Questions Bank Geometry
Second Class

Chapter 5

Exercise 5: Prove the followings

1. The 3 medians of $\triangle ABC$ are concurrent. Their common point, denoted by G, is called the **centroid** of $\triangle ABC$.
2. The 3 altitudes of $\triangle ABC$ are concurrent. Their common point, denoted by H, is called the **orthocentre** of $\triangle ABC$.
3. The internal bisectors of the 3 angles of $\triangle ABC$ are concurrent. Their common point, denoted by I, is called the **incentre** of $\triangle ABC$.
4. The internal bisector of $\angle A$ and the external bisectors of the other two angles of $\triangle ABC$ are concurrent. Their common point, denoted by I_a, is called the **excentre** of $\triangle ABC$. Similarly, there are excentres I_b and I_c.
5. The three perpendicular bisectors of a triangle $\triangle ABC$ are concurrent. Their common point, denoted by O is called the **circumcentre** of $\triangle ABC$.
6. The cevians where the feet are the tangency points of the incircle (or excircle) of a triangle are concurrent. This common point is called the **Gergonne** point. Thus there are 4 Gergonne points for a triangle.

Exercise 5.1 Let $ABCD$ be a trapezium with AB parallel to CD. Let M and N be the midpoints of AB and CD respectively. Prove that MN, AC and BD are concurrent.

Exercise 5.2 Suppose a circle cuts the sides of a triangle $A_1A_2A_3$ at the points $X_1, Y_1, X_2, Y_2, X_3, Y_3$. Show that if A_1X_1, A_2X_2, A_3X_3 are concurrent, then A_1Y_1, A_2Y_2, A_3Y_3 are concurrent.

[Hint: Observe that $X_1A_2 \cdot Y_1A_2 = X_3A_2 \cdot Y_3A_2$.]

Exercise 5.3 Let P be a point inside the triangle ABC. The bisector of $\angle BPC$, $\angle CPA$, and $\angle APB$ meet BC, CA and AB at X, Y and Z, respectively. Prove that AX, BY, CZ are concurrent.

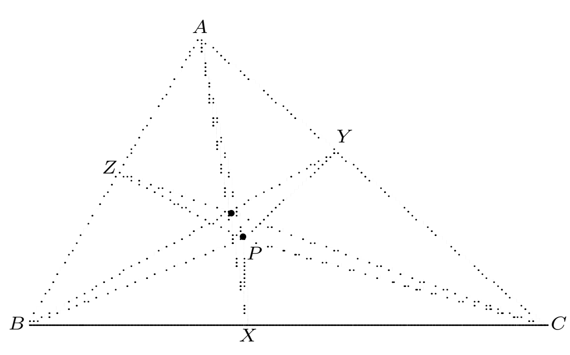

Figure 5.10: AX, BY, CZ are concurrent

Exercise 5.4 Let Γ be a circle with center I, the incentre of triangle ABC. Let D, E, F be points of intersection of Γ with the lines from I that are perpendicular to the sides BC, CA, AB respectively. Prove that AD, BE, CF are concurrent.

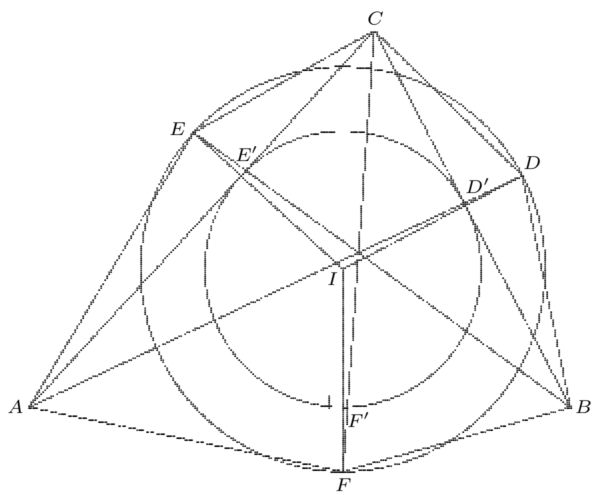

Figure 5.11: A generalization of the Gergonne point

[Hint: Let the intersection of AD, BE, CF with BC, CA, AB be D', E', F' respectively. It is easy to establish that $\angle FAF' = \angle EAE'$, $FBF' = \angle DBD'$, $\angle DCD' = \angle ECE'$. Also $AE = AF$, $BF = BD$, $CD = CE$. The ratio $AF'/F'B$ equals to the ratio of the altitudes from A and B on CF of the triangles AFC and BFC and hence equals to the ratio of their areas. Now apply Ceva's theorem.]

Exercise 5.5 Let A_1, B_1 and C_1 be points in the interiors of the sides BC, CA and AB of a triangle ABC respectively. Prove that the perpendiculars at the points A_1, B_1, C_1 are concurrent if and only if $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$. This is known as Carnot's lemma.

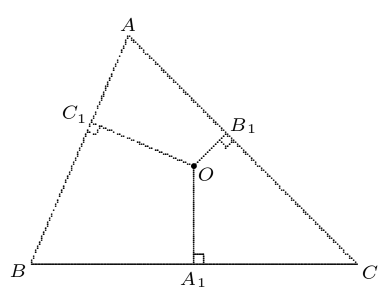

Figure 5.12: Carnot's lemma

Chapter 6

Exercise 6.1 Prove the converse of Desargues' theorem: Let ABC and $A_1B_1C_1$ be two triangles such that BC intersects B_1C_1 at L, CA intersects C_1A_1 at M and AB intersects A_1B_1 at N. Suppose L, M, N are collinear. Then AA_1, BB_1 and CC_1 are concurrent.

[Hint: Refer to figure 6.6. Let AA_1 intersect BB_1 at O. It suffices to prove O, C, C_1 are collinear. To do so, apply Desargues' theorem to the triangles MAA_1 and LBB_1 which are perspective from the point N.]

Exercise 6.2 Prove that the interior angle bisectors of two angles of a non-isosceles triangle and the exterior angle bisector of the third angle meet the opposite sides in three collinear points.

Exercise 6.3 (Monge's Theorem) Prove that the three pairs of common external tangents to three circles, taken two at a time, meet in three collinear points.

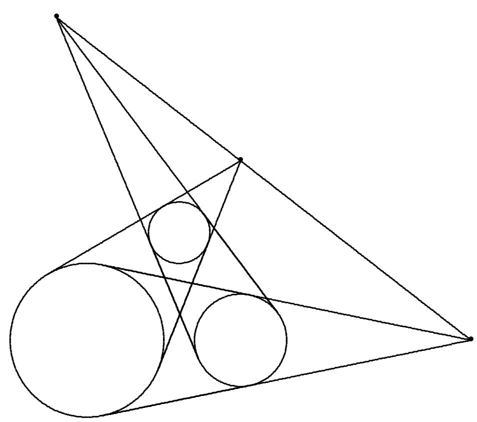

Figure 6.8: Monge's theorem

Exercise 6.4 Let I be the centre of the inscribed circle of the non-isosceles triangle ABC, and let the circle touch the sides BC, CA, AB at the points A_1, B_1, C_1 respectively. Prove that the centres of the circumcircles of $\triangle AIA_1, \triangle BIB_1$ and $\triangle CIC_1$ are collinear.

[Hint: Let the line perpendicular to CI and passing through C meet AB at C_2. By analogy, we have the points A_2 and B_2. It is obvious that the centres of the circumcircles of $\triangle AIA_1$, $\triangle BIB_1$ and $\triangle CIC_1$ are the midpoints of A_2I, B_2I and C_2I, respectively. So it is sufficient to prove that A_2, B_2 and C_2 are collinear.]

Chapter 7

Exercise6: Prove the followings

1. Let AB and CD be two chords in a circle. The followings are equivalent.

(i) $\overset{\frown}{AB}=\overset{\frown}{CD}$, where $\overset{\frown}{AB}$ is the length arc of AB.
(ii) $AB = CD$.
(iii) $\angle AOB = \angle COD$.
(iv) $OE = OF$.

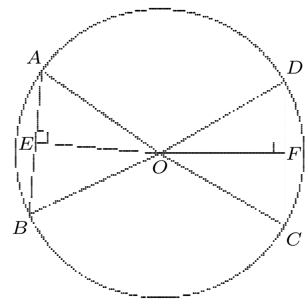

Figure 7.1

2. Let AB and CD be two chords in a circle. The followings are equivalent.

(i) $\overset{\frown}{AB} > \overset{\frown}{CD}$
(ii) $AB > CD$.
(iii) $\angle AOB > \angle COD$.
(iv) $OE < OF$.

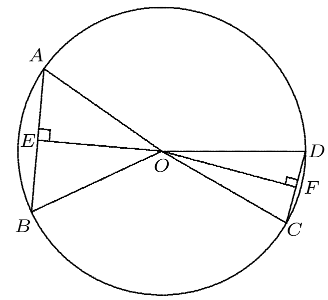

Figure 7.2

3. Let D be a point on the arc AB. The followings are equivalent.

(i) $\overset{\frown}{AD} = \overset{\frown}{DB}$.
(ii) $AC = CB$.
(iii) $\angle AOD = \angle BOD$.
(iv) $OD \perp AB$.

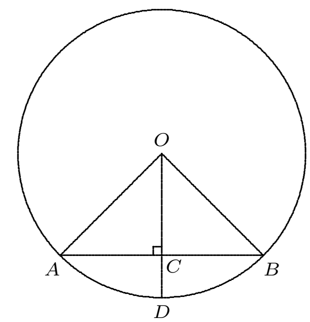

Figure 7.3

4. The angle subtended by an arc BC at a point A on a circle is half the angle subtended by the arc BC at the centre of the circle.

That is $\angle BOC = 2\angle BAC$.

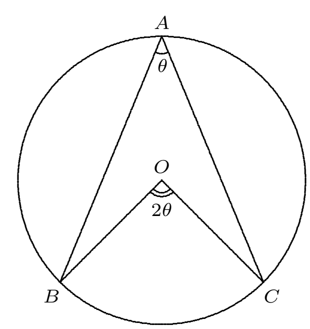

Figure 7.4

5. The angle subtended by the same segment at any point on the circle is constant.

That is $\angle BAC = \angle BDC$.

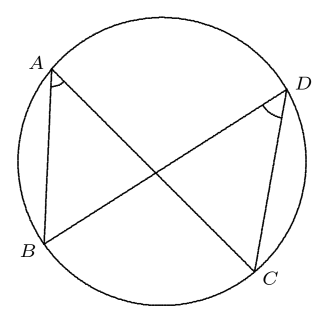

Figure 7.5

6. A chord BC is a diameter if and only if the angle subtended by it at point on the circle is a right angle.

That is $\angle BAC = 90°$ for any point $A \neq B$ or C on the circle.

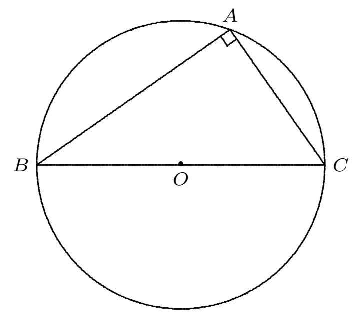

Figure 7.6

7. Let $ABCD$ be a convex quadrilateral. The followings are equivalent.

(i) $ABCD$ is a cyclic quadrilateral
(ii) $\angle BAC = \angle BDC$.
(iii) $\angle A + \angle C = 180°$.
(iv) $\angle ABE = \angle D$.

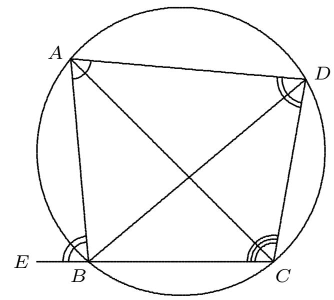

Figure 7.7

8. Alternate Segment Theorem. Let A, B, C be three points on a circle. Let TA be a line through A with T and B lying on the same side of the line AC. Then the followings the equivalent.

(i) AT is tangent to the circle at A.
(ii) $OA \perp AT$.
(iii) $\angle BAT = \angle BCA$.

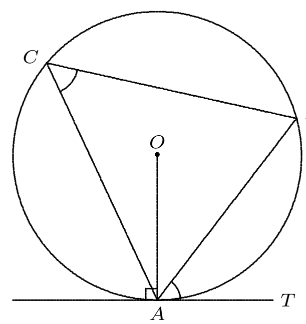

Figure 7.8

9. Let PS and PT be tangents to the circle. Then

(i) $PS = PT$,
(ii) OP bisects $\angle SPT$
(iii) OP bisects $\angle SOP$
(iv) OP is the perpendicular bisector of the segment ST.

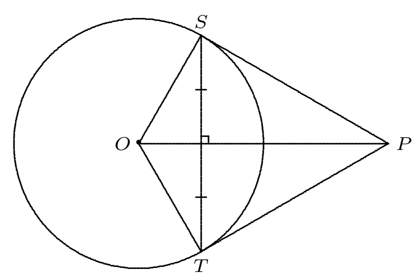

Figure 7.9

The power of a point P with respect to the circle centred at O with radius R is defined as $OP^2 - R^2$.

(i) If P is outside the circle, then
the Power of P
= $OP^2 \quad R^2$
= $PT^2 = PA \cdot PB$,
which is positive.

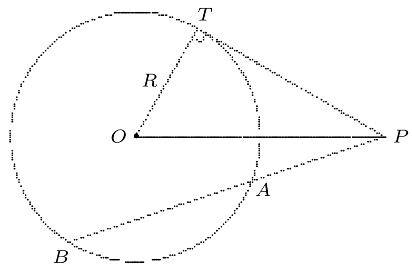

Figure 7.11

(ii) If P lies on the circumference, then
the power of $P = OP^2 \quad R^2 = 0$.

(iii) If P is inside the circle, then
the power of P
= $OP^2 - R^2 = -PZ^2$
= $-PX \cdot PY$
= $-PA \cdot PB$,
which is negative.

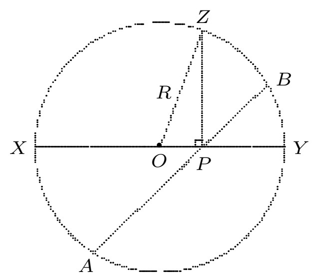

Figure 7.12

Exercise 7.1 Let D, E and F be three points on the sides BC, CA and AB of a triangle ABC respectively. Show that the circumcircles of the triangles AEF, BDF and CDE meet a common point. This point is called the Miquel point.

Corollary 7.3 $R \geq 2r$. *Equality holds if and only if ABC is equilateral.*

Exercise 7.2 Prove the *isoperimetric inequality* $s^2 \geq 3\sqrt{3}A$, where A is the area and s is the semi-perimeter of the triangle. Show that equality holds if and only if the triangle is equilateral.

Theorem 7.4 *The power of a point $P(a, b)$ with respect to a circle $C = 0$ is also given by $C(a, b)$.*

Exercise 7.3 Show that the radical axis of 2 circles is perpendicular to the line joining the centres of the 2 circles.

Exercise 7.4 Consider the pencil of circles $x^2 + y^2 \quad 2ax + c = 0$, where c is fixed and a is the parameter. (If $c > 0$, a varies in the range $\mathbb{R} \setminus (\quad \sqrt{c}, \sqrt{c})$.) Any two of its members have the same line of centres and the same radical axis. Hence it is a pencil of coaxal circles. Prove the following.

(a) If $c < 0$, each circle in the pencil meets the y-axis at the same two points $(0, \pm\sqrt{\quad c})$, and the pencil consists of circles through these two points.

(b) If $c = 0$, the pencil consists of circles touching the y-axis at the origin.

(c) If $c > 0$, the pencil consists of non-intersecting circles. Also when $a = \pm\sqrt{c}$ ($c > 0$), the circle degenerates into a point at $(\pm\sqrt{c}, 0)$.

Exercise 7.5 Consider the two pencils of circles $\mathcal{P}_1 :\ x^2 + y^2 \quad 2ax + c = 0$ and $\mathcal{P}_2 :\ x^2 + y^2$ $2by \quad c = 0$ where $c > 0$ is fixed, a and b are the parameters.

(a) Show that $\mathcal{P}_1$ consists of non-intersecting circles, and $\mathcal{P}_2$ consists of intersecting circles all passing through the points $(\pm\sqrt{c}, 0)$.

(b) Show that each circle in $\mathcal{P}_1$ is orthogonal to each circle in $\mathcal{P}_2$.

Theorem 7.9 *Let AD, BE and CF be the altitudes of the triangle ABC. The circle with diameter AB passes through D and E. Hence $HA \cdot HD = HB \cdot HE$. Similarly, $HB \cdot HE = HC \cdot HF$.*

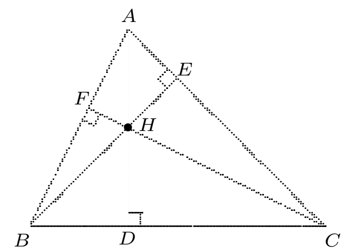

Figure 7.19

Theorem 7.10 *If X, Y, Z are any points on the respective sides BC, CA, AB of a triangle ABC, then the circles constructed on the cevians AX, BY, CZ as diameters will pass through the feet of the altitudes D, E, F respectively.*

Theorem 7.11 *If circles are constructed on 2 cevians of a triangle as diameters, then their radical axis passes through the orthocentre of the triangle.*

Theorem 7.12 *For any 3 non-coaxal circles having cevians of a triangle ABC as diameters, their radical centre is the orthocentre of $\triangle ABC$.*

Theorem 7.13 *If circles are constructed having the medians, (or altitudes or angle bisectors) of $\triangle ABC$ as diameters, then their radical centre is the orthocentre of $\triangle ABC$.*

Exercise 7.6 Let ABC be a triangle, and draw isosceles triangles BCD,CAE,ABF externally to ABC, with BC,CA,AB as their respective bases. Prove that the lines through A,B,C perpendicular to lines EF, FD, DE, respectively, are concurrent.

[Hint: Draw three circles with centres D, E, F and radii DB, EC and AF respectively.]

Exercise 7.7 A convex quadrilateral $ABCD$ is inscribed in a circle centred at O. The diagonals AC and BD meet at P. Points E and F, distinct from A, B, C, D, are chosen on this circle. The circle determined by A, P, F and the circle determined by B, P, E meet at a point Q distinct from P. Prove that the lines PQ, CE and DF are either all parallel or concurrent.

[Hint: Let R be the intersection of AF and BE. Apply Pascal's theorem to the crossed hexagon $AFDBEC$.]

Chapter 8

Exercise7: show that

1. **Ratio formula**. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. If P is the point that divides the line segment AB in the ratio $r : s$, (i.e. $AP : PB = r : s$), then the coordinates of P is given by

$$(\frac{sa_1 + rb_1}{r+s}, \frac{sa_2 + rb_2}{r+s}).$$

2. **Incentre**. Let the coordinates of the vertices of a triangle ABC be (x_A, y_A), (x_B, y_B), (x_C, y_C) respectively. The coordinates of the incentre I of $\triangle ABC$ are

$$x_I = \frac{ax_A + bx_B + cx_C}{a+b+c} \quad \text{and} \quad y_I = \frac{ay_A + by_B + cy_C}{a+b+c}.$$

3. **Family of lines**. If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two lines intersecting at a point P (i.e. $a_1b_2 \neq a_2b_1$), then the family of lines passing though P can be expressed as

$$\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) = 0.$$

4. **Area**. The algebraic area of a triangle with vertices $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ is given by $\frac{1}{2}(y_A + y_B)(x_A - x_B) + \frac{1}{2}(y_B + y_C)(x_B - x_C) + \frac{1}{2}(y_C + y_A)(x_C - x_A) = \frac{1}{2}(x_B y_C - x_C y_B + x_C y_A - x_A y_C + x_A y_B - x_B y_A)$ which can be expressed as a determinant

$$\frac{1}{2}\begin{matrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{matrix}.$$

This is only the algebraic area. If the ordering of the vertices of the triangle ABC is changed to ACB, then the value of this area changes by a sign. Thus (ABC) is the absolute value of this determinant.

The determinant can also be expressed as

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_B - x_A & y_B - y_A & 0 \\ x_C \quad x_A & y_C \quad y_A & 0 \end{vmatrix},$$

which is just $\frac{1}{2}(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{k}$.

5. **Tangent to a circle**. Let C be the circle with equation $x^2 + y^2 + 2fx + 2gy + h = 0$ and $P = (x_0, y_0)$ be a point on C. The equation of the tangent line to the circle C at P is given by

$$x_0x + y_0y + f(x + x_0) + g(y + y_0) + h = 0.$$

Proof. The center of the circle is $(\quad f, \quad g)$. Thus if (x, y) is a point on the tangent line, then $\langle (x \quad x_0, y \quad y_0), (x_0 + f, y_0 + g) \rangle = 0$. Using $x_0^2 + y_0^2 + 2fx_0 + 2gy_0 + h = 0$, the result follows.

6. **Coaxal circles**. The standard equation of a circle is of the form

$$C(x, y) = x^2 + y^2 + 2fx + 2gy + h = 0.$$

The power of a point $P(a, b)$ with respect to a circle $C = 0$ is also given by $C(a, b)$.

The locus of the points having equal power with respect to C_1 and C_2 is called the *radical axis* of C_1 and C_2. For any 2 circles $C_1 = 0$ and $C_2 = 0$, the radical axis is given by

$$C_1 \quad C_2 = 0$$

The collection of all circles of the form $C_3 = \lambda C_1 + \mu C_2$, where $\lambda + \mu = 1$, forms a so-called *pencil of circles*. Any two such circles have the same radical axes, and they are called *coaxal circles*.

Exercise 8.1 Let $P = (a, b)$ be a point outside the unit circle $x^2 + y^2 = 1$ and let PT_1 and PT_2 be the tangents to it. Show that the coordinates of T_1 and T_2 are given by

$$\left(\frac{a \quad b\sqrt{a^2 + b^2 \quad 1}}{a^2 + b^2}, \frac{b + a\sqrt{a^2 + b^2 \quad 1}}{a^2 + b^2} \right), \left(\frac{a + b\sqrt{a^2 + b^2 \quad 1}}{a^2 + b^2}, \frac{b \quad a\sqrt{a^2 + b^2 \quad 1}}{a^2 + b^2} \right).$$

Exercise 8.2 Let C be the circle with equation $x^2 + y^2 + 2ax + 2by + f = 0$ and $P = (x_0, y_0)$ a point outside C. The tangents from P touch C at the points X and Y. Show that the equation of the line XY is given by

$$x_0x + y_0y + a(x + x_0) + b(y + y_0) + f = 0.$$

Exercise 8.3 Show that the equation of the circle passing through the points $(p_1, p_2), (q_1, q_2), (r_1, r_2)$ is given by

$$\begin{array}{ccc} x \quad p_1 & y \quad p_2 & p_1^2 + p_2^2 \quad x^2 \quad y^2 \\ p_1 \quad q_1 & p_2 \quad q_2 & q_1^2 + q_2^2 \quad p_1^2 \quad p_2^2 \\ q_1 \quad r_1 & q_2 \quad r_2 & r_1^2 + r_2^2 \quad q_1^2 \quad q_2^2 \end{array} \Bigg| = 0.$$

[Hint: The form of this determinant shows that it is an equation of a circle. The substitution of the coordinates of each of the three points clearly makes the determinant zero. Consider the 4 points: $(x, y), (p_1, p_2), (q_1, q_2), (r_1, r_2)$ on the circle, the perpendicular bisectors of any three of the chords among these 4 points must concur at the centre of the circle. Thus 4 points are concyclic if and only if the above determinant is zero.]

Chapter 8

Theorem 8.1 *Let* $[MA_2A_3] = \mu_1$, $[MA_3A_1] = \mu_2$, $[MA_1A_2] = \mu_3$ *and* $[A_1A_2A_3] = 1$ *so that* $\mu_1 + \mu_2 + \mu_3 = 1$. *Then*
1. $A_3N_2 : N_2A_1 = \mu_1 : \mu_3$, *etc.*
2. $A_1M : MN_1 = (\mu_2 + \mu_3) : \mu_1$.
3. $\mathbf{A_2M} = \mu_3\mathbf{A_2A_3} + \mu_1\mathbf{A_2A_1}$.

Prove the follwings

1. The barycentric coordinates of a point are homogeneous. That is $(\mu_1 : \mu_2 : \mu_3) = (k\mu_1 : k\mu_2 : k\mu_3)$ for any nonzero real number k. As such, it can also be identified with the homogeneous coordinates of the point.

2. For the points A_1, A_2 and A_3, we have $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0)$ and $A_3 = (0 : 0 : 1)$ respectively.

3. Let the Cartesian coordinates of A, B, C be (x_A, y_A), $(x_B, y_B), (x_C, y_C)$ respectively. If the barycentric coordinates of M is $(\mu_1 : \mu_2 : \mu_3)$, then the Cartesian coordinates of M is $\left(\frac{\mu_1x_A+\mu_2x_B+\mu_3x_C}{\mu_1+\mu_2+\mu_3}, \frac{\mu_1y_A+\mu_2y_B+\mu_3y_C}{\mu_1+\mu_2+\mu_3}\right)$.

4. The centroid of $\triangle A_1A_2A_3$ is the point $G = (1 : 1 : 1)$.

5. The circumcentre of $\triangle A_1A_2A_3$ is the point $O = (\sin 2A_1 : \sin 2A_2 : \sin 2A_3)$.

6. Suppose $A_1M_3/M_3A_2 = m_1$ and $A_2M_1/M_1A_3 = m_2$. Then $M = (1 : m_1 : m_1m_2)$.

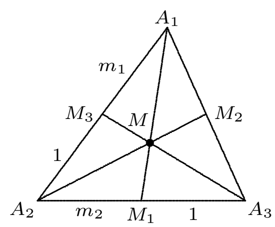

Figure 8.4: The barycentric coordinates of M

7. The incentre of $\triangle A_1A_2A_3$ is the point $(a_1 : a_2 : a_3)$, where a_1, a_2, a_3 are lengths of the sides $\triangle A_1A_2A_3$. This follows from the angle bisector theorem.

8. For the excentres of $\triangle A_1A_2A_3$, we have

$$I_1 = (-a_1 : a_2 : a_3), \quad I_2 = (a_1 : -a_2 : a_3), \quad I_3 = (a_1 : a_2 : -a_3).$$

9. The orthocentre of $\triangle A_1A_2A_3$ is the point

$$H = (\tan A_1 : \tan A_2 : \tan A_3) = (\frac{1}{-a_1^2 + a_2^2 + a_3^2} : \frac{1}{a_1^2 - a_2^2 + a_3^2} : \frac{1}{a_1^2 + a_2^2 - a_3^2}).$$

10. The Gergonne point of $\triangle A_1A_2A_3$ is the point

$$(\frac{1}{s - a_1} : \frac{1}{s - a_2} : \frac{1}{s - a_3}).$$

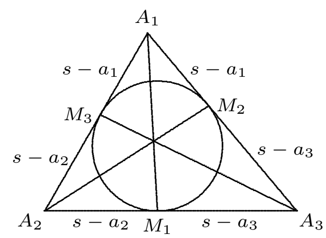

Figure 8.5: Gergonne point

11. The Nagel point of $\triangle A_1A_2A_3$ is the point $N = (s - a_1 : s - a_2 : s - a_3)$.

12. The equation of the line passing through the points $(a_1 : a_2 : a_3)$ and $(b_1 : b_2 : b_3)$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} x_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} x_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} x_3 = 0.$$

This is a linear relation of the homogeneous coordinates of a point. In general the equation of a straight line in homogeneous coordinates is of the form

$$\ell : \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 = 0.$$

Usually, the coefficients are used to denote such a line. In notation, we write

$$\ell = [\mu_1 : \mu_2 : \mu_3].$$

Thus the line passing through $(a_1 : a_2 : a_3)$ and $(b_1 : b_2 : b_3)$ is given by

$$\ell = [a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1].$$

13. Three points $A = (a_1 : a_2 : a_3), B = (b_1 : b_2 : b_3), C = (c_1 : c_2 : c_3)$ are collinear if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

14. The intersection of the lines $\ell_1 = [a_1 : a_2 : a_3]$ and $\ell_2 = [b_1 : b_2 : b_3]$ is given by

$$P = (a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1).$$

15. Three lines $\ell = [a_1 : a_2 : a_3], m = [b_1 : b_2 : b_3], n = [c_1 : c_2 : c_3]$ are concurrent if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Proof. The intersection of ℓ and m is $(a_2b_3 - a_3b_2 : -a_1b_3 + a_3b_2 : a_1b_2 - a_2b_1)$. It lies on n if and only if $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = 0.$

16. Let $P = (p_1 : p_2 : p_3)$ and $Q = (q_1 : q_2 : q_3)$ with $p_1+p_2+p_3 = 1$ and $q_1+q_2+q_3 = 1$. If M divides PQ in the ratio $PM : MQ = \beta : \alpha$, then the point M has homogeneous coordinates $(\alpha p_1 + \beta q_1 : \alpha p_2 + \beta q_2 : \alpha p_3 + \beta q_3)$.

17. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ be three non-collinear points on the plane. Let the homogeneous coordinates of A, B and C be $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ respectively. Show that for any point $P = (\lambda_1 : \lambda_2 : \lambda_3)$, its cartesian coordinates is given by

$$\left(\frac{\lambda_1 a_1 + \lambda_2 b_1 + \lambda_3 c_1}{\lambda_1 + \lambda_2 + \lambda_3}, \frac{\lambda_1 a_2 + \lambda_2 b_2 + \lambda_3 c_2}{\lambda_1 + \lambda_2 + \lambda_3}\right).$$

Exercise 8.4 Prove that in any triangle the 3 lines each of which joins the midpoint of a side to the midpoint of the altitude to that side are concurrent.

[Hint. Take $A_1 = (1 : 0 : 0), A_2 = (0 : 1 : 0), A_3 = (0 : 0 : 1)$. Let F_1, F_2 and F_3 be the midpoints of the altitudes A_1N_1, A_2N_2 and A_3N_3 respectively. If M_1, M_2 and M_3 are the midpoints of the sides A_2A_3, A_3A_1 and A_1A_2 respectively, show that $M_1F_1 = [\tan A_3 \quad \tan A_2 : \tan A_2 + \tan A_3 : \quad \tan A_2 \quad \tan A_3]$, $M_2F_2 = [\quad \tan A_1 \quad \tan A_3 : \tan A_1 \quad \tan A_3 : \tan A_1 + \tan A_3]$, and $M_3F_3 = [\tan A_1 + \tan A_2 : \quad \tan A_1 \quad \tan A_2 : \tan A_2 \quad \tan A_1]$.]

Exercise 8.5 (Euler line) Prove that the circumcentre, the centroid and the orthocentre of a triangle are collinear.

Exercise 8.6 (Newton line) In a quadrilateral $ABCD$, AB intersects CD at E, AD intersects BC at F. Let L, M and N be the midpoints of AC, BD and EF respectively. Prove that L, M, N are collinear.

[Hint. Let $A = (1 : 0 : 0), B = (0 : 1 : 0), C = (0 : 0 : 1)$ and $D = (u : v : w)$ with $u+v+w = 1$. Show that $N = (u \quad u^2 : v + v^2 : w \quad w^2)$.]

Exercise 8.7 Using the result of Example 8.4, deduce the Butterfly theorem 8.4.

Exercise 8.8 In Figure 8.10, the point O is the midpoint of BC. Prove that $OX = OY$.

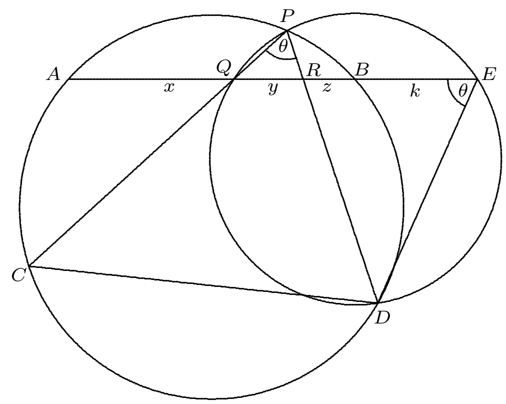

Figure 8.9: Haruki's lemma

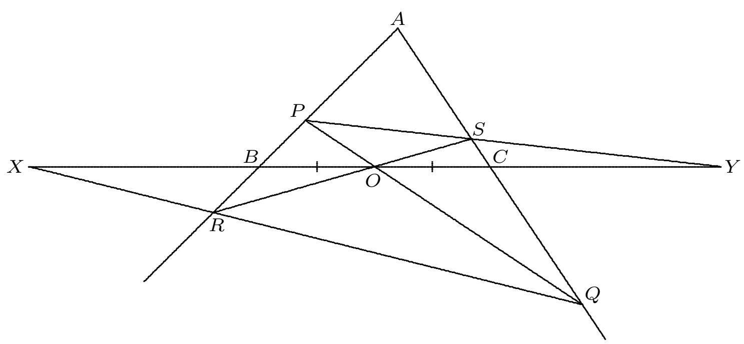

Figure 8.10: Butterfly theorem for 2-straight lines

Exercise 8.9 Let A, B, C, D, E, F be 6 points on the plane such that AB intersects DE at L, BC intersects EF at N and CD intersects FA at M. Prove that if L, N, M are collinear, then there is a conic passing through A, B, C, D, E, F.

[Hint: Use the fact that for any 5 points in general position, there is a conic passing through them. Let α be a conic passing through A, B, C, D, E. Let EN meet α at F' and let the intersection of AF' and CD be M'. By Pascal's theorem which also holds for six points on a conic, there is a conic passing through A, B, C, D, E, F'. Show that $M' = M$ and hence $F' = F$. This is the converse of Pascal's theorem.]

Exercise 8.10 Show that the Butterfly theorem holds for any quadratic curve $ax^2 + bxy + cy^2 + dx + ey + f = 0$.

[Hint: Position the chord PQ of the quadratic curve so that P and Q lie on the x-axis with the origin as their midpoint. Show that in this coordinate system, the coefficient $d = 0$. Then follow the proof of theorem 8.4.]

Exercise 8.11 (**A generalized Butterfly theorem**) Let AB be a chord of a circle with midpoint P, and let the chords XW and ZY intersect AB at M and N respectively. Let AB intersects XY at C and ZW at D. Prove that if $MP = PN$, then $CP = PD$.

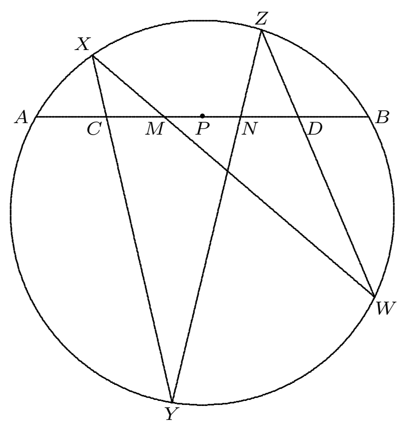

Figure 8.12: A generalized Butterfly theorem

[Hint: Use Haruki's lemma.]