Some special distributions:

In this section two additional distribution, known as T or t and F, are defined

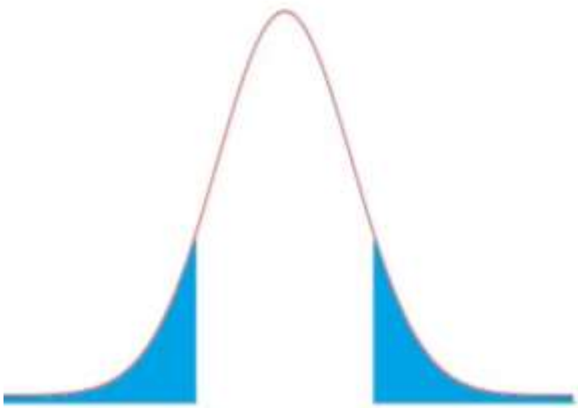
1. **T-distribution**:

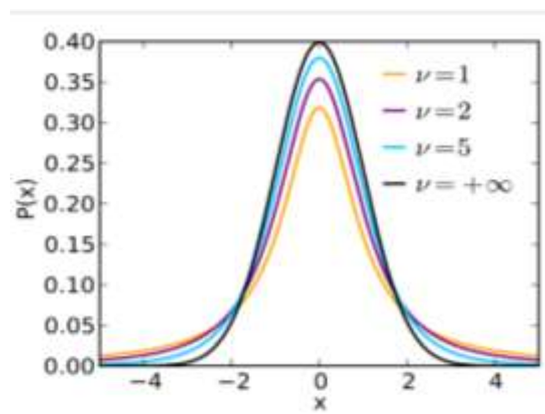
What Is a T Distribution?

The T distribution, also known as the Student's t-distribution, is a type of probability distribution that is similar to the normal distribution with its bell shape but has heavier tails. T distributions have a greater chance for extreme values than normal distributions, hence the fatter.

What Does a T Distribution Tell You?

Tail heaviness is determined by a parameter of the T distribution called degrees of freedom, with smaller values giving heavier tails, and with higher values making the T distribution look alike a standard normal distribution with a mean of 0, and a standard deviation of 1. The T distribution is also known as "Student's T Distribution."





The t Distributions: (Hogg – p.143)

Let W denote a random variable that is $N(0,1)$; let V denote a random variable that is $\chi^2_{(r)}$; and let W and V be stochastically independent, Then the joint p.d.f. of W and V, say $\varphi(w,V)$, is the product of the p.d.f. of W and that of V or

$$\varphi(w,v) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma(r/2)2^{r/2}} v^{r/2-1} e^{-v/2}, \quad -\infty < w < \infty, 0 < v < \infty,$$
$$= 0 \text{ elsewhere.}$$

Define a new random variable T, by writing $$T = \frac{W}{\sqrt{V/r}}.$$

The change-of-variable technique will be used to obtain the p.d.f. $g_1(t)$ of T. The equations

$$t = \frac{w}{\sqrt{v/r}} \quad \text{and} \quad u = v$$

define a one-to-one transformation that maps $\mathbf{A} = \{(w, v)\,;\, -\infty < w < \infty,\ 0 < v < \infty\}$ onto $\mathscr{B}= \{(t, u)\,;\,;\, -\infty < t < \infty,\ 0 < u < \infty\}$, Since

$w = t\frac{\sqrt{u}}{\sqrt{r}}$, $v = u$, the absolute value of the Jacobian of the transformation is $|J| = \frac{\sqrt{u}}{\sqrt{r}}$. Accordingly, the joint p.d.f. of T and $U = V$ is given by

$$\begin{aligned} g(t, u) &= \varphi\left(\frac{t\sqrt{u}}{\sqrt{r}}, u\right)|J| \\ &= \frac{1}{\sqrt{2\pi}\Gamma(r/2)2^{r/2}} u^{r/2-1} \exp\left[-\frac{u}{2}\left(1 + \frac{t^2}{r}\right)\right]\frac{\sqrt{u}}{\sqrt{r}}, \\ &\qquad\qquad -\infty < t < \infty, 0 < u < \infty, \\ &= 0 \text{ elsewhere.} \end{aligned}$$

The marginal p.d.f. of T is then

$$\begin{aligned} g_1(t) &= \int_{-\infty}^{\infty} g(t, u)\, du \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi r}\Gamma(r/2)2^{r/2}} u^{(r+1)/2-1} \exp\left[-\frac{u}{2}\left(1 + \frac{t^2}{r}\right)\right] du. \end{aligned}$$

In this integral let $= u\left[1 + \left(\frac{t^2}{r}\right)\right]/2$, and it is seen that

$$\begin{aligned} g_1(t) &= \int_0^{\infty} \frac{1}{\sqrt{2\pi r}\Gamma(r/2)2^{r/2}} \left(\frac{2z}{1 + t^2/r}\right)^{(r+1)/2-1} e^{-z}\left(\frac{2}{1 + t^2/r}\right) dz \\ &= \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\Gamma(r/2)} \frac{1}{(1 + t^2/r)^{(r+1)/2}}, \qquad -\infty < t < \infty. \end{aligned}$$

Thus, if W is $N(0, 1)$, if V is $\chi^2_{(r)}$, and if W and V are stochastically independent, then

$$T = \frac{W}{\sqrt{V/r}}$$

The distribution of the random variable T is usually called a t *distribution.* It should be observed that a t distribution is completely determined by the parameter r, the number of degrees of freedom of the random variable that has the
chi-square distribution. Some approximate values of can be found in Table 1.

$$\Pr(T \leq t) = \int_{-\infty}^{t} g_1(w)\, dw$$

for selected values of r and t, can be found in the graph above (r is the degree of freedom).

Notes: The Difference Between a T Distribution and a Normal Distribution :

Normal distributions are used when the population distribution is assumed to be normal. The T distribution is similar to the normal distribution, just with fatter tails. Both assume a normally distributed population. T distributions have higher kurtosis than normal distributions. The probability of getting values very far from the mean is larger with a T distribution than a normal distribution. Difference between using a normal and T distribution is relatively small.

Examples:

Table 1: t distribution:

t-Distribution

The following table presents selected quantiles of the *t*-distribution, i.e., the values t such that

$$P(T \leq t) = \int_{-\infty}^{t} \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma(r/2)(1+w^2/r)^{(r+1)/2}}\,dw,$$

for selected degrees of freedom r. The last row gives the standard normal quantiles.

	$P(T \leq t)$					
r	0.900	0.950	0.975	0.990	0.995	0.999
1	3.078	6.314	12.706	31.821	63.657	318.309
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.610
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
∞	1.282	1.645	1.960	2.326	2.576	3.090

$$\varphi(u, v) = \frac{1}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} u^{r_1/2-1} v^{r_2/2-1} e^{-(u+v)/2},$$

$$0 < u < \infty,\ 0 < v < \infty,$$

$$= 0 \text{ elsewhere.}$$

Expected value

The expected value of a standard Student's t random variable X is well-defined only for $n > 1$ and it is equal to

$$E[X] = 0$$

Proof

It follows from the fact that the density function is symmetric around 0:

$$\begin{aligned}
&E[X] \\
&= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{-\infty}^{0} x f_X(x) dx + \int_{0}^{\infty} x f_X(x) dx \\
&= -\int_{\infty}^{0} (-t) f_X(-t) dt + \int_{0}^{\infty} x f_X(x) dx \quad \text{(change of variable in the first integral: } t = -x\text{)} \\
&= \int_{\infty}^{0} t f_X(-t) dt + \int_{0}^{\infty} x f_X(x) dx \\
&= -\int_{0}^{\infty} t f_X(-t) dt + \int_{0}^{\infty} x f_X(x) dx \quad \text{(exchanging the bounds of integration)} \\
&= -\int_{0}^{\infty} x f_X(-x) dx + \int_{0}^{\infty} x f_X(x) dx \\
&= -\int_{0}^{\infty} x f_X(x) dx + \int_{0}^{\infty} x f_X(x) dx \quad \text{(since } f_X(-x) = f_X(x)\text{)} \\
&= 0
\end{aligned}$$

Note:

A Student's t random variable does not possess a moment generating function. i.e : m.g.f. for t-dis cannot be determined as the integral diverges.

<u>The variance for t-dis:</u>

We need to evaluate $E(X^2)$:

$E(X^2)$ is:

$$= \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

$$= \int_{-\infty}^{0} x^2 f_X(x)dx + \int_0^{\infty} x^2 f_X(x)dx$$

$$= -\int_{\infty}^{0} t^2 f_X(-t)dt + \int_0^{\infty} x^2 f_X(x)dx$$ (change of variable in the first integral: $t = -x$)

$$= \int_0^{\infty} t^2 f_X(-t)dt + \int_0^{\infty} x^2 f_X(x)dx$$ (exchanging the bounds of integration)

$$= \int_0^{\infty} t^2 f_X(t)dt + \int_0^{\infty} x^2 f_X(x)dx$$ (since $f_X(-t) = f_X(t)$)

$$= 2\int_0^{\infty} x^2 f_X(x)dx$$

$$= 2c\int_0^{\infty} x^2 \left(1 + \frac{x^2}{n}\right)^{-\frac{1}{2}(n+1)} dx$$

$$= 2c\int_0^{\infty} nt(1+t)^{-n/2-1/2} \frac{\sqrt{n}}{2} \frac{1}{\sqrt{t}} dt$$ (by a change of variable: $t = \frac{x^2}{n}$)

$$= cn^{3/2} \int_0^{\infty} t^{3/2-1}(1+t)^{-3/2-(n/2-1)} dt$$

$$= cn^{3/2} B\left(\frac{3}{2}, \frac{n}{2} - 1\right)$$ (integral representation of the Beta function)

$$= \frac{1}{\sqrt{n}} \frac{1}{B\left(\frac{n}{2}, \frac{1}{2}\right)} n^{3/2} B\left(\frac{1}{2} + 1, \frac{n}{2} - 1\right)$$ (by the definition of c)

$$= n \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2} + 1\right)\Gamma\left(\frac{n}{2} - 1\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}$$ (by the definition of Beta function)

$$= n \frac{\Gamma\left(\frac{1}{2} + 1\right)\Gamma\left(\frac{n}{2} - 1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$= n \frac{\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{n}{2}\right)\frac{2}{n-2}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$ (because $\Gamma(z) = \Gamma(z-1)(z-1)$)

$$= \frac{n}{n-2}$$

$$\mathrm{E}[X]^2 = 0$$

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 = \frac{n}{n-2}$$

F- distribution:

We define the new random variable F (Hogg – p.143)

$$F = \frac{U/r_1}{V/r_2}$$

Where U and V are iid with Chi-square dis with r_1, r_2 degrees of freedom respectively.
The joint p.d.f is given as
and we propose finding the p.d.f. $g_1(f)$ of F. The equations

$$f = \frac{u/r_1}{v/r_2}, \qquad z = v,$$

define a one-to-one transformation that maps **A** $= \{(u,v)\,;0 < u < \infty,\ 0 < v < \infty\}$ onto $\mathscr{B} = \{(f,z)\,;;0 < f < \infty,\ 0 < z < \infty\}$, Since
$u = (\frac{r_1}{r_2})zf$, $v = z$, the absolute value of the Jacobian of the transformation is $|J| = (\frac{r_1}{r_2})z$. The joint p.d.f. $g(f,z)$ of F and $z = V$ is then

$$g(f,z) = \frac{1}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} \left(\frac{r_1 z f}{r_2}\right)^{r_1/2-1} z^{r_2/2-1} \times \exp\left[-\frac{z}{2}\left(\frac{r_1 f}{r_2}+1\right)\right]\frac{r_1 z}{r_2},$$

provided that $(f,z) \in \mathscr{B}$, and zero elsewhere. The marginal p.d.f. $g_1(f)$ of F is then

$$g_1(f) = \int_{-\infty}^{\infty} g(f,z)\,dz$$

$$= \int_0^{\infty} \frac{(r_1/r_2)^{r_1/2}(f)^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} z^{(r_1+r_2)/2-1} \exp\left[-\frac{z}{2}\left(\frac{r_1 f}{r_2}+1\right)\right] dz.$$

If we change the variable of integration by writing

$$y = \frac{z}{2}\left(\frac{r_1 f}{r_2}+1\right)$$

it can be seen that

$$g_1(f) = \int_0^{\infty} \frac{(r_1/r_2)^{r_1/2}(f)^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} \left(\frac{2y}{r_1 f/r_2+1}\right)^{(r_1+r_2)/2-1} e^{-y} \times \left(\frac{2}{r_1 f/r_2+1}\right) dy$$

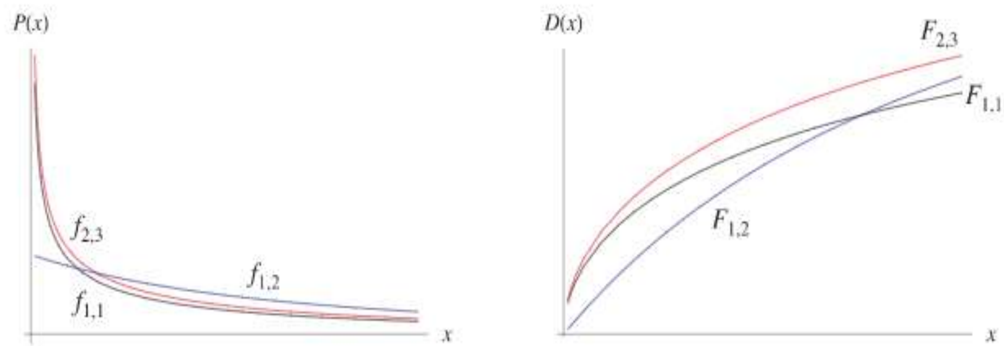
$$= \frac{\Gamma[(r_1+r_2)/2](r_1/r_2)^{r_1/2}}{\Gamma(r_1/2)\Gamma(r_2/2)} \frac{(f)^{r_1/2-1}}{(1+r_1 f/r_2)^{(r_1+r_2)/2}}, \qquad 0 < f < \infty,$$

$$= 0 \text{ elsewhere.}$$

F-distribution:

F-distribution is a useful probability distribution for studying population variances. Some important properties for F-dis.

- The F-distribution is a family of distributions. This means that there is an infinite number of different F-distributions. The particular F-distribution that we define depends upon the number of degrees of freedom . This feature of the F-distribution is similar to both the *t*-distribution and the chi-square distribution.
- The F-distribution is either zero or positive, so there are no negative values for *F*. This feature of the F-distribution is similar to the chi-square distribution.
- The F-distribution is skewed to the right. Thus, this probability distribution is nonsymmetrical. This feature of the F-distribution is similar to the chi-square distribution.



$$F = \frac{U/r_1}{V/r_2}$$

It should be observed that an F distribution is completely determined by the two parameters r_1 and r_2 Table 2 gives some approximate values of

$$\Pr(F \leq f) = \int_0^f g_1(w)\, dw$$

for selected values of r_1, r_2, and *f*.

See Table 2:

Exercises:

1. Let T have a t distribution with 10 degrees of freedom. Find Pr ($|T| > 2.228$) from Table 1.

2. Let T have a t distribution with 14 degrees of freedom. Determine b so that $\Pr(-b < T < b) = 0.90$).

3. Let F have an F distribution with parameters r_1 and r_2 Prove that $1/F$ has an F distribution with parameters r_1 and r_2.

4. If F has an F distribution with parameters $r_1 = 5$ and $r_2 = 10$, find a and b so that $\Pr(F \le a) = 0.05$) and $\Pr(F \le b) = 0.95$), and, accordingly,
$\Pr(a < F < b) = 0.90$). *Hint.* Write $\Pr(F \le a) = \Pr\left(\frac{1}{F} \ge \frac{1}{a}\right) = 1 - \left(\frac{1}{F} \le \frac{1}{a}\right)$.

5. Let $= W/\sqrt{V/r}$, where the stochastically independent variables W and V are, respectively, normal with mean zero and variance 1 and chi-square with r degrees of freedom. Show that T^2 has an F distribution with parameters $r_1 = 1$ and $r_2 = r$. *Hint.* What is the distribution of the numerator of T^2?

6. Show that the t distribution with r =1 degree of freedom and the Cauchy distribution are the same.

7. Show that

$$Y = \frac{1}{1 + (r_1/r_2)F}$$

where F has an F distribution with parameters r_1 and r_2, has a beta distribution.

8. Let X_1 *and* X_2 be a random sample from a distribution having the p.d.f. $f(x) = e^{-x}$,$0 < x < \infty$,, zero elsewhere. Show that $Z = X_1/X_2$ has an F distribution.

Table 2:

F-Distribution, Continued
Upper 0.05 Critical Points

$F_{0.05}(r_1, r_2)$									
	r_1								
r_2	10	15	20	25	30	40	60	120	∞
1	241.88	245.95	248.01	249.26	250.10	251.14	252.20	253.25	254.31
2	19.40	19.43	19.45	19.46	19.46	19.47	19.48	19.49	19.50
3	8.79	8.70	8.66	8.63	8.62	8.59	8.57	8.55	8.53
4	5.96	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.62	4.56	4.52	4.50	4.46	4.43	4.40	4.36
6	4.06	3.94	3.87	3.83	3.81	3.77	3.74	3.70	3.67
7	3.64	3.51	3.44	3.40	3.38	3.34	3.30	3.27	3.23
8	3.35	3.22	3.15	3.11	3.08	3.04	3.01	2.97	2.93
9	3.14	3.01	2.94	2.89	2.86	2.83	2.79	2.75	2.71
10	2.98	2.85	2.77	2.73	2.70	2.66	2.62	2.58	2.54
11	2.85	2.72	2.65	2.60	2.57	2.53	2.49	2.45	2.40
12	2.75	2.62	2.54	2.50	2.47	2.43	2.38	2.34	2.30
13	2.67	2.53	2.46	2.41	2.38	2.34	2.30	2.25	2.21
14	2.60	2.46	2.39	2.34	2.31	2.27	2.22	2.18	2.13
15	2.54	2.40	2.33	2.28	2.25	2.20	2.16	2.11	2.07
16	2.49	2.35	2.28	2.23	2.19	2.15	2.11	2.06	2.01
17	2.45	2.31	2.23	2.18	2.15	2.10	2.06	2.01	1.96
18	2.41	2.27	2.19	2.14	2.11	2.06	2.02	1.97	1.92
19	2.38	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.20	2.12	2.07	2.04	1.99	1.95	1.90	1.84
21	2.32	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.15	2.07	2.02	1.98	1.94	1.89	1.84	1.78
23	2.27	2.13	2.05	2.00	1.96	1.91	1.86	1.81	1.76
24	2.25	2.11	2.03	1.97	1.94	1.89	1.84	1.79	1.73
25	2.24	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.07	1.99	1.94	1.90	1.85	1.80	1.75	1.69
27	2.20	2.06	1.97	1.92	1.88	1.84	1.79	1.73	1.67
28	2.19	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.03	1.94	1.89	1.85	1.81	1.75	1.70	1.64
30	2.16	2.01	1.93	1.88	1.84	1.79	1.74	1.68	1.62
40	2.08	1.92	1.84	1.78	1.74	1.69	1.64	1.58	1.51
60	1.99	1.84	1.75	1.69	1.65	1.59	1.53	1.47	1.39
120	1.91	1.75	1.66	1.60	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.67	1.57	1.51	1.46	1.39	1.32	1.22	1.00