars at the points A_1, B_1, C_1 are concurrent if and only if $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$. This is known as Carnot's lemma.

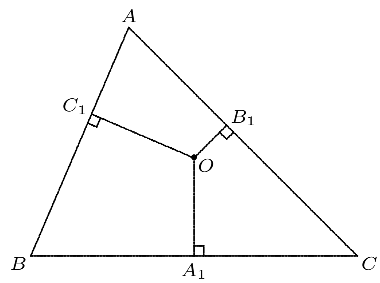


Figure 5.12: Carnot's lemma

Solution. Suppose the three perpendiculars concur at a point O. Note that O is inside the triangle ABC. As $BA_1^2 - A_1C^2 = (OB^2 - OA_1^2) - (OC^2 - OA_1^2) = OB^2 - OC^2$, $CB_1^2 - B_1A^2 = (OC^2 - OB_1^2) - (OA^2 - OB_1^2) = OC^2 - OA^2$, and $AC_1^2 - C_1B^2 = (OA^2 - OC_1^2) - (OB^2 - OC_1^2) = OA^2 - OB^2$, we thus have $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$.
Conversely, suppose $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$. Let the perpendiculars at B_1 and C_1 meet at a point O. Note that O is inside the triangle ABC. Drop the perpendicular OA' from O onto BC. We want to prove $A' = A_1$. By the proven forward implication, we know that $BA'^2 - A'C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$. Together with the given relation, we obtain $BA'^2 - A'C^2 = BA_1^2 - A_1C^2$. That is $(BA' + A'C)(BA' - A'C) = (BA_1 + A_1C)(BA_1 - A_1C)$. As $BA' + A'C = BC = BA_1 + A_1C$, we have $BA' - A'C = BA_1 - A_1C$. From these equations, we deduce that $BA' = BA_1$ and $A'C = A_1C$. Thus $A' = A_1$ and the three perpendiculars are concurrent.

Chapter 6

Collinearity

Problems on collinearity of points and concurrence of lines are very common in elementary plane geometry. To prove that 3 points A, B, C are collinear, the most straightforward technique is to verify that one of the angles $\angle ABC, \angle ACB$ or $\angle BAC$ is $180°$. We could also try to verify that the given points all lie on a specific line which is known to us. These methods have been applied in earlier chapters to prove that the Simson line and the Euler line are lines of collinearity of certain special points of a triangle. In this chapter, we shall explore more results such as Desargues' theorem, Menelaus' theorem and Pappus' theorem which give conditions on when three points are collinear.

The concept of collinearity and concurrence are dual to each other. For instance, suppose we wish to prove that 3 lines PQ, MN, XY are concurrent. Let PQ intersect MN at Z. Now it reduces to prove that X, Y, Z are collinear. Conversely, to prove that X, Y, Z are collinear, it suffices to show that the 3 lines PQ, MN, XY are concurrent.

6.1 Menelaus' theorem

Theorem 6.1 (Menelaus) *The three points P, Q, R on the sides AC, AB and BC respectively of a triangle ABC are collinear if and only if*

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1,$$

where directed segments are used. That is either 1 or 3 points among P, Q, R are outside the triangle.

Proof. Suppose that P, Q, R are collinear. Construct a line through C parallel to AB intersecting the line containing P, Q, R at a point D. See figure 6.1. Since $\triangle DCR \sim \triangle QBR$ and $\triangle PDC \sim \triangle PQA$, we have

$$\frac{QB \cdot RC}{BR} = DC = \frac{AQ \cdot CP}{PA}.$$

From this, the result follows.

Conversely, suppose

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1.$$

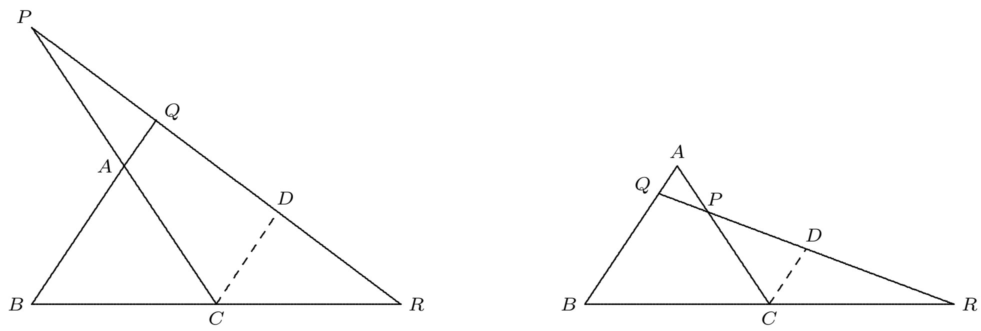


Figure 6.1: Menelaus' theorem

Let the line containing R and Q intersect AC at P'. Now P', Q, R are collinear. Hence,

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP'}{P'A} = -1.$$

Therefore, $CP'/P'A = CP/PA$. This implies that P and P' must coincide.

Definition 6.1 *The line PQR that cuts the sides of a triangle is called a transversal of the triangle.*

Example 6.1 The side AB of a square $ABCD$ is extended to P so that $BP = 2AB$. Let M be the midpoint of CD and Q the point of intersection between AC and BM. Find the position of the point R on BC such that P, R, Q are collinear.

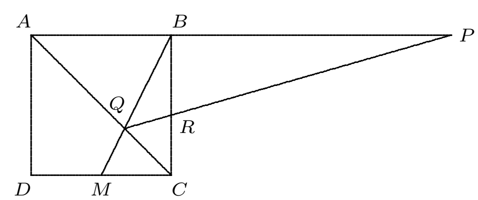


Figure 6.2

Solution. First we know that $AP : PB = 3 : -2$. Next we have $\triangle ABQ \sim \triangle CMQ$. Hence, $CQ : QA = CM : AB = \frac{1}{2}$. By Menelaus' theorem applied to triangle ABC, the points P, R, Q are collinear if and only if

$$\frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{CQ}{QA} = -1.$$

That is $BR : RC = 4 : 3$.

Example 6.2 In the figure, a line intersects each of the three sides of a triangle ABC at D, E, F. Let X, Y, Z be the midpoints of the segments AD, BE, CF respectively. Prove that X, Y, Z are collinear.

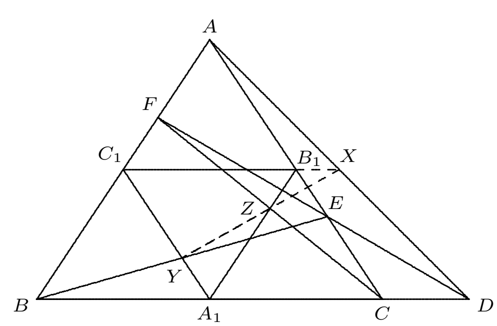


Figure 6.3

Solution. Let A_1, B_1 and C_1 be the midpoints of BC, AC and AB respectively. Then B_1C_1 is parallel to BC and B_1, C_1 and X are collinear. Hence, $BD/DC = C_1X/XB_1$. Similarly, $CE/EA = A_1Y/YC_1$ and $AF/FB = B_1Z/ZA_1$. Now apply Menelaus' theorem to $\triangle ABC$ and the straight line DEF. We have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1,$$

That is

$$\frac{C_1X}{XB_1} \cdot \frac{B_1Z}{ZA_1} \cdot \frac{A_1Y}{YC_1} = -1.$$

Then, by Menelaus' theorem applied to $\triangle A_1B_1C_1$ and the points X, Y, Z, the points X, Y, Z are collinear.
(*The line XYZ is called the Gauss line.*)

Example 6.3 A line through the centroid G of $\triangle ABC$ cuts the sides AB at M and AC at N. Prove that

$$AM \cdot NC + AN \cdot MB = AM \cdot AN.$$

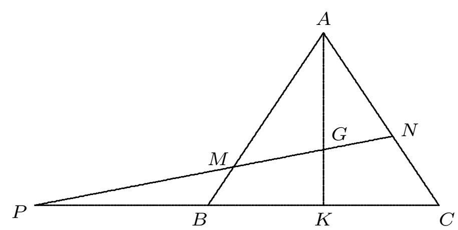


Figure 6.4

Solution. The above relation is equivalent to $NC/AN + MB/AM = 1$. If MN is parallel to BC, then $NC/AN = MB/AM = GK/AK = \frac{1}{2}$. Therefore the result is true.
Next consider the case where MN meets BC at a point P. Apply Menelaus' theorem to $\triangle AKB$ and the line PMG. We have $(BP/PK) \cdot (KG/GA) \cdot (AM/MB) = 1$ in absolute value. As $KG/GA = \frac{1}{2}$, we have $BP = (2MB \cdot PK)/AM$. Similarly, by applying Menelaus' theorem to $\triangle ACK$ and the line PGN, we have $PC = (2CN \cdot KP)/NA$.
Note that $PC - PK = KC = BK = PK - PB$. Substituting the above relations into this equation, we obtain the desired expression.

Theorem 6.2 *In the convex quadrilateral $ACGE$, AG intersects CE at H, the extension of AE intersects the extension of CG at I, the extension of EG intersects the extension of AC at D, and the line IH meets EG at F and AD at B. Then*
(i) $AB/BC = -AD/DC$,
(ii) $EF/FG = -ED/DG$.
Here directed line segments are used.

Proof. (i) Refer to Figure 6.5. Applying Ceva's Theorem to $\triangle ACI$, we have

$$\frac{IE}{EA}\frac{AB}{BC}\frac{CG}{GI} = 1.$$

Next by Menelaus' Theorem applied to $\triangle ACI$ with transversal EGD, we have

$$\frac{AD}{DC}\frac{CG}{GI}\frac{IE}{EA} = -1.$$

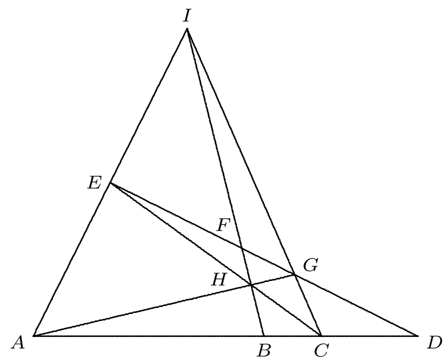


Figure 6.5: A complete quadrilateral

Thus, $AB/BC = -AD/DC$.

(ii) To prove the second assertion, apply Ceva's Theorem to $\triangle IEG$ with cevians IF, EC and GA. They concur at H. Thus, we have

$$\frac{IA}{AE}\frac{EF}{FG}\frac{GC}{CI} = 1.$$

By Menelaus' Theorem applied to $\triangle IEG$ with transversal ACD,

$$\frac{ED}{DG}\frac{GC}{CI}\frac{IA}{AE} = -1.$$

Thus $EF/FG = -ED/DG$.

6.2 Desargues' theorem

Theorem 6.3 (Desargues) *Let ABC and $A_1B_1C_1$ be two triangles such that AA_1, BB_1, CC_1 meet at a point O. (The two triangles are said to be perspective from the point O.) Let L be the intersection of BC and B_1C_1, M the intersection of CA and C_1A_1 and N the intersection of AB and A_1B_1. Then L, M and N are collinear.*

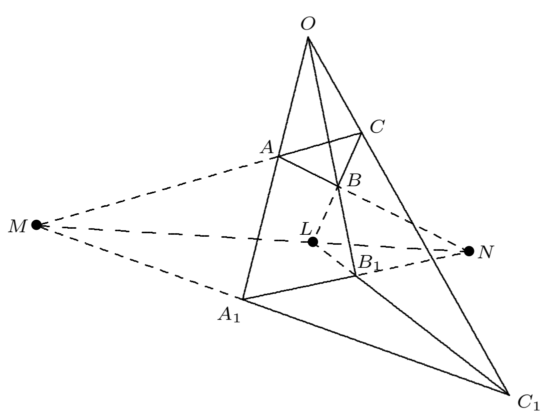


Figure 6.6: Two triangles in perspective from a point

Proof. The line LB_1C_1 cuts $\triangle OBC$ at L, B_1 and C_1. By Menelaus' theorem,

$$\frac{BL}{LC}\cdot\frac{CC_1}{C_1O}\cdot\frac{OB_1}{B_1B}=-1.$$

Similarly, the lines MA_1C_1 and NB_1A_1 cut $\triangle OCA$ and $\triangle OAB$ respectively. By Menelaus' theorem, we have

$$\frac{CM}{MA}\cdot\frac{AA_1}{A_1O}\cdot\frac{OC_1}{C_1C}=-1 \text{ and } \frac{AN}{NB}\cdot\frac{BB_1}{B_1O}\cdot\frac{OA_1}{A_1A}=-1.$$

Multiplying these together, we obtain

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=-1.$$

By Menelaus' theorem applied to $\triangle ABC$, the points L, M and N are collinear.

Exercise 6.1 Prove the converse of Desargues' theorem: Let ABC and $A_1B_1C_1$ be two triangles such that BC intersects B_1C_1 at L, CA intersects C_1A_1 at M and AB intersects A_1B_1 at N. Suppose L, M, N are collinear. Then AA_1, BB_1 and CC_1 are concurrent.

[Hint: Refer to figure 6.6. Let AA_1 intersect BB_1 at O. It suffices to prove O, C, C_1 are collinear. To do so, apply Desargues' theorem to the triangles MAA_1 and LBB_1 which are perspective from the point N.]

6.3 Pappus' theorem

Theorem 6.4 (Pappus) *If A, C, E are three points on one line, B, D, F on another, and if the three lines AB, CD, EF meet DE, FA, BC respectively at points L, M, N, then L, M, N are collinear.*

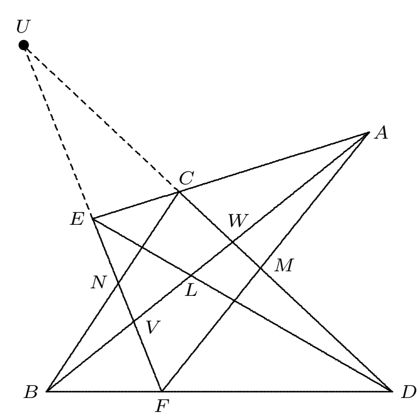


Figure 6.7

Proof. Extend FE and DC meeting at a point U as in the figure. If FE and DC are parallel, then the point U is at infinity. The proof is still valid if the problem is suitably translated in terms of projective geometry. Let's not worry about this situation as this would take us too far in the direction of projective geometry. We may as well consider the intersection point between BC and FA if they are not parallel. The case where $FE \parallel DC$ and $BC \parallel FA$ can be proved directly. The reader is invited to try by himself or herself.

Apply Menelaus' theorem to the five triads of points L, D, E; A, M, F; B, C, N; A, C, E; B, D, F on the sides of the triangle UVW. We obtain

$$\frac{VL}{LW} \cdot \frac{WD}{DU} \cdot \frac{UE}{EV} = -1, \frac{VA}{AW} \cdot \frac{WM}{MU} \cdot \frac{UF}{FV} = -1, \frac{VB}{BW} \cdot \frac{WC}{CU} \cdot \frac{UN}{NV} = -1,$$

$$\frac{VA}{AW} \cdot \frac{WC}{CU} \cdot \frac{UE}{EV} = -1, \frac{VB}{BW} \cdot \frac{WD}{DU} \cdot \frac{UF}{FV} = -1.$$

Dividing the product of the first three expressions by the product of the last two, we have

$$\frac{VL}{LW} \cdot \frac{WM}{MU} \cdot \frac{UN}{NV} = -1.$$

By Menelaus' theorem, N, L, M are collinear.

Exercise 6.2 Prove that the interior angle bisectors of two angles of a non-isosceles triangle and the exterior angle bisector of the third angle meet the opposite sides in three collinear points.

Exercise 6.3 (Monge's Theorem) Prove that the three pairs of common external tangents to three circles, taken two at a time, meet in three collinear points.

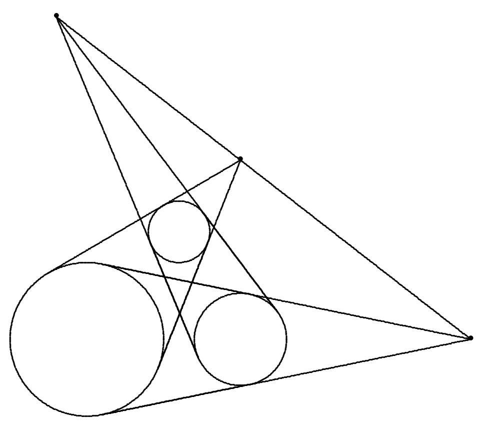


Figure 6.8: Monge's theorem

Exercise 6.4 Let I be the centre of the inscribed circle of the non-isosceles triangle ABC, and let the circle touch the sides BC, CA, AB at the points A_1, B_1, C_1 respectively. Prove that the centres of the circumcircles of $\triangle AIA_1, \triangle BIB_1$ and $\triangle CIC_1$ are collinear.

[Hint: Let the line perpendicular to CI and passing through C meet AB at C_2. By analogy, we have the points A_2 and B_2. It is obvious that the centres of the circumcircles of $\triangle AIA_1$, $\triangle BIB_1$ and $\triangle CIC_1$ are the midpoints of A_2I, B_2I and C_2I, respectively. So it is sufficient to prove that A_2, B_2 and C_2 are collinear.]

Chapter 7

Circles

A circle consists of points on the plane which are of fixed distance r from a given point O. Here O is the centre and r is the radius of the circle. It has long been known to the Pythagoreans such as Antiphon and Eudoxus that the area of the circle is proportional to the square of its radius. Inevitably the value of the proportionality π is of great importance to science and mathematics. Many ancient mathematicians spent tremendous effort in computing its value. Archimedes was the first to calculate the value of π to 4 decimal places by estimating the perimeter of a 96-gon inscribed in the circle. He obtained $223/71 < \pi < 22/7$. Around 265AD, Liu Hui in China came up with a simple and rigorous iterative algorithm to calculate π to any degree of accuracy. He himself carried out the calculation to 3072-gon and obtained $\pi = 3.1416$. The Chinese mathematician Zu Chongzhi (429-500) gave the incredible close rational approximation $\frac{355}{113}$ to π, which is often referred to as "Milu".

7.1 Basic properties

Circles are the most symmetric plane figures and they possess remarkable geometric properties. In this chapter, we shall explore some of these results as well as coaxal families of circles. In addition, figures inscribed in a circle or circumscribing a circle also enjoy interesting properties. We begin with some basic results about circles which we will leave them for the readers to supply the proofs.

1. Let AB and CD be two chords in a circle. The followings are equivalent.

(i) $\widehat{AB}$=$\widehat{CD}$, where $\widehat{AB}$ is the length arc of AB.
(ii) $AB = CD$.
(iii) $\angle AOB = \angle COD$.
(iv) $OE = OF$.

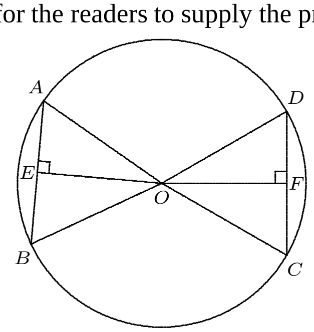


Figure 7.1

2. Let AB and CD be two chords in a circle. The followings are equivalent.

(i) $\overset{\frown}{AB} > \overset{\frown}{CD}$

(ii) $AB > CD$.

(iii) $\angle AOB > \angle COD$.

(iv) $OE < OF$.

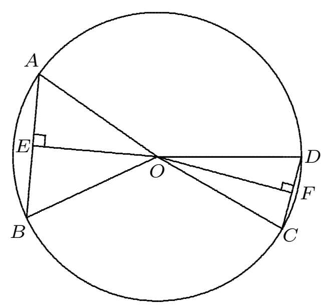


Figure 7.2

3. Let D be a point on the arc AB. The followings are equivalent.

(i) $\overset{\frown}{AD} = \overset{\frown}{DB}$.

(ii) $AC = CB$.

(iii) $\angle AOD = \angle BOD$.

(iv) $OD \perp AB$.

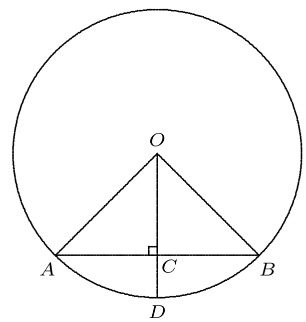


Figure 7.3

4. The angle subtended by an arc BC at a point A on a circle is half the angle subtended by the arc BC at the centre of the circle.

That is $\angle BOC = 2\angle BAC$.

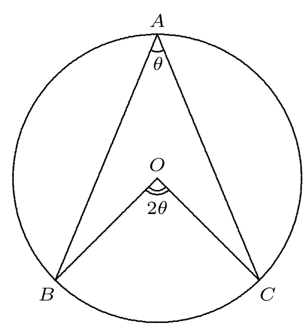


Figure 7.4

5. The angle subtended by the same segment at any point on the circle is constant.

That is $\angle BAC = \angle BDC$.

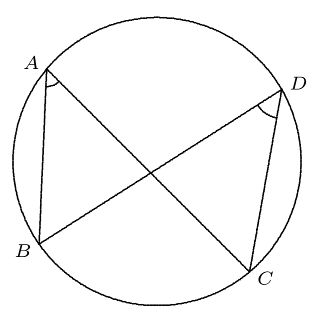


Figure 7.5

6. A chord BC is a diameter if and only if the angle subtended by it at point on the circle is a right angle.

That is $\angle BAC = 90°$ for any point $A \neq B$ or C on the circle.

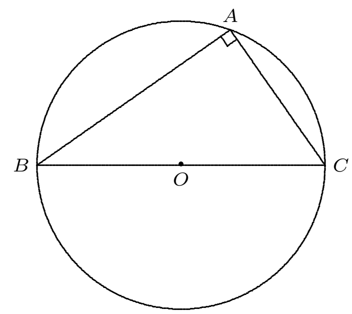


Figure 7.6

7. Let $ABCD$ be a convex quadrilateral. The followings are equivalent.

(i) $ABCD$ is a cyclic quadrilateral
(ii) $\angle BAC = \angle BDC$.
(iii) $\angle A + \angle C = 180°$.
(iv) $\angle ABE = \angle D$.

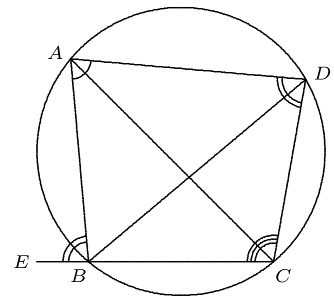


Figure 7.7

8. Alternate Segment Theorem. Let A, B, C be three points on a circle. Let TA be a line through A with T and B lying on the same side of the line AC. Then the followings the equivalent.

(i) AT is tangent to the circle at A.
(ii) $OA \perp AT$.
(iii) $\angle BAT = \angle BCA$.

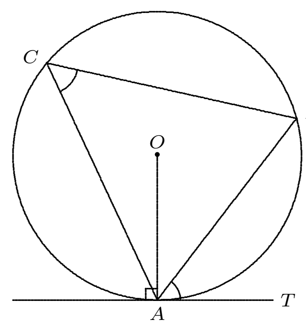


Figure 7.8

9. Let PS and PT be tangents to the circle. Then

(i) $PS = PT$,
(ii) OP bisects $\angle SPT$
(iii) OP bisects $\angle SOP$
(iv) OP is the perpendicular bisector of the segment ST.

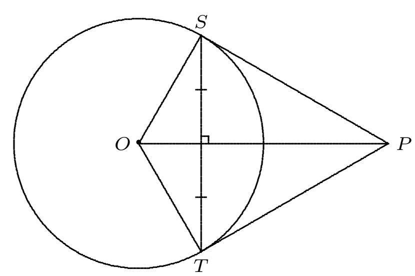


Figure 7.9

Definition 7.1 *Four points are concyclic if they lie on a circle.*

Theorem 7.1 (Euclid's theorem) *Let A, B, C, D be 4 points on the plane such AB and CD or their extensions intersect at the point P. Then A, B, C, D are concyclic if and only if*

$$PA \cdot PB = PC \cdot PD.$$

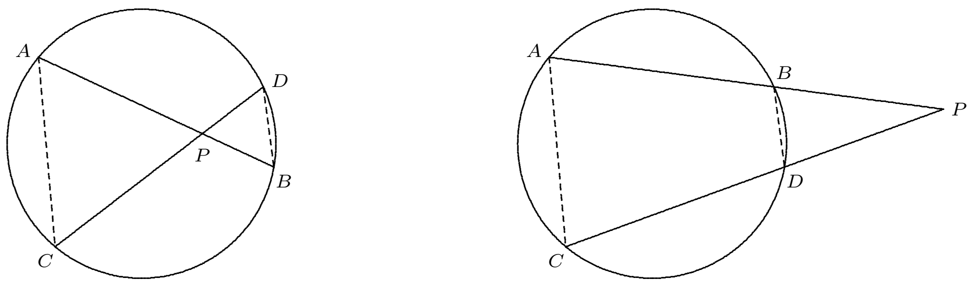


Figure 7.10: Euclid's theorem

Proof. The result follows from the fact the triangles APC and DBC are similar.

Definition 7.2 *The power of a point P with respect to the circle centred at O with radius R is defined as $OP^2 - R^2$.*

(i) If P is outside the circle, then
the Power of P
= $OP^2 - R^2$
= $PT^2 = PA \cdot PB$,
which is positive.

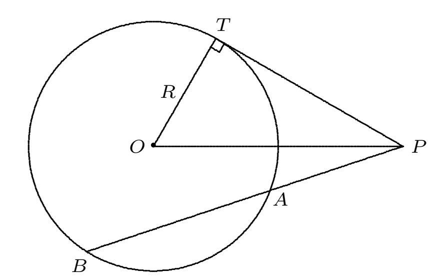


Figure 7.11

(ii) If P lies on the circumference, then
the power of $P = OP^2 - R^2 = 0$.

(iii) If P is inside the circle, then
the power of P
= $OP^2 - R^2 = -PZ^2$
= $-PX \cdot PY$
= $-PA \cdot PB$,
which is negative.

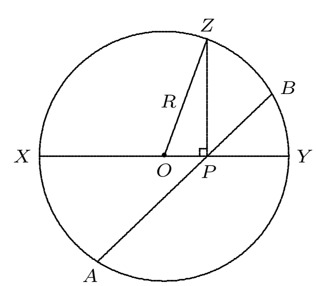


Figure 7.12

Exercise 7.1 Let D, E and F be three points on the sides BC, CA and AB of a triangle ABC respectively. Show that the circumcircles of the triangles AEF, BDF and CDE meet a common point. This point is called the Miquel point.

Theorem 7.2 (Euler's formula for OI)
Let O and I be the circumcentre and the incentre, respectively, of $\triangle ABC$ with circumradius R and inradius r. Then

$$OI^2 = R^2 - 2rR.$$

Proof. As $\angle CBQ = \frac{1}{2}\angle A$, it follows that $\angle QBI = \angle QIB$ and $QB = QI$. The absolute value of the power of I with respect to the circumcircle of ABC is $R^2 - OI^2$, which is also equal to $IA \cdot QI = IA \cdot QB = \frac{r}{\sin\frac{A}{2}} \cdot 2R\sin\frac{A}{2} = 2Rr$.

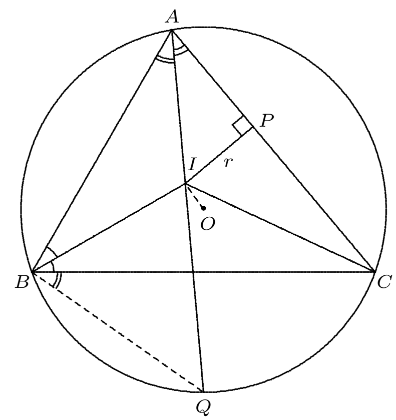


Figure 7.13

Corollary 7.3 $R \geq 2r$. *Equality holds if and only if ABC is equilateral.*

Exercise 7.2 Prove the *isoperimetric inequality* $s^2 \geq 3\sqrt{3}A$, where A is the area and s is the semi-perimeter of the triangle. Show that equality holds if and only if the triangle is equilateral.

7.2 Coaxal circles

Let C be a circle and P a point. Suppose AA' and BB' are two chords of C intersecting at P. Then $PA \cdot PA' = PB \cdot PB'$. Let R be the radius of C and d the distance from P to the centre of C. We have $PA \cdot PA' = d^2 - R^2$ or $R^2 - d^2$, depending on whether P is outside or inside C. Recall that the quantity $d^2 - R^2$ is called the *power* of P with respect to the circle C. Note that the power of P with respect to C is positive if and only if P is outside C.

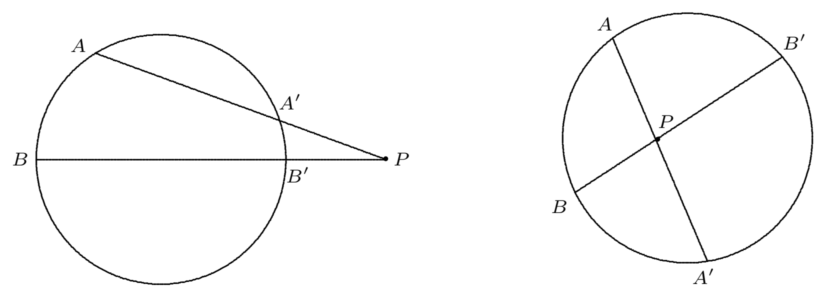


Figure 7.14: The power of a point with respect to a circle

If P is outside C and PT is a tangent to C at T, then the power of P with respect to C is PT^2. The power of P with respect to C can also be expressed in terms of the equation of C. (The coefficients of x^2 and y^2 are both 1.)
The standard equation of a circle centred at $(-f, -g)$ is of the form

$$C(x, y) = x^2 + y^2 + 2fx + 2gy + h = 0.$$

Theorem 7.4 *The power of a point $P(a, b)$ with respect to a circle $C = 0$ is also given by $C(a, b)$.*

Definition 7.3 *The locus of the points having equal power with respect to two non-concentric circles C_1 and C_2 is called the radical axis of C_1 and C_2.*

Theorem 7.5 *For any two non-concentric circles $C_1 = 0$ and $C_2 = 0$, the radical axis is given by*

$$C_1 - C_2 = 0.$$

Proof. If $P(a, b)$ is on the radical axis, then $C_1(a, b) = C_2(a, b)$, i.e, P is on the line $C_1 - C_2 = 0$. Conversely, any point $P(a, b)$ on the line has equal power with respect to the two circles.

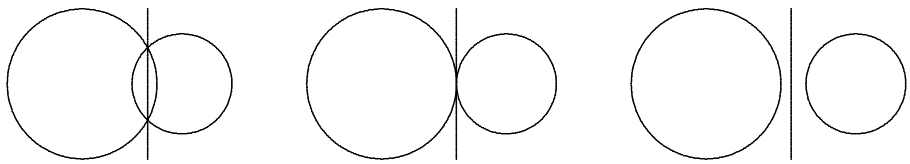


Figure 7.15: Radical axis

Exercise 7.3 Show that the radical axis of 2 circles is perpendicular to the line joining the centres of the 2 circles.

Theorem 7.6 *Let $C_3 = \lambda C_1 + \mu C_2 = 0$, where $\lambda + \mu = 1$.*
(i) Any point $P(a, b)$ on the line $C_1(x, y) - C_2(x, y) = 0$ has equal power with respect to the three circles C_1, C_2, C_3.
(ii) For any point $Q(c, d)$ on C_3, the ratio of the powers of Q w.r.t C_1 and C_2 is $-\mu/\lambda$, which is a constant.

Proof. (i) The power of P with respect to C_1 and C_2 are equal to $k = C_1(a, b) = C_2(a, b)$. Its power with respect to C_3 is

$$\lambda C_1(a, b) + \mu C_2(a, b) = (\lambda + \mu)k = k.$$

(ii) Since Q is on C_3, we have $\lambda C_1(c, d) + \mu C_2(c, d) = 0$ or $C_1(c, d)/C_2(c, d) = -\mu/\lambda$.

Definition 7.4 *The collection of all circles of the form $C_3 = \lambda C_1 + \mu C_2$, where $\lambda + \mu = 1$, forms a so-called pencil of circles. Any two such circles have the same radical axis, and they are called coaxal circles.*

Theorem 7.7 *Suppose C_1, C_2, C_3 are three circles such that for any point $P(a, b)$ on C_3, the ratio of the powers of P w.r.t to C_1, C_2 is a constant $k(\neq 1)$, then $C_3 = \lambda C_1 + \mu C_2$, where $\mu = k/(k-1)$ and $\lambda = -1/(k - 1)$.*

Proof. We have $C_1(a, b)/C_2(a, b) = k$. So $C_1(a, b) - kC_2(a, b) = 0$. Thus $C_3 = \lambda C_1 + \mu C_2$.
Note that for the above statement to be true we need the condition to hold for 3 points on C_3 because 3 points determine a unique circle, i.e. if $C_1(a_i, b_i)/C_2(a_i, b_i) = k$ for 3 distinct points (a_i, b_i), $i = 1, 2, 3$, then C_3 above is the circumcircle of the triangle whose vertices are (a_i, b_i).

Example 7.1 Let C_1 and C_2 be two circles tangent at a point M. If A is any point on C_1, with AP as the tangent to C_2, then AP/AM is a constant as A varies on C_1.

Solution. Regard M as a circle of 0 radius. Then the 3 circles C_1, M, C_2 are coaxal with the tangent at M the radical axis. Thus, AP/AM is the ratio of the powers of A with respect to C_2 and M which is constant.

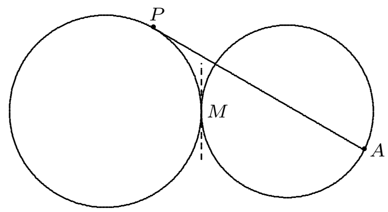


Figure 7.16: AP/AM is a constant as A varies on C_1

Theorem 7.8 *The three radical axes of three non-concentric circles C_1, C_2, C_3, taken in pairs, are either parallel or concurrent.*

Proof. The three radical axes are $C_1 - C_2 = 0$, $C_2 - C_3 = 0$, $C_3 - C_1 = 0$. Any point that satisfies two of the equations must satisfy the third. Thus if two of the lines intersect, then the third must also pass through the point of intersection, i.e., they are concurrent. Otherwise, they are pairwise parallel.

Definition 7.5 *The point of concurrence of the 3 radical axes of 3 circles is called the radical centre of the 3 circles.*

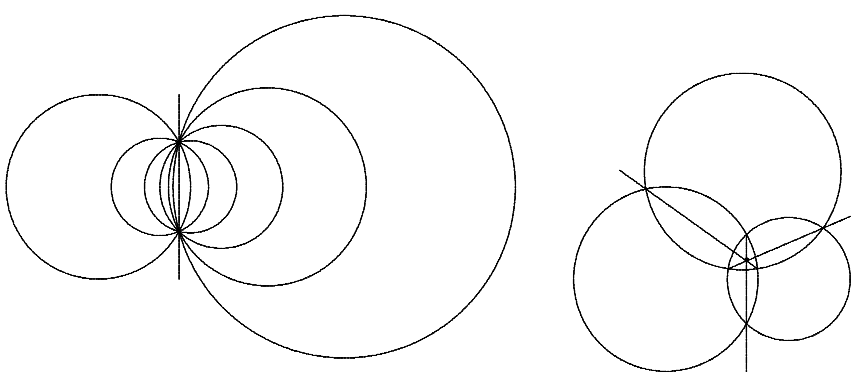


Figure 7.17: Coaxal circles and the radical centre of three non-coaxal circles

Exercise 7.4 Consider the pencil of circles $x^2 + y^2 - 2ax + c = 0$, where c is fixed and a is the parameter. (If $c > 0$, a varies in the range $\mathbb{R} \setminus (-\sqrt{c}, \sqrt{c})$.) Any two of its members have the same line of centres and the same radical axis. Hence it is a pencil of coaxal circles. Prove the following.

(a) If $c < 0$, each circle in the pencil meets the y-axis at the same two points $(0, \pm\sqrt{-c})$, and the pencil consists of circles through these two points.

(b) If $c = 0$, the pencil consists of circles touching the y-axis at the origin.

(c) If $c > 0$, the pencil consists of non-intersecting circles. Also when $a = \pm\sqrt{c}$ ($c > 0$), the circle degenerates into a point at $(\pm\sqrt{c}, 0)$.

7.3 Orthogonal pair of pencils of circles

Two non-intersecting circles give rise to a pencil of non-intersecting coaxal circles together with two degenerate circles, called the *limit points* of the pencil. For any point on the radical axis of this pencil of circles, the tangents to these circles are all of the same length. Therefore, the circle centred at that point with radius equal to the length of the tangent is orthogonal to all the circles in this pencil. All such circles form another pencil and any two of them uniquely determine the original pencil. Moreover, each circle in one pencil is orthogonal to each circle of the other pencil.

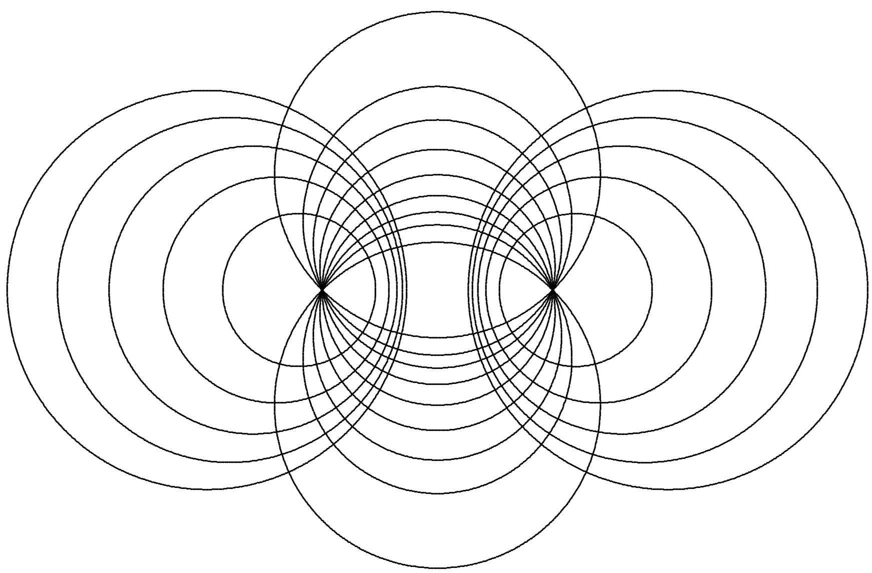


Figure 7.18: Two orthogonal pencils of coaxal circles

Exercise 7.5 Consider the two pencils of circles $\mathcal{P}_1 : x^2 + y^2 - 2ax + c = 0$ and $\mathcal{P}_2 : x^2 + y^2 - 2by - c = 0$ where $c > 0$ is fixed, a and b are the parameters.

(a) Show that $\mathcal{P}_1$ consists of non-intersecting circles, and $\mathcal{P}_2$ consists of intersecting circles all passing through the points $(\pm\sqrt{c}, 0)$.

(b) Show that each circle in $\mathcal{P}_1$ is orthogonal to each circle in $\mathcal{P}_2$.

7.4 The orthocentre

Theorem 7.9 *Let AD, BE and CF be the altitudes of the triangle ABC. The circle with diameter AB passes through D and E. Hence $HA \cdot HD = HB \cdot HE$. Similarly, $HB \cdot HE = HC \cdot HF$.*

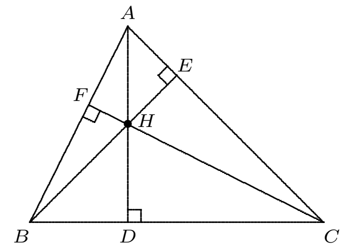


Figure 7.19

Theorem 7.10 *If X, Y, Z are any points on the respective sides BC, CA, AB of a triangle ABC, then the circles constructed on the cevians AX, BY, CZ as diameters will pass through the feet of the altitudes D, E, F respectively.*

Theorem 7.11 *If circles are constructed on 2 cevians of a triangle as diameters, then their radical axis passes through the orthocentre of the triangle.*

Theorem 7.12 *For any 3 non-coaxal circles having cevians of a triangle ABC as diameters, their radical centre is the orthocentre of $\triangle ABC$.*

Theorem 7.13 *If circles are constructed having the medians, (or altitudes or angle bisectors) of $\triangle ABC$ as diameters, then their radical centre is the orthocentre of $\triangle ABC$.*

7.5 Pascal's theorem and Brianchon's theorem

Theorem 7.14 (Pascal) *If all 6 vertices of a hexagon lie on a circle and the 3 pairs of opposite sides intersect, then the three points of intersection are collinear.*

Theorem 7.15 (Brianchon) *If all 6 sides of a hexagon touch a circle, then the three diagonals are concurrent (or possibly parallel).*

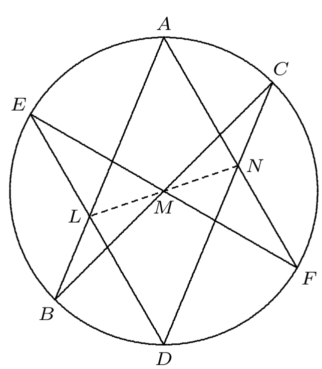


Figure 7.20: Pascal's Theorem

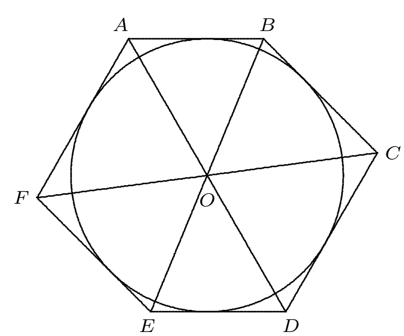


Figure 7.21: Brianchon's theorem

Proof of Pascal's theorem. We assume the lines AB, CD, EF form a triangle. Let AB intersect CD at W. The intersection points between various lines are shown in the figure.

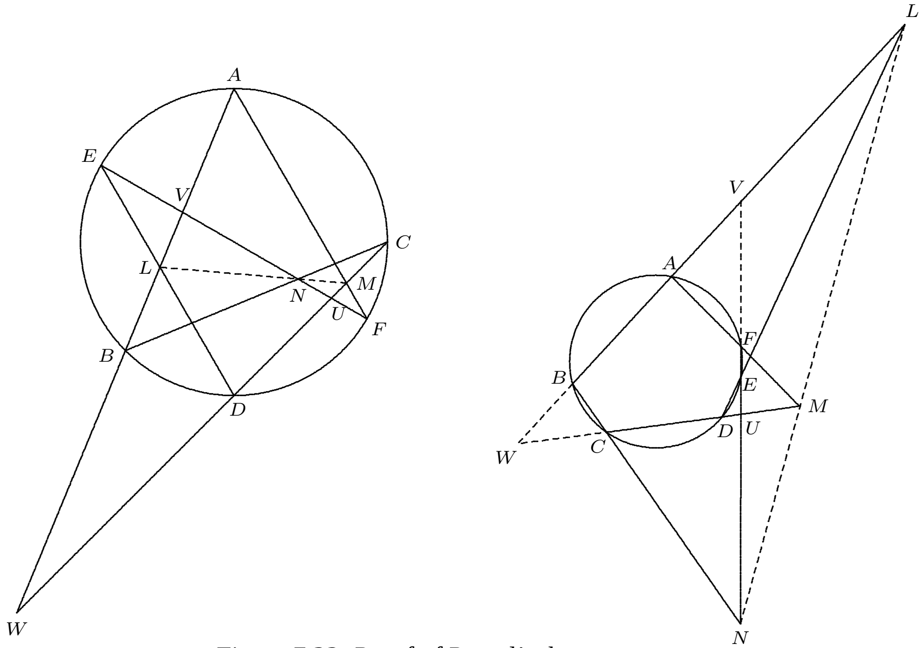


Figure 7.22: Proof of Pascal's theorem

Apply Menelaus' theorem to the transversals ELD, AMF, BNC with respect to $\triangle UVW$. We have

$$\frac{VL}{LW}\frac{WD}{DU}\frac{UE}{EV} = -1, \frac{VA}{AW}\frac{WM}{MU}\frac{UF}{FV} = -1, \frac{VB}{BW}\frac{WC}{CU}\frac{UN}{NV} = -1.$$

Therefore

$$\frac{VL}{LW}\frac{WM}{MU}\frac{UN}{NV} = \frac{DU}{WD}\frac{EV}{UE}\cdot\frac{AW}{VA}\frac{FV}{UF}\cdot\frac{BW}{VB}\frac{CU}{WC} = -1,$$

since $DU \cdot CU = UE \cdot UF$, $EV \cdot FV = VA \cdot VB$ and $AW \cdot BW = WC \cdot WD$. By Menelaus' theorem, L, N, M are collinear.

Note that the 3 equations obtained by applying Menelaus' theorem to the transversals ELD, AMF, BNC with respect to $\triangle UVW$ are the same as those in the proof of Pappus' theorem. In Pappus' theorem, there are two more such equations arising from the 2 original lines which are also transversals to $\triangle UVW$. In Pascal's theorem, these are replaced by the 3 equations arising from the condition on equality of powers of the three vertices of $\triangle UVW$ with respect to the circle.

Proof of Brianchon's theorem. Let R, Q, T, S, P, U be the points of contact of the six tangents AB, BC, CD, DE, EF, FA, as in the figure. For simplicity, we assume the hexagon is convex so that all three diagonals AD, BE, CF are chords of the inscribed circles and they are not parallel. On the lines, FE, BC, BA, DE, DC, FA extended, take points P', Q', R', S', T', U' so that

$$PP' = QQ' = RR' = SS' = TT' = UU',$$

with any convenient length, and construct circles I touching PP' and QQ' at P' and Q', II touching RR' and SS' at R' and S', and III touching TT' and UU' at T' and U'. This is possible because $ABCDEF$ has an incircle.

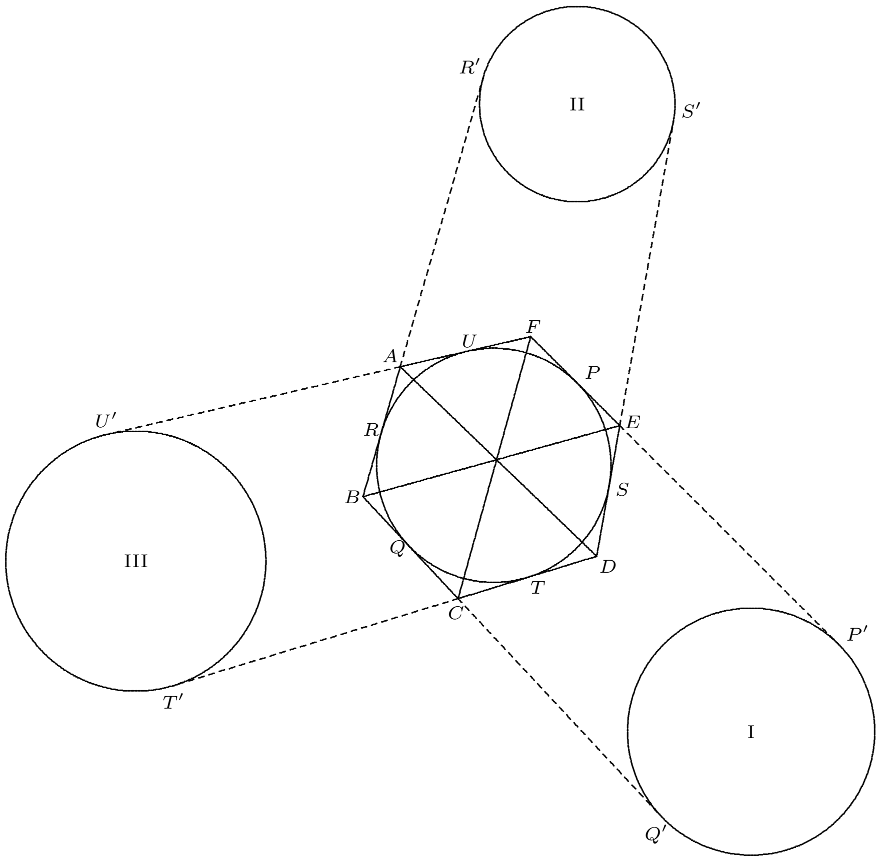


Figure 2.23: Proof of Brianchon's theorem

Now $AU' = UU' - AU = RR' - AR = AR'$ and $DT' = DT + TT' = DS + SS' = DS$ so that A and D are of equal power with respect to the circles II and III. Thus AD is the radical axes of II and III. Similarly, BE is the radical axis of I and II, and CF is the radical axis of I and III. Consequently, AD, BE and CF are concurrent.

Example 7.2 Tangents to the circumcircle of $\triangle ABC$ at points A, B, C meet sides $BC, AC,$ and AB at points P, Q and R respectively. Prove that points P, Q and R are collinear.

Solution. As $\triangle RCA$ is similar to $\triangle RBC$, we have $RB/RC = RC/RA = BC/AC$. Hence, $RB/RA = (RC/RA)^2 = (BC/AC)^2$. Similarly, we have $QA/QC = (BA/BC)^2$ and PC/PB=$(AC/BA)^2$. Consequently, $(BR/RA) \cdot (AQ/QC) \cdot (CP/PB) = 1$. Therefore, by Menelaus' theorem, P, Q, R are collinear.

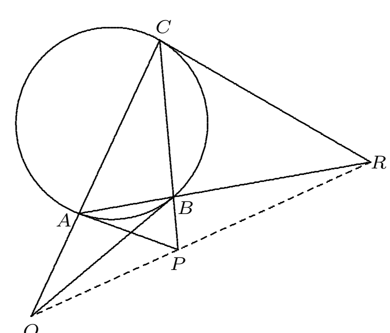


Figure 2.24

2nd solution. Alternatively, the result can be proved by applying Pascal's theorem to the degenerate 'hexagon' $AABBCC$.

Example 7.3 Let ABC be any triangle and P any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from P to the sides AC and BC. Draw AP and BP and from C drop perpendiculars to AP and BP. Let Q_1 and Q_2 be the feet of these perpendiculars. Prove that the lines Q_1P_2, Q_2P_1 and AB are concurrent.

Solution. Since $\angle CP_1P$, $\angle CP_2P$, $\angle CQ_2P$, $\angle CQ_1P$ are all right angles, one sees that the points C, Q_1, P_1, P, P_2, Q_2 lie on a circle with CP as diameter. CP_1 and Q_1P intersect at A and Q_2P and CP_2 intersect at B. If we apply Pascal's Theorem to the crossed hexagon $CP_1Q_2PQ_1P_2$, we see that P_2Q_1 and P_1Q_2 intersect at a point X on the line AB.

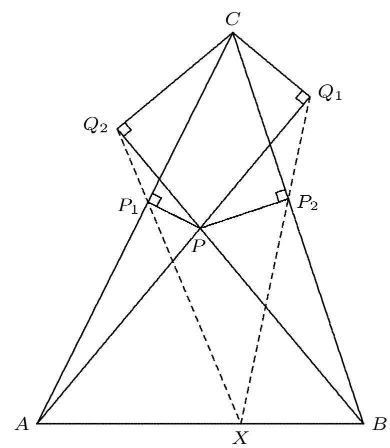


Figure 2.25

Example 7.4 A, E, B, D are points on a circle in a clockwise sense. The tangents at E and B meet at a point N, lines AE and DB meet at M and the diagonals AB and DE meet at L. Prove that L, N, M are collinear.

Solution. Apply Pascal's theorem to the degenerate hexagon $ABCDEF$ with $B = C$ and $E = F$. The sides BC and EF degenerate into the tangents at B and E respectively.

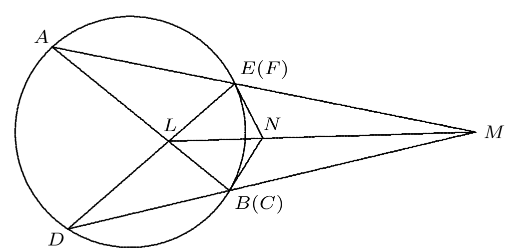


Figure 2.26

Example 7.5 Prove that the lines joining the tangency point of the incircle of a triangle to its opposite vertices concur at a common point.

Solution. The result is obvious by Ceva's theorem. Alternatively, the result follows by applying Brianchon's theorem to the hexagon $AC'BA'CB'$, where A', B', C' are the tangency points of the incircle of $\triangle ABC$ to its sides. This point is called the Gergonne point of $\triangle ABC$. See also example 5.1 in chapter 5.

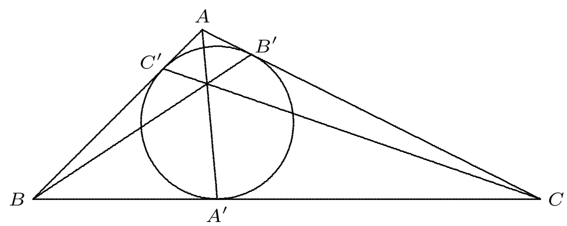


Figure 2.27

Example 7.6 Suppose $ABCD$ has an inscribed circle. Show that the lines joining the points of tangency of the inscribed circle on opposite sides are concurrent with the two diagonals.

Solution. The proof is by a degenerate case of Brianchon's theorem. For example, by taking the hexagon $ABYCDW$, we see that AC, BD, YW are concurrent; and by taking the hexagon $AXBCZD$, AC, BD, XZ are concurrent. Consequently, AC, BD, YW and XZ are concurrent. Moreover, WZ, AC, XY (same for WX, ZY, DB) are concurrent by suitably applying Brianchon's theorem.

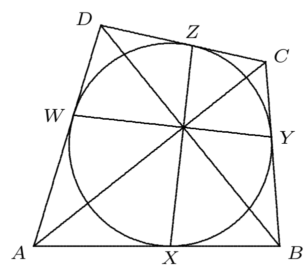


Figure 2.28

Exercise 7.6 Let ABC be a triangle, and draw isosceles triangles BCD,CAE,ABF externally to ABC, with BC,CA,AB as their respective bases. Prove that the lines through A,B,C perpendicular to lines EF, FD, DE, respectively, are concurrent.

[Hint: Draw three circles with centres D, E, F and radii DB, EC and AF respectively.]

Exercise 7.7 A convex quadrilateral $ABCD$ is inscribed in a circle centred at O. The diagonals AC and BD meet at P. Points E and F, distinct from A, B, C, D, are chosen on this circle. The circle determined by A, P, F and the circle determined by B, P, E meet at a point Q distinct from P. Prove that the lines PQ, CE and DF are either all parallel or concurrent.

[Hint: Let R be the intersection of AF and BE. Apply Pascal's theorem to the crossed hexagon $AFDBEC$.]

Chapter 8

Using Coordinates

Coordinate geometry is invented and developed by Ren Descartes (1596-1650). First a coordinate system in which two mutually perpendicular axes intersecting at the origin is set up. In such a system, points are denoted by ordered pairs of real numbers while lines are represented by linear equations. Other objects such as circles can be represented by algebraic equations. Finding intersections between lines and curves reduces to solving equations. It has the advantage of translating geometry into purely algebra. For instance, concurrence of lines and collinearity of points can also be expressed in terms of algebraic conditions.

8.1 Basic coordinate geometry

In this section, we shall review some basic formulas in coordinate geometry.

1. **Ratio formula**. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. If P is the point that divides the line segment AB in the ratio $r : s$, (i.e. $AP : PB = r : s$), then the coordinates of P is given by

$$\left(\frac{sa_1 + rb_1}{r+s}, \frac{sa_2 + rb_2}{r+s}\right).$$

2. **Incentre**. Let the coordinates of the vertices of a triangle ABC be $(x_A, y_A), (x_B, y_B), (x_C, y_C)$ respectively. The coordinates of the incentre I of $\triangle ABC$ are

$$x_I = \frac{ax_A + bx_B + cx_C}{a+b+c} \quad \text{and} \quad y_I = \frac{ay_A + by_B + cy_C}{a+b+c}.$$

Proof. Let the sides BC, AC, AB of $\triangle ABC$ be a, b, c respectively. Let BI meet AC at B'. Then using the Angle Bisector Theorem, $AB' : B'C = c : a$, and $BI : IB' = (a + c) : b$. (For the second ratio, extend AB to AB_1 so that $BB_1 = a$ and extend AI to meet B_1C at I'. Then B_1C is parallel to BB'. Hence $BI : IB' = B_1I' : I'C = (a + c) : b$. From this, we obtain the coordinates of I.

3. **Family of lines**. If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two lines intersecting at a point P (i.e. $a_1b_2 \neq a_2b_1$), then the family of lines passing though P can be expressed as

$$\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) = 0.$$

4. **Area**. The algebraic area of a triangle with vertices $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ is given by $\frac{1}{2}(y_A + y_B)(x_A - x_B) + \frac{1}{2}(y_B + y_C)(x_B - x_C) + \frac{1}{2}(y_C + y_A)(x_C - x_A) = \frac{1}{2}(x_By_C - x_Cy_B + x_Cy_A - x_Ay_C + x_Ay_B - x_By_A)$ which can be expressed as a determinant

$$\frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

This is only the algebraic area. If the ordering of the vertices of the triangle ABC is changed to ACB, then the value of this area changes by a sign. Thus (ABC) is the absolute value of this determinant.

The determinant can also be expressed as

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix},$$

which is just $\frac{1}{2}(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{k}$.

5. **Tangent to a circle**. Let C be the circle with equation $x^2 + y^2 + 2fx + 2gy + h = 0$ and $P = (x_0, y_0)$ be a point on C. The equation of the tangent line to the circle C at P is given by

$$x_0x + y_0y + f(x + x_0) + g(y + y_0) + h = 0.$$

Proof. The center of the circle is $(-f, -g)$. Thus if (x, y) is a point on the tangent line, then $\langle (x - x_0, y - y_0), (x_0 + f, y_0 + g) \rangle = 0$. Using $x_0^2 + y_0^2 + 2fx_0 + 2gy_0 + h = 0$, the result follows.

6. **Coaxal circles**. The standard equation of a circle is of the form

$$C(x, y) = x^2 + y^2 + 2fx + 2gy + h = 0.$$

The power of a point $P(a, b)$ with respect to a circle $C = 0$ is also given by $C(a, b)$.

The locus of the points having equal power with respect to C_1 and C_2 is called the *radical axis* of C_1 and C_2. For any 2 circles $C_1 = 0$ and $C_2 = 0$, the radical axis is given by

$$C_1 - C_2 = 0$$

The collection of all circles of the form $C_3 = \lambda C_1 + \mu C_2$, where $\lambda + \mu = 1$, forms a so-called *pencil of circles*. Any two such circles have the same radical axes, and they are called *coaxal circles*.

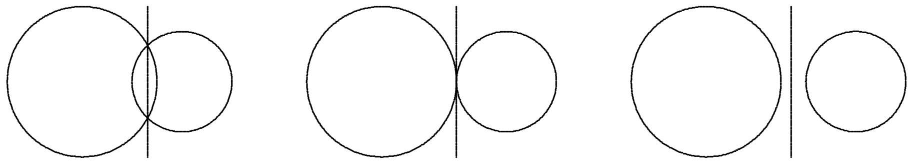


Figure 8.1: Coaxal circles

Example 8.1 Let $C_1 : x^2 + y^2 = 10$ and $C_2 : x^2 + y^2 - 2x + y = 10$. Find the equation of the circle passing through the points of intersection of C_1 and C_2 and the point $(5, 5)$.

Solution. The radical axis of C_1 and C_2 has the equation given by $(x^2 + y^2 - 10) - (x^2 + y^2 - 2x + y - 10) = 0$. That is $y - 2x = 0$. Thus the equation of the required circle is of the form $(x^2 + y^2 - 10) + \lambda(y - 2x) = 0$. Since it passes through the point $(5, 5)$, we find that $\lambda = 8$. Consequently, the equation is $x^2 + y^2 - 10 + 8y - 16x = 0$.

Example 8.2 Let ℓ be a line outside a circle $\mathcal{C}$. Take any point T on ℓ. Let TA and TB be the two tangents from T to $\mathcal{C}$. Prove that the chord AB passes through a fixed point.

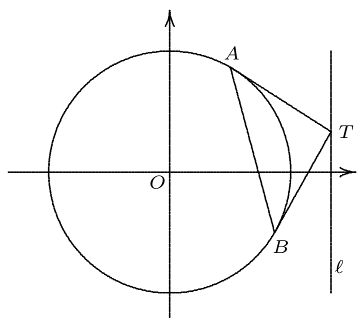


Figure 8.2: The chord AB passes through a fixed point

Solution. Let the centre O of the circle be the origin. Choose coordinate axes so that ℓ is parallel to the y-axis. Let r be the radius of the circle and (c, t) the coordinates of T. Here $r < c$. The equation of the circle is $x^2 + y^2 = r^2$.

Next, we wish to find the equation of the chord AB. To do this, it is not necessary to find the coordinates of A and B. Let the coordinates of A be (x_A, y_A). The equation of the tangent line TA is $x_A x + y_A y = r^2$. (It is a straight line passing through A and perpendicular to OA.) As it passes through T, we have $x_A c + y_A t = r^2$. Therefore, A lies on the straight line

$$cx + ty = r^2. \tag{8.1.1}$$

Similarly, B lies on (8.1.1). So (8.1.1) is the equation of AB! Clearly, the line defined by (8.1.1) passes through the point $(r^2/c, 0)$ which is independent of t.

There is also another easy way to find the equation of the line AB. Observe that O, A, T, B lie on a circle $\mathcal{C}'$ with diameter OT. The equation of this circle is $(x - c)x + (y - t)y = 0$. (Take a point

X on $\mathcal{C}'$. Then OX is perpendicular to TX. The scalar product gives the equation satisfied by X.) Now $\mathcal{C}$ and $\mathcal{C}'$ both pass through A and B. Hence the difference of their equations is the equation of AB. Note that the line AB is the radical axis of $\mathcal{C}$ and $\mathcal{C}'$.

Exercise 8.1 Let $P = (a, b)$ be a point outside the unit circle $x^2 + y^2 = 1$ and let PT_1 and PT_2 be the tangents to it. Show that the coordinates of T_1 and T_2 are given by

$$\left(\frac{a - b\sqrt{a^2 + b^2 - 1}}{a^2 + b^2}, \frac{b + a\sqrt{a^2 + b^2 - 1}}{a^2 + b^2}\right), \left(\frac{a + b\sqrt{a^2 + b^2 - 1}}{a^2 + b^2}, \frac{b - a\sqrt{a^2 + b^2 - 1}}{a^2 + b^2}\right).$$

Exercise 8.2 Let C be the circle with equation $x^2 + y^2 + 2ax + 2by + f = 0$ and $P = (x_0, y_0)$ a point outside C. The tangents from P touch C at the points X and Y. Show that the equation of the line XY is given by

$$x_0x + y_0y + a(x + x_0) + b(y + y_0) + f = 0.$$

Exercise 8.3 Show that the equation of the circle passing through the points $(p_1, p_2), (q_1, q_2), (r_1, r_2)$ is given by

$$\begin{vmatrix} x - p_1 & y - p_2 & p_1^2 + p_2^2 - x^2 - y^2 \\ p_1 - q_1 & p_2 - q_2 & q_1^2 + q_2^2 - p_1^2 - p_2^2 \\ q_1 - r_1 & q_2 - r_2 & r_1^2 + r_2^2 - q_1^2 - q_2^2 \end{vmatrix} = 0.$$

[Hint: The form of this determinant shows that it is an equation of a circle. The substitution of the coordinates of each of the three points clearly makes the determinant zero. Consider the 4 points: $(x, y), (p_1, p_2), (q_1, q_2), (r_1, r_2)$ on the circle, the perpendicular bisectors of any three of the chords among these 4 points must concur at the centre of the circle. Thus 4 points are concyclic if and only if the above determinant is zero.]

8.2 Barycentric and homogeneous coordinates

Let $A_1A_2A_3$ be a triangle on the plane. For any point M, the ratio of the (signed) areas

$$[MA_2A_3] : [MA_3A_1] : [MA_1A_2]$$

is called the *barycentric coordinates* or *areal coordinates* of M.
Here $[MA_2A_3]$ is the signed area of the triangle MA_2A_3. It is positive, negative or zero according to both M and A_1 lie on the same side, opposite side, or on the line A_2A_3. Generally, we use $(\mu_1 : \mu_2 : \mu_3)$ to denote the barycentric coordinates of a point M.

Theorem 8.1 *Let $[MA_2A_3] = \mu_1$, $[MA_3A_1] = \mu_2$, $[MA_1A_2] = \mu_3$ and $[A_1A_2A_3] = 1$ so that $\mu_1 + \mu_2 + \mu_3 = 1$. Then*
1. $A_3N_2 : N_2A_1 = \mu_1 : \mu_3$, etc.
2. $A_1M : MN_1 = (\mu_2 + \mu_3) : \mu_1$.
3. $\mathbf{A_2M} = \mu_3\mathbf{A_2A_3} + \mu_1\mathbf{A_2A_1}$.

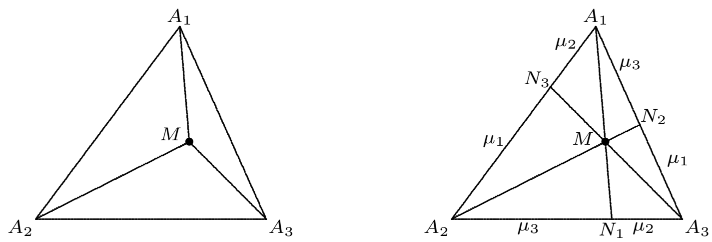


Figure 8.3: Barycentric coordinates

Proof. Let prove 2. Let $[MN_1A_3] = \alpha, [MA_2N_1] = \beta$. Then $\frac{A_1M}{MN_1} = \frac{\mu_2}{\alpha}$ and $\frac{A_1M}{MN_1} = \frac{\mu_3}{\beta}$. Thus

$$\frac{A_1M}{MN_1} = \frac{\mu_2 + \mu_3}{\alpha + \beta} = \frac{\mu_2 + \mu_3}{\mu_1}.$$

Properties

1. The barycentric coordinates of a point are homogeneous. That is $(\mu_1 : \mu_2 : \mu_3) = (k\mu_1 : k\mu_2 : k\mu_3)$ for any nonzero real number k. As such, it can also be identified with the homogeneous coordinates of the point.

2. For the points A_1, A_2 and A_3, we have $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0)$ and $A_3 = (0 : 0 : 1)$ respectively.

3. Let the Cartesian coordinates of A, B, C be (x_A, y_A), $(x_B, y_B), (x_C, y_C)$ respectively. If the barycentric coordinates of M is $(\mu_1 : \mu_2 : \mu_3)$, then the Cartesian coordinates of M is $\left(\frac{\mu_1 x_A + \mu_2 x_B + \mu_3 x_C}{\mu_1 + \mu_2 + \mu_3}, \frac{\mu_1 y_A + \mu_2 y_B + \mu_3 y_C}{\mu_1 + \mu_2 + \mu_3}\right)$.

4. The centroid of $\triangle A_1A_2A_3$ is the point $G = (1 : 1 : 1)$.

5. The circumcentre of $\triangle A_1A_2A_3$ is the point $O = (\sin 2A_1 : \sin 2A_2 : \sin 2A_3)$.

6. Suppose $A_1M_3/M_3A_2 = m_1$ and $A_2M_1/M_1A_3 = m_2$. Then $M = (1 : m_1 : m_1m_2)$.

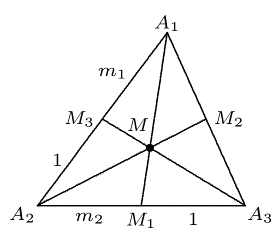


Figure 8.4: The barycentric coordinates of M

Proof. As $[MA_2A_3] : [MA_3A_1] = M_3A_2 : A_1M_3 = 1 : m_1$, and $[MA_3A_1] : [MA_1A_2] = 1 : m_2 = m_1 : m_1m_2$, we have $[MA_2A_3] : [MA_3A_1] : [MA_1A_2] = 1 : m_1 : m_1m_2$.

7. The incentre of $\triangle A_1A_2A_3$ is the point $(a_1 : a_2 : a_3)$, where a_1, a_2, a_3 are lengths of the sides $\triangle A_1A_2A_3$. This follows from the angle bisector theorem.

8. For the excentres of $\triangle A_1A_2A_3$, we have

$$I_1 = (-a_1 : a_2 : a_3), \quad I_2 = (a_1 : -a_2 : a_3), \quad I_3 = (a_1 : a_2 : -a_3).$$

9. The orthocentre of $\triangle A_1A_2A_3$ is the point

$$H = (\tan A_1 : \tan A_2 : \tan A_3) = (\frac{1}{-a_1^2 + a_2^2 + a_3^2} : \frac{1}{a_1^2 - a_2^2 + a_3^2} : \frac{1}{a_1^2 + a_2^2 - a_3^2}).$$

10. The Gergonne point of $\triangle A_1A_2A_3$ is the point

$$(\frac{1}{s-a_1} : \frac{1}{s-a_2} : \frac{1}{s-a_3}).$$

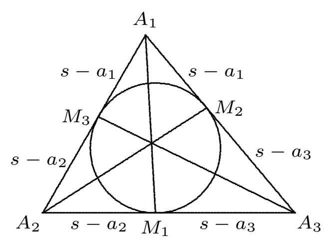


Figure 8.5: Gergonne point

11. The Nagel point of $\triangle A_1A_2A_3$ is the point $N = (s - a_1 : s - a_2 : s - a_3)$.

12. The equation of the line passing through the points $(a_1 : a_2 : a_3)$ and $(b_1 : b_2 : b_3)$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} x_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} x_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} x_3 = 0.$$

This is a linear relation of the homogeneous coordinates of a point. In general the equation of a straight line in homogeneous coordinates is of the form

$$\ell : \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 = 0.$$

Usually, the coefficients are used to denote such a line. In notation, we write

$$\ell = [\mu_1 : \mu_2 : \mu_3].$$

Thus the line passing through $(a_1 : a_2 : a_3)$ and $(b_1 : b_2 : b_3)$ is given by

$$\ell = [a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1].$$

13. Three points $A = (a_1 : a_2 : a_3), B = (b_1 : b_2 : b_3), C = (c_1 : c_2 : c_3)$ are collinear if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

14. The intersection of the lines $\ell_1 = [a_1 : a_2 : a_3]$ and $\ell_2 = [b_1 : b_2 : b_3]$ is given by

$$P = (a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1).$$

15. Three lines $\ell = [a_1 : a_2 : a_3], m = [b_1 : b_2 : b_3], n = [c_1 : c_2 : c_3]$ are concurrent if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Proof. The intersection of ℓ and m is $(a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1)$. It lies on n if and only if $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = 0.$

16. Let $P = (p_1 : p_2 : p_3)$ and $Q = (q_1 : q_2 : q_3)$ with $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$. If M divides PQ in the ratio $PM : MQ = \beta : \alpha$, then the point M has homogeneous coordinates $(\alpha p_1 + \beta q_1 : \alpha p_2 + \beta q_2 : \alpha p_3 + \beta q_3)$.

17. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ be three non-collinear points on the plane. Let the homogeneous coordinates of A, B and C be $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ respectively. Show that for any point $P = (\lambda_1 : \lambda_2 : \lambda_3)$, its cartesian coordinates is given by

$$\left(\frac{\lambda_1 a_1 + \lambda_2 b_1 + \lambda_3 c_1}{\lambda_1 + \lambda_2 + \lambda_3}, \frac{\lambda_1 a_2 + \lambda_2 b_2 + \lambda_3 c_2}{\lambda_1 + \lambda_2 + \lambda_3}\right).$$

Example 8.3 In any triangle $A_1A_2A_3$, the centroid G, the incentre I and the Nagel point N are collinear.

Proof. This is because $\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ s - a_1 & s - a_2 & s - a_3 \end{vmatrix} = 0$. In fact G divides the segment IN in the ratio 1:2.

Theorem 8.2 (Menelaus) *In the triangle $A_1A_2A_3$, points B_1, B_2, and B_3 are on the sides A_2A_3, A_3A_1 and A_1A_2 respectively such that $A_2B_1 : B_1A_3 = \alpha_1 : \beta_1$, $A_3B_2 : B_2A_1 = \alpha_2 : \beta_2$ and $A_1B_3 : B_3A_2 = \alpha_3 : \beta_3$. Then B_1, B_2 and B_3 are collinear if and only if $\alpha_1\alpha_2\alpha_3 = -\beta_1\beta_2\beta_3$.*

Proof. Take $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0), A_3 = (0 : 0 : 1)$. Then $B_1 = (0 : \beta_1 : \alpha_1)$, $B_2 = (\alpha_2 : 0 : \beta_2)$ and $B_3 = (\beta_3 : \alpha_3 : 0)$. Thus

$$\begin{vmatrix} 0 & \beta_1 & \alpha_1 \\ \alpha_2 & 0 & \beta_2 \\ \beta_3 & \alpha_3 & 0 \end{vmatrix} = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3.$$

Therefore, B_1, B_2 and B_3 are collinear if and only if $\alpha_1\alpha_2\alpha_3 = -\beta_1\beta_2\beta_3$.

Theorem 8.3 *Let B_1 and C_1, B_2 and C_2, B_3 and C_3 be respective pairs of points on the sides A_2A_3, A_3A_1, A_1A_2 or their extensions of $\triangle A_1A_2A_3$ such that*

$$\frac{A_2B_1}{B_1A_3} = \lambda_1,\ \frac{A_3B_2}{B_2A_1} = \lambda_2,\ \frac{A_1B_3}{B_3A_2} = \lambda_3,$$

$$\frac{A_3C_1}{C_1A_2} = \mu_1,\ \frac{A_1C_2}{C_2A_3} = \mu_2,\ \frac{A_2C_3}{C_3A_1} = \mu_3.$$

Then B_1C_2, B_2C_3, B_3C_1 are concurrent if and only if

$$\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3 + \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 - 1 = 0.$$

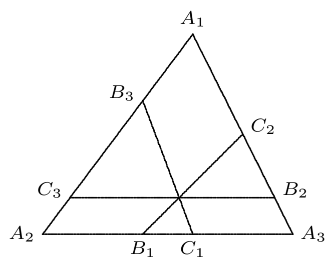


Figure 8.6: A generalization of Ceva's theorem

Remark 8.1 Suppose $C_1 = A_3, C_2 = A_1, C_3 = A_2$ so that $\mu_1 = \mu_2 = \mu_3 = 0$. The conclusion is that B_1A_2, B_2A_3, B_3A_1 are concurrent if and only if $\lambda_1\lambda_2\lambda_3 = 1$, which is Ceva's Theorem.

Proof. Take $A_1 = (1:0:0), A_2 = (0:1:0), A_3 = (0:0:1)$. Then

$$B_1 = (0:1:\lambda_1), B_2 = (\lambda_2:0:1), B_3 = (1:\lambda_3:0)$$

and

$$C_1 = (0:\mu_1:1), C_2 = (1:0:\mu_2), C_3 = (\mu_3:1:0).$$

The line B_1C_2 is given by $\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & 1 & \lambda_1 \\ 1 & 0 & \mu_2 \end{vmatrix} = 0$. That is $B_1C_2 = [\mu_2 : \lambda_1 : -1]$. Similarly, $B_2C_3 = [-1:\mu_3:\lambda_2]$ and $B_3C_1 = [\lambda_3 : -1 : \mu_1]$. They are concurrent if and only if

$$\begin{vmatrix} \mu_2 & \lambda_1 & -1 \\ -1 & \mu_3 & \lambda_2 \\ \lambda_3 & -1 & \mu_1 \end{vmatrix} = 0,$$

which is the required expression.

Exercise 8.4 Prove that in any triangle the 3 lines each of which joins the midpoint of a side to the midpoint of the altitude to that side are concurrent.

[Hint. Take $A_1 = (1:0:0), A_2 = (0:1:0), A_3 = (0:0:1)$. Let F_1, F_2 and F_3 be the midpoints of the altitudes A_1N_1, A_2N_2 and A_3N_3 respectively. If M_1, M_2 and M_3 are the midpoints of the sides A_2A_3, A_3A_1 and A_1A_2 respectively, show that $M_1F_1 = [\tan A_3 - \tan A_2 : \tan A_2 + \tan A_3 : -\tan A_2 - \tan A_3]$, $M_2F_2 = [-\tan A_1 - \tan A_3 : \tan A_1 - \tan A_3 : \tan A_1 + \tan A_3]$, and $M_3F_3 = [\tan A_1 + \tan A_2 : -\tan A_1 - \tan A_2 : \tan A_2 - \tan A_1]$.]

Exercise 8.5 (Euler line) Prove that the circumcentre, the centroid and the orthocentre of a triangle are collinear.

Exercise 8.6 (Newton line) In a quadrilateral $ABCD$, AB intersects CD at E, AD intersects BC at F. Let L, M and N be the midpoints of AC, BD and EF respectively. Prove that L, M, N are collinear.

[Hint. Let $A = (1:0:0), B = (0:1:0), C = (0:0:1)$ and $D = (u:v:w)$ with $u+v+w = 1$. Show that $N = (u - u^2 : v + v^2 : w - w^2)$.]

8.3 Projective plane

The real projective plane usually denoted by $\mathbb{P}^2$ consists of all lines in $\mathbb{R}^3$ passing through the origin. That is

$$\mathbb{P}^2 = \{L \ : \ L \text{ is a line through } O \text{ in } \mathbb{R}^3\}.$$

We can represent each line L through O by any non-zero vector $\mathbf{OA}$ along L. This suggests we can represent L by homogeneous coordinates consisting of a triple of three numbers $(\alpha : \beta : \gamma)$. (That is $(\alpha : \beta : \gamma) = (k\alpha : k\beta : k\gamma)$ for any non-zero k.) Thus

$$\mathbb{P}^2 = \{(\alpha : \beta : \gamma) \ : \ \alpha, \beta, \gamma \in \mathbb{R} \text{ and not all } \alpha, \beta, \gamma = 0\}.$$

For any two distinct lines L_1 and L_2 through O, it determines a plane $ax + by + xz = 0$ through O. We can represent this plane by the three coefficients a, b, c. As any non-zero multiple of $ax + by + xz = 0$ represents the same plane, this plane can be represented by the homogeneous coordinates $[a : b : c]$. Furthermore, the vector $\langle a, b, c\rangle$ is a normal vector to this plane. Thus if $L_1 = (\alpha_1 : \beta_1 : \gamma_1)$ and $L_2 = (\alpha_2 : \beta_2 : \gamma_2)$, the plane ℓ determined by L_1 and L_2 has a normal vector given by the *cross product* of $\langle \alpha_1 : \beta_1 : \gamma_1\rangle$ and $\langle \alpha_2 : \beta_2 : \gamma_2\rangle$. That is the homogeneous coordinates of the plane ℓ is

$$\left[\begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} : -\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix} : \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}\right].$$

If we denote the collection of all planes through the origin by $\mathbb{P}^{2^*}$, then

$$\mathbb{P}^{2^*} = \{[a : b : c] \ : \ a, b, c \in \mathbb{R} \text{ and not all } a, b, c = 0\}.$$

There is a one-to-one correspondence between $\mathbb{P}^2$ and $\mathbb{P}^{2^*}$ given by associating a line L the plane perpendicular to L.

Consider the plane $\rho : z = 1$, or any plane not containing the origin. Any element L of $\mathbb{P}^2$ not contained in the xy plane intersects ρ in a unique point P_L. See figure 8.7. In this way we can

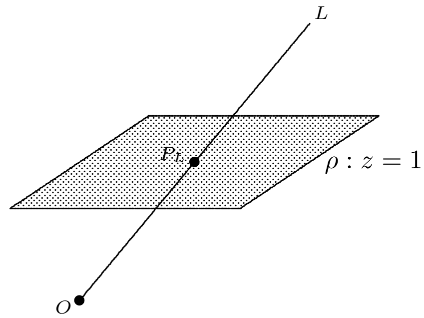


Figure 8.7: The projective plane

think of $\mathbb{R}^2 \equiv \rho$ lying inside $\mathbb{P}^2$. Any plane containing two distinct lines L_1 and L_2 in $\mathbb{P}^2$ (both L_1 and L_2 are not contained in the xy-plane) intersects ρ in a line ℓ joining P_{L_1} and P_{L_2}. Thus we can represents a point in $\mathbb{R}^2 \equiv \rho$ by the homogeneous coordinates $(\alpha : \beta : \gamma)$, and a line in $\mathbb{R}^2 \equiv \rho$ by $[a : b : c]$.

In fact we can think of $\mathbb{P}^2$ as $\mathbb{R}^2$ with a "line" ω added at infinity. This "line" ω corresponds to the xy-plane. With this correspondence, every "line" in $\mathbb{P}^2$ meets ω in a unique point, and any two "lines" in $\mathbb{P}^2$ meet.

If S^2 denotes the unit sphere in $\mathbb{R}^3$, then every line in $\mathbb{R}^3$ intersects S^2 in a pair of antipodal (diametrically opposite) points. In this way, we can regard $\mathbb{P}^2$ as the *space* obtained by identifying antipodal points of the unit sphere.

Though the geometry of $\mathbb{P}^2$ is different from $\mathbb{R}^2$, the properties of concurrence and collinearity are equivalent in both $\mathbb{P}^2$ and $\mathbb{R}^2$. Thus many of the results involving concurrence and collinearity in $\mathbb{R}^2$ can be stated and proved in $\mathbb{P}^2$.

8.4 Quadratic curves

A quadratic curve (or a conic) is a curve with equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Thus the general equation of a quadratic curve is determined by 6 coefficients. So it only requires 5 points to determine a quadratic curve. Quadratic curves are classified into the following types: parabola, circle, ellipse, hyperbola, and 2-straight line. They are the possible cross-sections obtained by slicing a double cone with a plane, thus they are also called conics. If $F_1(x, y) = 0$ and $F_2(x, y) = 0$ are two such curves, their intersection points are given by the roots of the system of these two equations. Since F_1 and F_2 are quadratic, there are generally 4 solutions for this system. Thus two quadratic curves generally intersect in 4 points (or less). Suppose $F_1(x, y) = 0$ and $F_2(x, y) = 0$ intersect in P_1, P_2, P_3, P_4. Then for any real numbers λ_1 and λ_2 not both equal to 0, $\lambda_1 F_1 + \lambda_2 F_2 = 0$ is also a quadratic curve, and it passes through P_1, P_2, P_3, P_4. Conversely, any quadratic curve passing through P_1, P_2, P_3, P_4 is of the form $\lambda_1 F_1 + \lambda_2 F_2 = 0$ for some suitable λ_1 and λ_2.

Theorem 8.4 (Butterfly theorem) *Through the midpoint O of a chord GH of a circle, two other chords AB and CD are drawn; chords AC and BD meet GH at E and F respectively. Then O is the midpoint of EF.*

Proof. Let the equation of the circle be $x^2+y^2-2by+f=0$. Let the equations of the lines AB and CD be $y=k_1x$ and $y=k_2x$ respectively. Therefore the pair of lines $(y-k_1x)(y-k_2x)=0$ passes through the 4 points A, B, C, D. Each quadratic curve passing through the 4 points A, B, C, D is represented by

$$x^2+y^2-2by+f+\lambda(y-k_1x)(y-k_2x)=0.$$

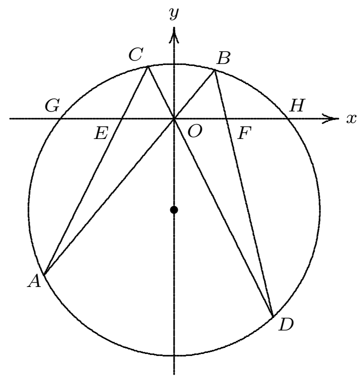


Figure 8.8: Butterfly theorem

In particular the pair of lines AC and BD is of this form for some suitable λ. Setting $y=0$ for the equation of this pair of lines, we get $(1+\lambda k_1k_2)x^2+f=0$. From this we see that the roots of this equation give the intercepts E and F of this pair of lines with the x-axis, and they are of equal magnitude but opposite sign. Thus $OE=OF$.

Remark 8.2 We can also take the lines AD and BC meeting the x-axis at E' and F' respectively. Then $OE'=OF'$.

Example 8.4 Suppose AB and CD are non-intersecting chords in a circle and that P is a point on the arc AB remote from C and D. Let PC and PD intersect AB at Q and R respectively. Prove that $AQ\cdot RB/QR$ is a constant independent of the position of P.

Solution. Let $AQ=x$, $QR=y$ and $RB=z$. Suppose we draw the circle through P, Q and D to cross AB extended at E. In this circle, the chord QD will subtend equal angles θ at P and E. Now, as P varies, $\angle CPD=\theta$ remains the same in the given circle, implying that, for all positions of P, this second circle through P, Q and D always goes through the same point E on AB extended. Consequently, the segment BE always has the same length k.
Therefore,$(x+y)z=PR\cdot RD=y(z+k)$ giving $xz=yk$, thus $xz/y=k$ is a constant. [This is called Haruki's lemma and can be used to prove the Butterfly Theorem and the double Butterfly Theorem. See Mathematics Magazine vol 63, No 4, October 1990, pp256.]

Exercise 8.7 Using the result of Example 8.4, deduce the Butterfly theorem 8.4.

Exercise 8.8 In Figure 8.10, the point O is the midpoint of BC. Prove that $OX=OY$.

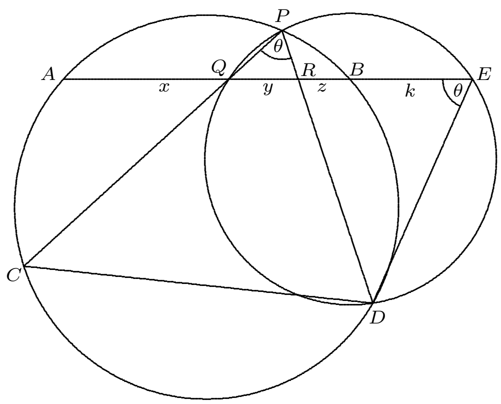


Figure 8.9: Haruki's lemma

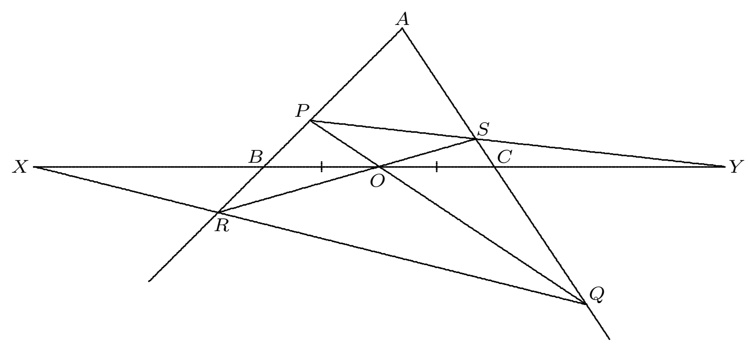


Figure 8.10: Butterfly theorem for 2-straight lines

Exercise 8.9 Let A, B, C, D, E, F be 6 points on the plane such that AB intersects DE at L, BC intersects EF at N and CD intersects FA at M. Prove that if L, N, M are collinear, then there is a conic passing through A, B, C, D, E, F.

[Hint: Use the fact that for any 5 points in general position, there is a conic passing through them. Let α be a conic passing through A, B, C, D, E. Let EN meet α at F' and let the intersection of AF' and CD be M'. By Pascal's theorem which also holds for six points on a conic, there is a conic passing through A, B, C, D, E, F'. Show that $M' = M$ and hence $F' = F$. This is the converse of Pascal's theorem.]

Exercise 8.10 Show that the Butterfly theorem holds for any quadratic curve $ax^2 + bxy + cy^2 + dx + ey + f = 0$.

[Hint: Position the chord PQ of the quadratic curve so that P and Q lie on the x-axis with the origin as their midpoint. Show that in this coordinate system, the coefficient $d = 0$. Then follow the proof of theorem 8.4.]

Remark 8.3 A direct proof of the Butterfly theorem is as follow.
In figure 8.11, M is the midpoint of GH. Let K and L be the midpoints of the chords AC and BD respectively. Join K, E, M, F, L to the centre O of the circle. Join KM and LM. Since the triangles

AMC and DMB are similar, we have $2CK/CM = CA/CM = BD/BM = 2BL/MB$ so that the triangles KCM and LBM are similar. Thus $\angle CKM = \angle BLM$.

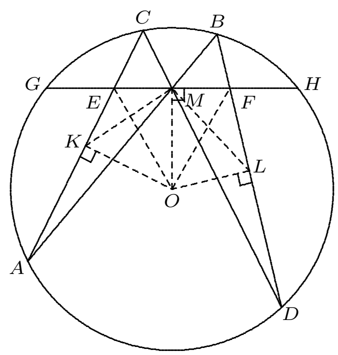


Figure 8.11: A direct proof of the Butterfly Theorem

As O, K, E, M and O, L, F, M are concyclic, we have $\angle EOM = \angle CKM = \angle BLM = \angle FOM$. Since OM is perpendicular to GH, we conclude that M is the midpoint of EF.

Exercise 8.11 (A generalized Butterfly theorem) Let AB be a chord of a circle with midpoint P, and let the chords XW and ZY intersect AB at M and N respectively. Let AB intersects XY at C and ZW at D. Prove that if $MP = PN$, then $CP = PD$.

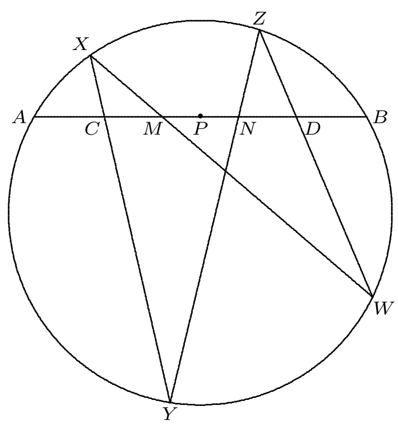


Figure 8.12: A generalized Butterfly theorem

[Hint: Use Haruki's lemma.]