The Steady Magnetic Field

Lec 5

Objective

To underestand:

- Biot-Savart law
- Ampere's law of force
- magnetic flux density
- applications of Ampere's law

Magnetostatics

- ***Magnetostatics*** is the branch of electromagnetics dealing with the effects of electric charges in steady motion (i.e, steady current or DC).
- The fundamental law of *magnetostatics* is ***Ampere's law of force***.
- *Ampere's law of force* is analogous to *Coulomb's law* in electrostatics.
- In magnetostatics, the magnetic field is produced by steady currents.

3

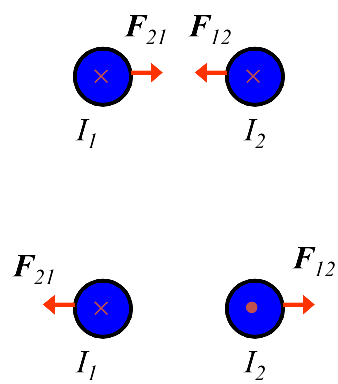
Ampere's Law of Force

- *Ampere's law of force* is the "law of action" between current carrying circuits.
- *Ampere's law of force* gives the magnetic force between two *current carrying circuits* in an otherwise empty universe.
- Ampere's law of force involves complete circuits since current must flow in closed loops.

4

Ampere's Law of Force (Cont'd)

- Experimental facts:
  - Two parallel wires carrying current in the same direction attract.
  - Two parallel wires carrying current in the opposite directions repel.



5

Ampere's Law of Force (Cont'd)

- Experimental facts:
  - The magnitude of the force is inversely proportional to the distance squared.
  - The magnitude of the force is proportional to the product of the currents carried by the two wires.

6

Ampere's Law of Force (Cont'd)

- The direction of the force established by the experimental facts can be mathematically represented by

unit vector in direction of current I_2

unit vector in direction of current I_1

$$\bar{a}_{F_{12}} = \bar{a}_2 \times \left(\bar{a}_1 \times \bar{a}_{R_{12}}\right)$$

unit vector in direction of force on I_2 due to I_1

unit vector in direction of I_2 from I_1

7

Ampere's Law of Force (Cont'd)

- The force acting on a current element $I_2\,d\boldsymbol{l}_2$ by a current element $I_1\,d\boldsymbol{l}_1$ is given by

$$\bar{F}_{12} = \frac{\mu_0}{4\pi}\,\frac{I_2 d\bar{l}_2 \times \left(I_1 d\bar{l}_1 \times \bar{a}_{R_{12}}\right)}{R_{12}^2}$$

Permeability of free space
$\mu_0 = 4\pi \times 10^{-7}$ H/m

8

Ampere's Law of Force (Cont'd)

- The total force acting on a circuit C_2 having a current I_2 by a circuit C_1 having current I_1 is given by

$$\boxed{\overline{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\overline{l}_2 \times \left(d\overline{l}_1 \times \overline{a}_{R_{12}} \right)}{R_{12}^2}}$$

9

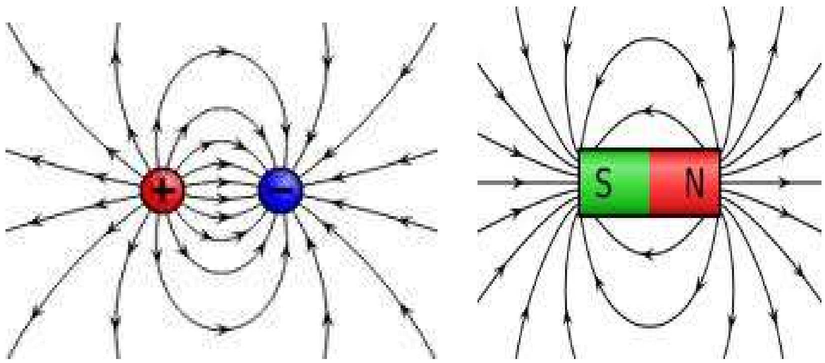
Ampere's Law of Force (Cont'd)

- The force on C_1 due to C_2 is equal in magnitude but opposite in direction to the force on C_2 due to C_1.

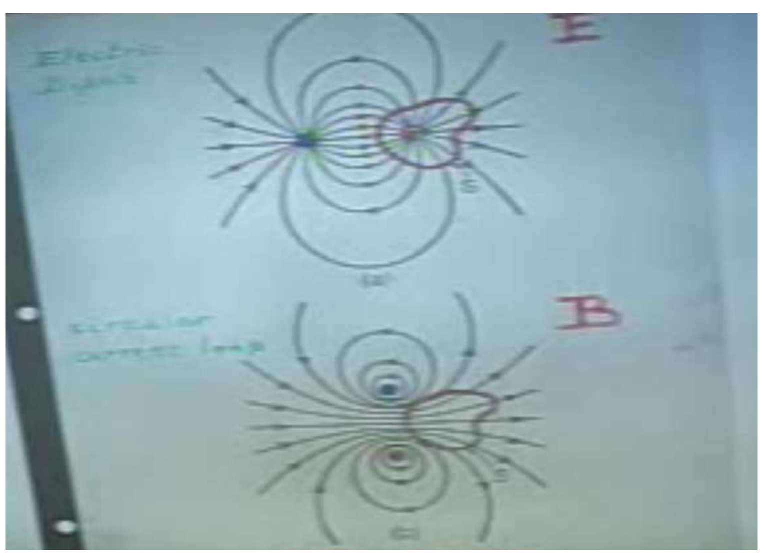
$$\overline{F}_{21} = -\overline{F}_{12}$$

10

E&B



E & B



Magnetic Flux Density

- Ampere's force law describes an "action at a distance" analogous to Coulomb's law.
- In Coulomb's law, it was useful to introduce the concept of an *electric field* to describe the interaction between the charges.
- In Ampere's law, we can define an appropriate field that may be regarded as the means by which currents exert force on each other.

13

Magnetic Flux Density (Cont'd)

- The ***magnetic flux density*** can be introduced by writing

$$\overline{F}_{12} = \oint_{C_2} I_2 d\overline{l}_2 \times \frac{\mu_0}{4\pi} \oint_{C_1} \frac{\left(I_1 d\overline{l}_1 \times \overline{a}_{R_{12}}\right)}{R_{12}^2}$$

$$= \oint_{C_2} I_2 d\overline{l}_2 \times \overline{B}_{12}$$

14

Magnetic Flux Density (Cont'd)

- where

$$\overline{B}_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 d\overline{l}_1 \times \overline{a}_{R_{12}}}{R_{12}^2}$$

the magnetic flux density at the location of $d\mathbf{l}_2$ due to the current I_1 in C_1

15

Magnetic Flux Density (Cont'd)

- Suppose that an infinitesimal current element $I d\mathbf{l}$ is immersed in a region of magnetic flux density $\mathbf{B}$. The current element experiences a force $d\mathbf{F}$ given by

$$d\overline{F} = I d\overline{l} \times \overline{B}$$

16

Magnetic Flux Density (Cont'd)

- The total force exerted on a circuit C carrying current I that is immersed in a magnetic flux density $\boldsymbol{B}$ is given by

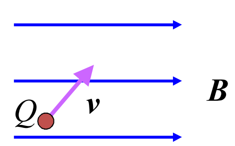
$$\boxed{\overline{F} = I\oint_C d\overline{l} \times \overline{B}}$$

17

Force on a Moving Charge

- A moving point charge placed in a magnetic field experiences a force given by

$$\overline{F}_m = Q\overline{v} \times \overline{B} \qquad I d\overline{l} \Leftrightarrow Q\overline{v}$$



The force experienced by the point charge is in the direction into the paper.

18

Lorentz Force

- If a point charge is moving in a region where both electric and magnetic fields exist, then it experiences a total force given by

$$\boxed{\overline{F} = \overline{F}_e + \overline{F}_m = q\left(\overline{E} + \overline{v} \times \overline{B}\right)}$$

- The Lorentz force equation is useful for determining the equation of motion for electrons in electromagnetic deflection systems such as CRTs.

19

The Biot-Savart Law

- The ***Biot-Savart law*** gives us the B-field arising at a specified point P from a given current distribution.
- It is a fundamental law of magnetostatics.

20

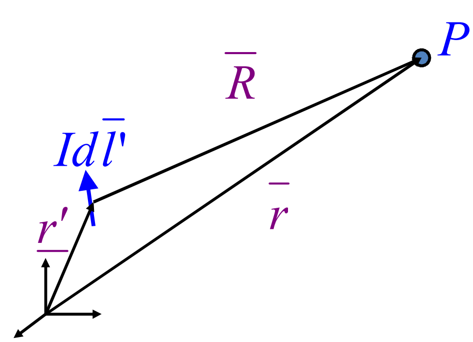
The Biot-Savart Law (Cont'd)

- The contribution to the B-field at a point P from a differential current element $Id\mathbf{l}'$ is given by

$$d\overline{B}(\overline{r}) = \frac{\mu_0 I}{4\pi}\frac{d\overline{l}'\times\overline{R}}{R^3}$$

21

The Biot-Savart Law (Cont'd)



22

The Biot-Savart Law (Cont'd)

- The total magnetic flux at the point P due to the entire circuit C is given by

$$\boxed{\overline{B}(\overline{r}) = \oint_C \frac{\mu_0 I}{4\pi} \frac{d\overline{l}' \times \overline{R}}{R^3}}$$

23

Types of Current Distributions

- ***Line current density*** (current) - occurs for infinitesimally thin filamentary bodies (i.e., wires of negligible diameter).
- ***Surface current density*** (current per unit width) - occurs when body is perfectly conducting.
- ***Volume current density*** (current per unit cross sectional area) - most general.

24

The Biot-Savart Law (Cont'd)

- For a surface distribution of current, the B-S law becomes

$$\boxed{\overline{B}(\overline{r}) = \int_{S'} \frac{\mu_0 \overline{J}_s\left(\overline{r}'\right) \times \overline{R}}{4\pi R^3} ds'}$$

- For a volume distribution of current, the B-S law becomes

$$\boxed{\overline{B}(\overline{r}) = \int_{V'} \frac{\mu_0 \overline{J}\left(\overline{r}'\right) \times \overline{R}}{4\pi R^3} dv'}$$

25

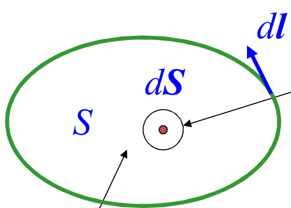
Ampere's Circuital Law in Integral Form

- ***Ampere's Circuital Law*** in integral form states that "the circulation of the magnetic flux density in free space is proportional to the total current through the surface bounding the path over which the circulation is computed."

$$\boxed{\oint_C \overline{B} \cdot d\overline{l} = \mu_0 I_{encl}}$$

26

Ampere's Circuital Law in Integral Form (Cont'd)



By convention, $d\mathbf{S}$ is taken to be in the direction defined by the right-hand rule applied to $d\mathbf{l}$.

$$I_{encl} = \int_S \overline{J} \cdot d\overline{s}$$

Since volume current density is the most general, we can write I_{encl} in this way.

27

Ampere's Law and Gauss's Law

- Just as Gauss's law follows from Coulomb's law, so Ampere's circuital law follows from Ampere's force law.
- Just as Gauss's law can be used to derive the electrostatic field from symmetric charge distributions, so Ampere's law can be used to derive the magnetostatic field from symmetric current distributions.

28

Applications of Ampere's Law

- Ampere's law in integral form is an *integral equation* for the unknown magnetic flux density resulting from a given current distribution.

$$\oint_C \underbrace{\overline{B}}_{\text{unknown}} \cdot d\overline{l} = \mu_0 \underbrace{I_{encl}}_{\text{known}}$$

Applications of Ampere's Law (Cont'd)

- In general, solutions to *integral equations* must be obtained using numerical techniques.
- However, for certain symmetric current distributions closed form solutions to Ampere's law can be obtained.

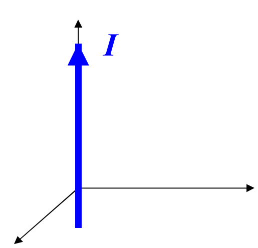
Applications of Ampere's Law (Cont'd)

- Closed form solution to Ampere's law relies on our ability to construct a suitable family of ***Amperian paths***.
- An *Amperian path* is a closed contour to which the magnetic flux density is tangential and over which equal to a constant value.

31

Magnetic Flux Density of an Infinite Line Current Using Ampere's Law

Consider an infinite line current along the z-axis carrying current in the +z-direction:



32

Magnetic Flux Density of an Infinite Line Current Using Ampere's Law (Cont'd)

(1) Assume from symmetry and the right-hand rule the form of the field

$$\overline{B} = \overline{a}_\phi \, B_\phi(\rho)$$

(2) Construct a family of Amperian paths

circles of radius ρ where

$$0 \le \rho \le \infty$$

33

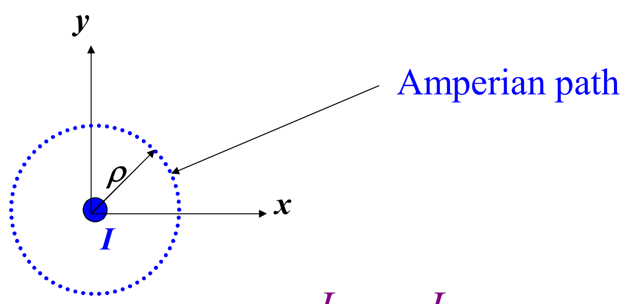
Magnetic Flux Density of an Infinite Line Current Using Ampere's Law (Cont'd)

(3) Evaluate the total current passing through the surface bounded by the Amperian path

$$I_{encl} = \oint_S \overline{J} \cdot d\overline{s}$$

34

Magnetic Flux Density of an Infinite Line Current Using Ampere's Law (Cont'd)



$$I_{encl} = I$$

Magnetic Flux Density of an Infinite Line Current Using Ampere's Law (Cont'd)

(4) For each Amperian path, evaluate the integral

$$\oint_C \overline{B} \cdot d\overline{l} = Bl$$

length of Amperian path.

magnitude of <u>B</u> on Amperian path.

$$\oint_C \overline{B} \cdot d\overline{l} = B_\phi(\rho) 2\pi \rho$$

Magnetic Flux Density of an Infinite Line Current Using Ampere's Law (Cont'd)

(5) Solve for ***B*** on each Amperian path

$$B = \frac{\mu_0 I_{encl}}{l}$$

$$\boxed{\overline{B} = \overline{a}_\phi \frac{\mu_0 I}{2\pi\rho}}$$

37

Applying Stokes's Theorem to Ampere's Law

$$\oint_C \overline{B} \cdot d\overline{l} = \int_S \nabla \times \overline{B} \cdot d\overline{s}$$

$$= \mu_0 I_{encl} = \mu_0 \int_S \overline{J} \cdot d\overline{s}$$

$\Rightarrow$ Because the above must hold for <u>any</u> surface *S*, we must have

$$\boxed{\nabla \times \overline{B} = \mu_0 \overline{J}}$$

Differential form of Ampere's Law

38

Ampere's Law in Differential Form

- Ampere's law in differential form implies that the B-field is *conservative* outside of regions where current is flowing.

Fundamental Postulates of Magnetostatics

- Ampere's law in differential form

$$\nabla \times \overline{B} = \mu_0 \overline{J}$$

- No isolated magnetic charges

$$\nabla \cdot \overline{B} = 0$$

B is solenoidal

Divergence of B-Field

- The B-field is ***solenoidal***, i.e. the divergence of the B-field is identically equal to zero:

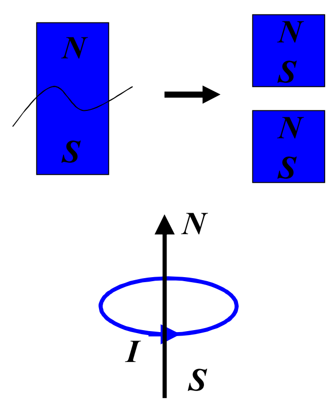
$$\nabla \cdot \overline{B} \equiv 0$$

- Physically, this means that magnetic charges (monopoles) do not exist.
- A magnetic charge can be viewed as an isolated magnetic pole.

41

Divergence of B-Field (Cont'd)

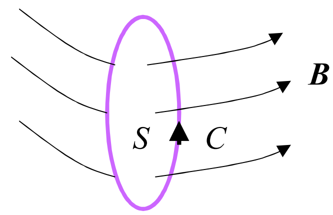
- No matter how small the magnetic is divided, it always has a north pole and a south pole.
- The elementary source of magnetic field is a magnetic dipole.



42

Magnetic Flux

- The magnetic flux crossing an open surface S is given by



$$\underset{\text{Wb}}{\Psi} = \int_S \underset{\text{Wb/m}^2}{\overline{B}} \cdot d\overline{s}$$

Magnetic Flux (Cont'd)

- From the divergence theorem, we have

$$\nabla \cdot \overline{B} = 0 \Rightarrow \int_V \nabla \cdot \overline{B}\, dv = 0 \Rightarrow \oint_S \overline{B} \cdot d\overline{s} = 0$$

- Hence, *the net magnetic flux leaving any closed surface is zero*. This is another manifestation of the fact that there are no magnetic charges.