VECTOR CALCULUS
Gradient of a Scalar
Field, Divergence of a Vector Field, Divergence Theorem, Gausses law, concept of flux

Lecture 2

1

Derivative

- Given a function f(x), what is it's derivative?

$$df = \frac{\partial f}{\partial x} dx$$

  - The derivative $\frac{\partial f}{\partial x}$ tells us how fast f varies when x varies.

→ The derivative is the proportionality factor between a change in x and a change in f.

- What if f=f(x,y,z)?

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

2

Gradient of a Scalar Field

$$\text{grad}\, f \equiv \nabla f \equiv \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Conclusions:

- ∇f measures how fast f(x,y,z) varies when x, y and z vary
- Logical extension of the concept of derivative!
- f is a scalar function but ∇f is a vector!

3

The "del" operator

Definition:

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Properties:

- It looks like a vector
- It works like a vector
- But it's not a real vector because it's meaningless by itself. It's an operator.

How it works:

It can act on both scalar and vector functions:

- Acting on a scalar function: gradient $\vec{\nabla} f$ (vector)
- Acting on a vector function with dot product: divergence $\vec{\nabla} \bullet \vec{f}$ (scalar)
- Acting on a scalar function with cross product: curl $\vec{\nabla} \times \vec{f}$ (vector)

4

Divergence

Given a vector function $\vec{v}(x,y,z)$

$$\vec{v}(x,y,z) \equiv v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \equiv (v_x, v_y, v_z)$$

we define its divergence as:

$$\text{div}\,\vec{v} \equiv \vec{\nabla}\bullet\vec{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Observations:

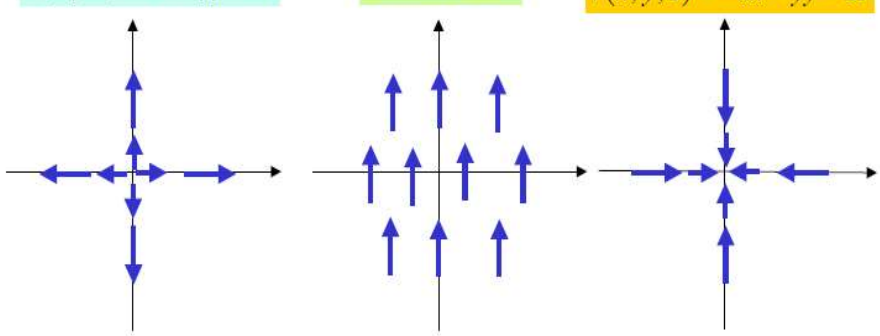
- The divergence is a scalar
- Geometrical interpretation: it measures how much the function $\vec{v}(x,y,z)$ "spreads around a point".

5

Divergence: interpretation

Calculate the divergence for the following functions:

$\vec{v}(x,y,z) = x\hat{x} + y\hat{y} + z\hat{z}$ | $\vec{v}(x,y,z) = \hat{z}$ | $\vec{v}(x,y,z) = -x\hat{x} - y\hat{y} - z\hat{z}$

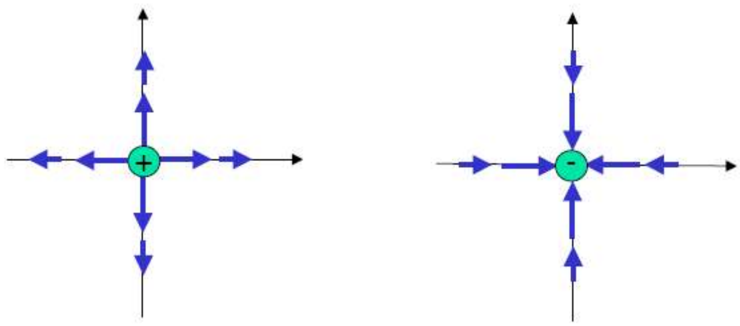


div v=3>0 (faucet) | div v=0 | div v = -3 (sink)

6

Divergence: (Cont'd)

Electric field around a charge has divergence .ne. 0 !



div E>0 for + charge: faucet

div E <0 for – charge: sink

7

Curl

Given a vector function $\vec{v}(x,y,z)$

$$\vec{v}(x,y,z) \equiv v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \equiv (v_x, v_y, v_z)$$

we define its curl as:

$$\vec{\nabla} \times \vec{v} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

Observations:

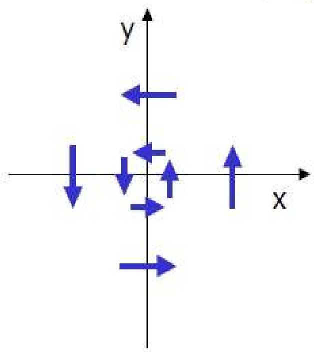
- The curl is a vector
- Geometrical interpretation: it measures how much the function $\vec{v}(x,y,z)$ "curls around a point".

8

Curl (Cont'd)

Calculate the curl for the following function:

$$\vec{v}(x,y,z) = -y\hat{x} + x\hat{y}$$

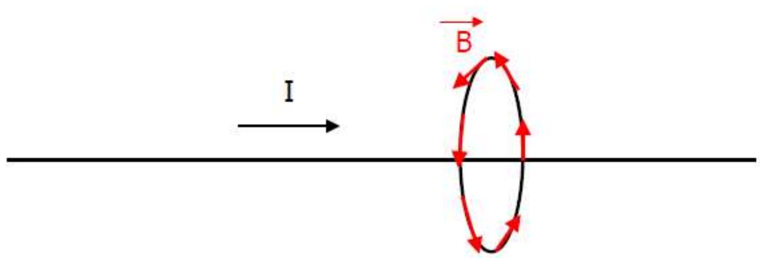


$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{k}$$

9

Curl (Cont'd)

Magnetic filed around a wire :



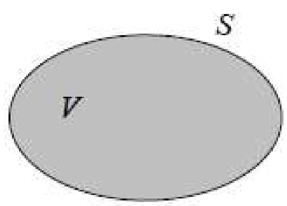
$$\vec{\nabla} \times \vec{B} \neq 0$$

10

Divergence (Gauss) Theorem

- The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_V \nabla \bullet \overline{A}\, dV = \oint_S \overline{A} \bullet d\overline{S}$$

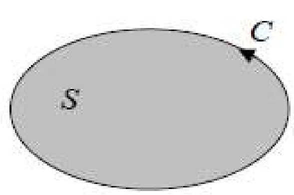


Proof follows from the definition of divergence.

Stokes's Theorem

- The surface integral of the curl of a vector field over an open surface is equal to the net circulation of the vector along the contour bounding the surface.

$$\int_S \left(\nabla \times \overline{A}\right) \bullet dS = \oint_C \overline{A} \bullet d\overline{l}$$



Proof follows from the definition of curl.

Null Identity: Curl of the Gradient

- Theorem:
  - The curl of the gradient of any scalar field is everywhere equal to zero.

$$\nabla \times (\nabla \Phi) \equiv 0$$

- Corollary:
  - If a vector field is conservative (or irrotational), then it can be written as the gradient of a scalar field.

$$\nabla \times \overline{A} \equiv 0 \quad \Rightarrow \quad \overline{A} = \nabla \Phi$$

13

Null Identity: Divergence of the Curl

- Theorem:
  - The divergence of the curl of any vector field is everywhere equal to zero.

$$\nabla \bullet (\nabla \times \overline{A}) \equiv 0$$

- Corollary:
  - If a vector field is solenoidal, then it can be written as the curl of another vector field.

$$\nabla \bullet \overline{B} \equiv 0 \quad \Rightarrow \quad \overline{B} = \nabla \times \overline{A}$$

14

Electricity and Magnetism
HISTORY…

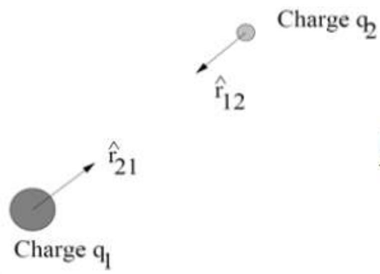
15

The electric charge

- The EM force acts on charges
  - 2 flavors: positive and negative
    - Positive: obtained rubbing glass with silk
    - Negative: obtained rubbing resin with fur
- Electric charge is quantized (Millikan)
  - Multiples of the **e** = elementary charge
    - **e** = 1.602 10^{-19} C (SI), 4.803 10^{-10} esu (cgs)
    - $Q_{electron}$= **-e**; Q_{proton}=**+e**
- Electric charge is conserved
  - In any isolated system, the total charge cannot change
    - If the total charge of a system changes, then it means the system is not isolated and charges came in or escaped.

16

Coulomb's law

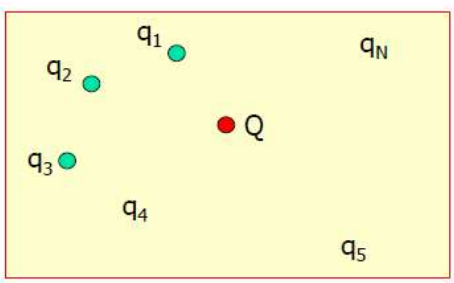


$$\vec{F}_2 = k \frac{q_1 q_2}{|r_{21}|^2} \hat{r}_{21}$$

- Where:
  - $\vec{F}_2$ is the force that the charge q_2 feels due to q_1
  - $\hat{r}_{21}$ is the unit vector going from q_1 to q_2
- Consequences:
  - Newton's third law: $\vec{F}_2 = -\vec{F}_1$
  - Like signs repel, opposite signs attract

17

The superposition principle: discrete charges



The force on the charge Q due to all the other charges is equal to the vector sum of the forces created by the individual charges:

$$\vec{F}_Q = \frac{q_1 Q}{|r_1|^2} \hat{r}_1 + \frac{q_2 Q}{|r_2|^2} \hat{r}_2 + \ldots + \frac{q_N Q}{|r_N|^2} \hat{r}_N = \sum_{i=1}^{i=N} \frac{q_i Q}{|r_i|^2} \hat{r}_i$$

18

continuous distribution of charges

- Charges are distributed inside a volume V:

$$\vec{F}_Q = \int_V \frac{\rho \; dV \; Q}{|r|^2} \; \hat{r}$$

- Charges are distributed on a surface A:

$$\vec{F}_Q = \int_A \frac{\sigma \; da \; Q}{|r|^2} \; \hat{r}$$

- Charges are distributed on a line L:

$$\vec{F}_Q = \int_L \frac{\lambda \; dl \; Q}{|r|^2} \; \hat{r}$$

Where:

- ρ = charge per unit volume: "volume charge density"
- σ = charge per unit area: "surface charge density"
- λ = charge per unit length: "line charge density"

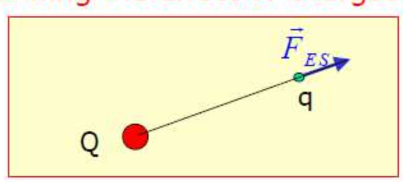
19

The electric field

what is the best way of describing the effect of charges?

- 1 charge in the Universe
- 2 charges in the Universe

$$\vec{F}_q = \frac{qQ}{|r|^2}\hat{r}$$



But: the force F depends on the test charge q... ☹

→ define a quantity that describes the effect of the charge Q on the surroundings: Electric Field

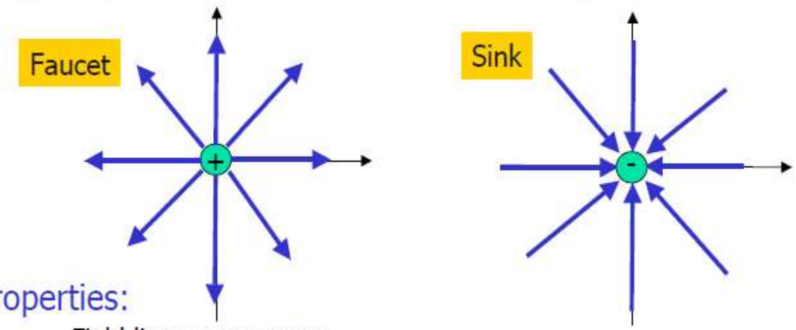
$$\vec{E} = \frac{\vec{F}_q}{q} = \frac{Q}{|r|^2}\hat{r}$$

20

Electric field lines

Visualize the direction and strength of the Electric Field:

- Direction: // to E, pointing towards – and away from +
- Magnitude: the denser the lines, the stronger the field.

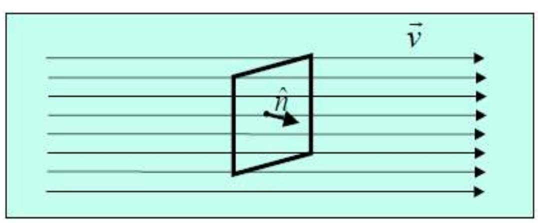


Properties:

- Field lines never cross
- They are orthogonal to equipotential surfaces (will see this later).

21

The concept of flux



- Consider the flow of water in a river
- The water velocity is described by

$$\vec{v}(x, y, z) \equiv v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \equiv (v_x, v_y, v_z)$$

- Immerse a squared wire loop of area A in the water (surface S)
- Define the loop area vector as $\vec{A} \equiv A\hat{n}$

Q: how much water will flow through the loop? E.g.:

What is the "flux of the velocity" through the surface S?

22

What is the flux of the velocity?

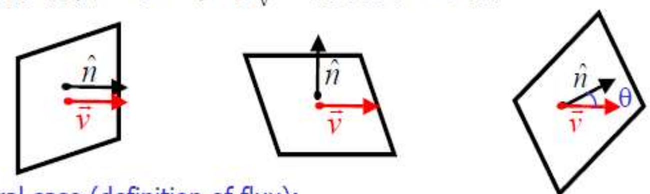
It depends on how the loop is oriented w.r.t. the water…

- Assuming constant velocity and plane loop:

  1) if $\vec{A} \perp \vec{v} \quad \rightarrow \quad \Phi_v = 0;$

  2) if $\vec{A} \parallel \vec{v} \quad \rightarrow \quad \Phi_v = vA;$

  3) if $\vec{A} \measuredangle \vec{v} = \theta \quad \rightarrow \quad \Phi_v = vA\cos\theta = \vec{v} \bullet \vec{A}.$



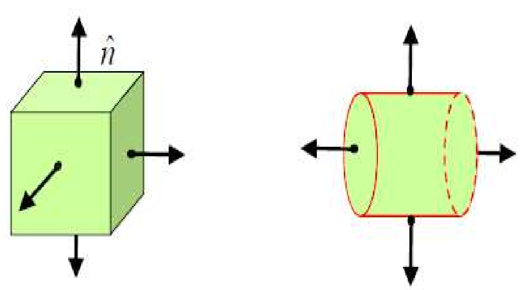
- General case (definition of flux):

$$\Phi_{\vec{v}} = \int_S \vec{v} \bullet d\vec{A}$$

23

What is the Direction of dA

- Defined unambiguously only for a 3d surface:
  - At any point in space, dA is perpendicular to the surface
  - It points towards the "outside" of the surface
- Examples:



24

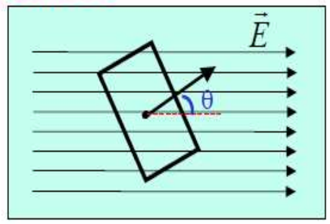
Flux of Electric Field

Definition: $\Phi_{\vec{E}} \equiv \Phi = \int_S \vec{E} \bullet d\vec{A}$

Example: uniform electric field + flat surface

- Calculate the flux:

$$\Phi = \int_S \vec{E} \bullet d\vec{A} = \vec{E} \bullet \vec{A} = EA\cos\theta$$



- Interpretation:

Represent E using field lines:

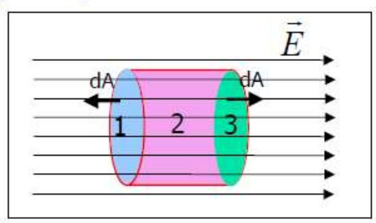
Φ_E is proportional to $N_{\text{field lines}}$ that go through the loop

25

Φ_E through closed (3d) surface

- Consider the total flux of E through a cylinder:

$$\Phi_{tot} = \Phi_1 + \Phi_2 + \Phi_3$$



- Calculate Φ_1, Φ_2, Φ_3
  - Cylinder axis is // to field lines
  - Φ_2=0 because $\vec{E} \perp \hat{n}$
  - $|\Phi_1|=|\Phi_3|$ but opposite sign since

$$\Phi = \int_S \vec{E} \bullet d\vec{A} = EA\cos\theta$$

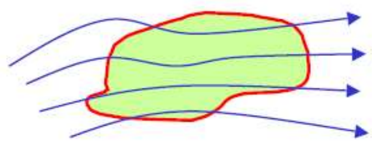
→ The total flux through the cylinder is zero!

26

Φ_E through closed empty surface

Q1: Is this a coincidence due to shape/orientation of the cylinder?

- Clue:
  - Think about interpretation of Φ_E: proportional # of field lines through the surface…
- Answer:
  - No: all field lines that get into the surface have to come out!



Conclusion:

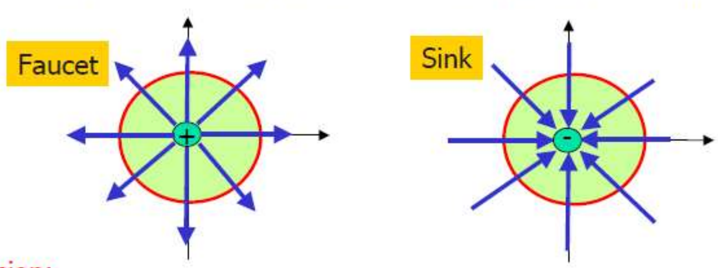
The electric flux through a closed surface that does not contain charges is zero.

27

Φ_E through surface containing Q

Q1: What if the surface contains charges?

- Clue:
  - Think about interpretation of Φ_E: the lines will either originate in the surface (positive flux) or terminate inside the surface (negative flux)



Conclusion:

The electric flux through a closed surface that does contain a net charge is non zero.

28

Thoughts on Gauss's law

$$\Phi = \oint_S \vec{E} \bullet d\vec{A} = 4\pi Q_{encl} \quad \text{(Gauss's law in integral form)}$$

- Why is Gauss's law so important?
  - Because it relates the electric field E with its sources Q
    - Given Q distribution → find E (integral form)
    - Given E → find Q (differential form, next week)
- Is Gauss's law always true?
  - Yes, no matter what E or what S, the flux is always = $4\pi Q$
- Is Gauss's law always useful?
  - No, it's useful only when the problem has symmetries

29