Maxwell's Equations

Lecture 6

Introduction

stationary charges → electrostatic fields
steady currents → magnetostatic fields
time-varying currents → electromagnetic fields (or waves)

Faraday’s Law

- Faraday discovered that the induced emf. Vemf(in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

$$V_{\text{emf}} = -N\frac{d\Psi}{dt}$$

- where *N is the number of turns in the circuit and V is the flux through each turn. The negative* sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as *Lenz's law.*

Faraday’s Law (Con,d)

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

By applying Stokes's theorem to the middle term in eq.

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

It shows that the time varying E field is not conservative $(\nabla \times \mathbf{E} \neq 0)$

DISPLACEMENT CURRENT

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J}$$

But the divergence of the curl of any vector field is identically zero
Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}$$

The continuity of current however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

DISPLACEMENT CURRENT

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d$$

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

DISPLACEMENT CURRENT

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial\mathbf{D}/\partial t$ is known as *displacement current density* and $\mathbf{J}$ is the conduction current density ($\mathbf{J} = \sigma\mathbf{E}$)

Maxwell's Equations

Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\dfrac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampere's circuit law

*This is also referred to as Gauss's law for magnetic fields.

ELECTROMAGNETIC WAVE PROPAGATION

- In general, **waves are means of transporting energy or information.**
- All forms of EM energy share three fundamental characteristics:
- they all travel at high velocity;
- in traveling, they assume the properties of waves;
- and they radiate outward from a source

Solution of Maxwell's Equations

- goal is to solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_0, \mu = \mu_r\mu_0$, or $\sigma \gg \omega\varepsilon$)

where w is the angular frequency of the wave

Solution of Maxwell's Equations

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$). Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's equations become

$$\nabla \cdot \mathbf{E}_s = 0$$

$$\nabla \cdot \mathbf{H}_s = 0$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s$$

Taking the curl of both sides of $\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s$

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu\, \nabla \times \mathbf{H}_s$$

Solution of Maxwell's Equations

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla(\cancelto{0}{\nabla \cdot \mathbf{E}_s}) - \nabla^2\mathbf{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}_s$$

or

$$\boxed{\nabla^2\mathbf{E}_s - \gamma^2\mathbf{E}_s = 0} \quad \textit{wave equation.}$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

By a similar pdure $\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0$ *wave equation.*

γ is called the *propagation constant* (in per meter) of the medium.

$$\gamma = \alpha + j\beta$$

$$-\text{Re}\, \gamma^2 = \beta^2 - \alpha^2 = \omega^2 \mu \varepsilon$$

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu \sqrt{\sigma^2 + \omega^2 \varepsilon^2}$$

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon} \right]^2} - 1 \right]}$$ α is known as the *attenuation constant*

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon} \right]^2} + 1 \right]}$$

β is phase shift per length and is called the *phase constant* or *wave number.*

assume that the wave propagates along $+\mathbf{a}_z$ and that $\mathbf{E}_s$ has only an x-component, then

$$\mathbf{E}_s = E_{xs}(z)\mathbf{a}_x$$

$$(\nabla^2 - \gamma^2)E_{xs}(z)$$

$$\underbrace{\cancel{\frac{\partial^2 E_{xs}(z)}{\partial x^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 E_{xs}(z)}{\partial y^2}}}_{0} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

$$\left[\frac{d^2}{dz^2} - \gamma^2\right]E_{xs}(z) = 0$$

$$E_{xs}(z) = E_o e^{-\gamma z} + E'_o e^{\gamma z}$$

where E_0 and E'_0 are constants. The fact that the field must be finite at infinity requires that

$E'_o = 0$

$$\mathbf{E}(z, t) = \text{Re}\,[E_{xs}(z)e^{j\omega t}\mathbf{a}_x] = \text{Re}\,(E_o e^{-\alpha z} e^{j(\omega t - \beta z)}\mathbf{a}_x)$$

$$\mathbf{E}(z, t) = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x$$

$$\mathbf{H}(z, t) = \text{Re}\,(H_o e^{-\alpha z} e^{j(\omega t - \beta z)}\,\mathbf{a}_y)$$

$$H_o = \frac{E_o}{\eta}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta = |\eta|e^{j\theta_\eta}$$

η is a complex quantity known as the *intrinsic impedance* (in ohms) of the medium.

the wave velocity u and wavelength λ are, respectively, given by

$$u = \frac{\omega}{\beta}, \qquad \lambda = \frac{2\pi}{\beta}$$

the ratio of the magnitude of the conduction current density $\mathbf{J}$ to that of the displacement current density $\mathbf{J}_d$ in a lossy medium

$$\frac{|\mathbf{J}_s|}{|\mathbf{J}_{ds}|} = \frac{|\sigma \mathbf{E}_s|}{|j\omega\varepsilon \mathbf{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta \qquad \tan\theta = \frac{\sigma}{\omega\varepsilon}$$

θ is known as the *loss tangent* and θ is the *loss angle* of the medium

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s = j\omega\varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]\mathbf{E}_s$$
$$= j\omega\varepsilon_c \mathbf{E}_s$$

$$\varepsilon_c = \varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]$$

$$\varepsilon_c = \varepsilon' - j\varepsilon''$$

and $\varepsilon' = \varepsilon$, $\varepsilon'' = \sigma/\omega$; ε_c is called the *complex permittivity* of the medium. We observe that the ratio of ε'' to ε' is the loss tangent of the medium; that is,

$$\tan\theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}$$

PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\varepsilon$

$$\sigma \simeq 0, \qquad \varepsilon = \varepsilon_o \varepsilon_r, \qquad \mu = \mu_o \mu_r$$

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0^\circ$$

and thus **E** and **H** are in time phase with each other.

PLANE WAVES IN FREE SPACE

$$\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o$$

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c}$$

$$u = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta}$$

$$\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\ \Omega$$

$$\mathbf{E} = E_o \cos(\omega t - \beta z)\, \mathbf{a}_x$$

$$\mathbf{H} = H_o \cos(\omega t - \beta z)\, \mathbf{a}_y = \frac{E_o}{\eta_o} \cos(\omega t - \beta z)\, \mathbf{a}_y$$

PLANE WAVES IN GOOD CONDUCTORS

$\sigma \gg \omega\varepsilon$ so that $\sigma/\omega\varepsilon \to \infty$

$$\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ$$

$$\mathbf{E} = E_o e^{-\alpha z}\cos(\omega t - \beta z)\,\mathbf{a}_x$$

$$\mathbf{H} = \frac{E_o}{\sqrt{\frac{\omega\mu}{\sigma}}}\, e^{-\alpha z}\cos(\omega t - \beta z - 45°)\,\mathbf{a}_y$$

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

$$\delta = \frac{1}{\alpha}$$

The distance δ through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called *skin depth* or *penetration depth* of the medium

- The **skin depth is a measure of the depth to which an EM wave can penetrate the** medium.

- The net electric flux through any closed (Gaussian) surface is proportional to the net charge inside the surface.
- The net Magnetic flux through any closed (Gaussian) surface is equal to zero, means there is no Magnetism Monopole.

- The line integral of the electric field vector around any closed path equals the rate of change in the magnetic flux through any surface bounded by that path.
- The circulation of the magnetic field vector around any amperian loop is proportional to the sum of the total conduction and displacement current through any surface bounded by the path.