Chapter One: Fourier Series

By: Maha George Zia

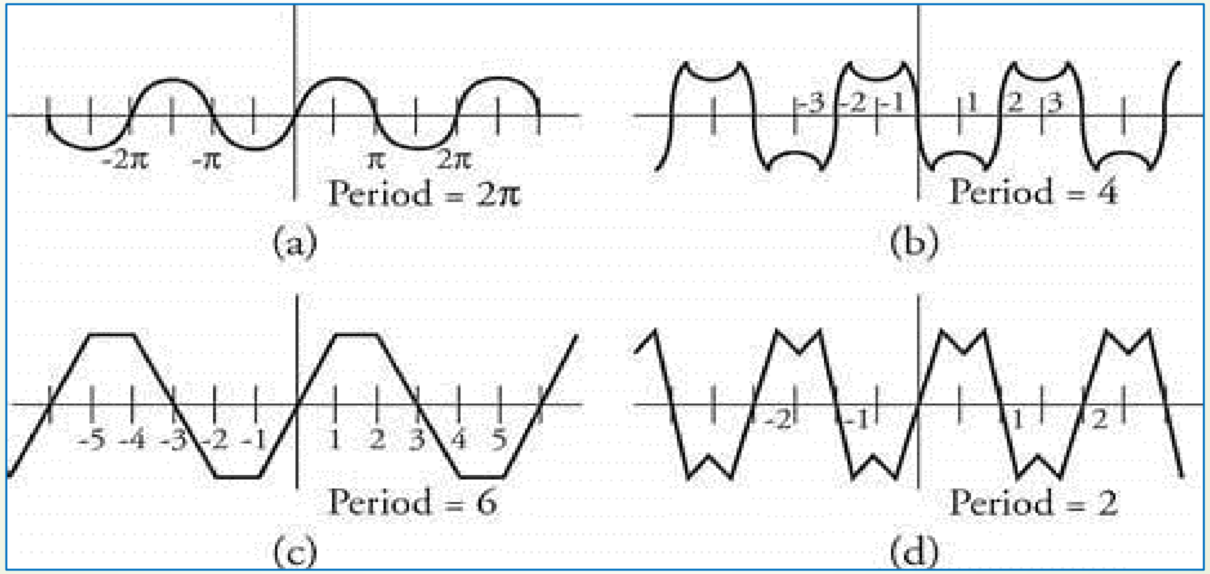
Assistant Professor

Electrical Engineering Department

Definitions:

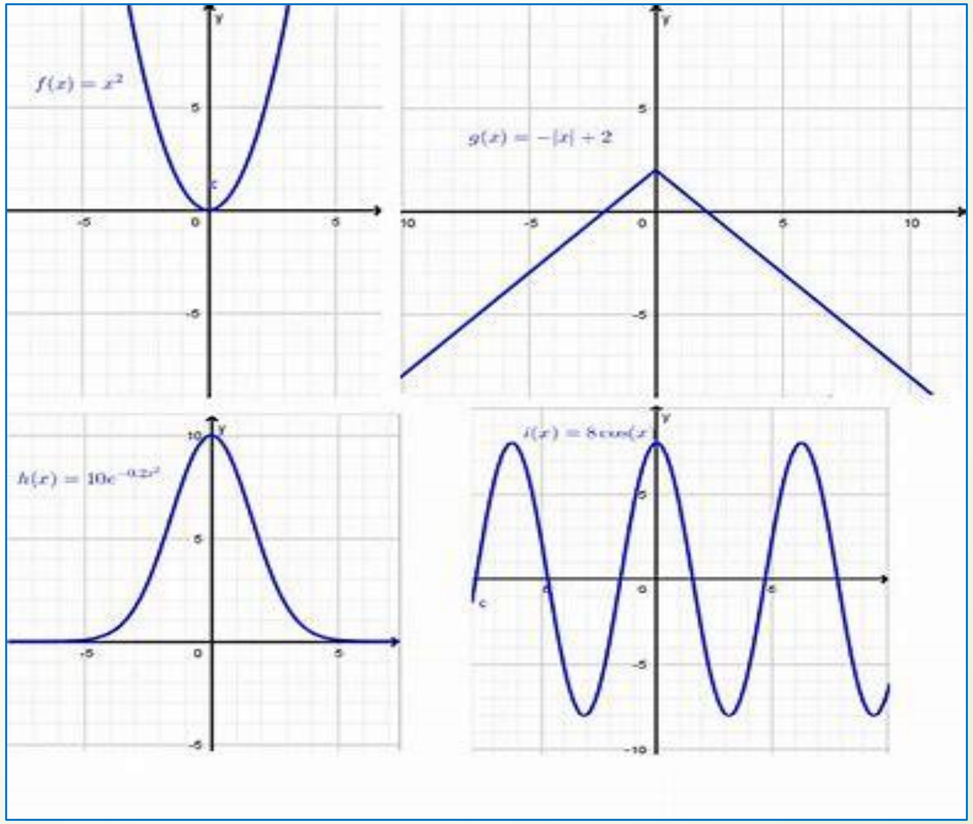
1. Periodic Function: A function $f(t)$ is said to be periodic if it is defined for all real t and if there is some positive number T such that:

$$f(t) = f(t + nT), \quad n = 0,1,2,.. \quad is\ an\ integer\ value$$

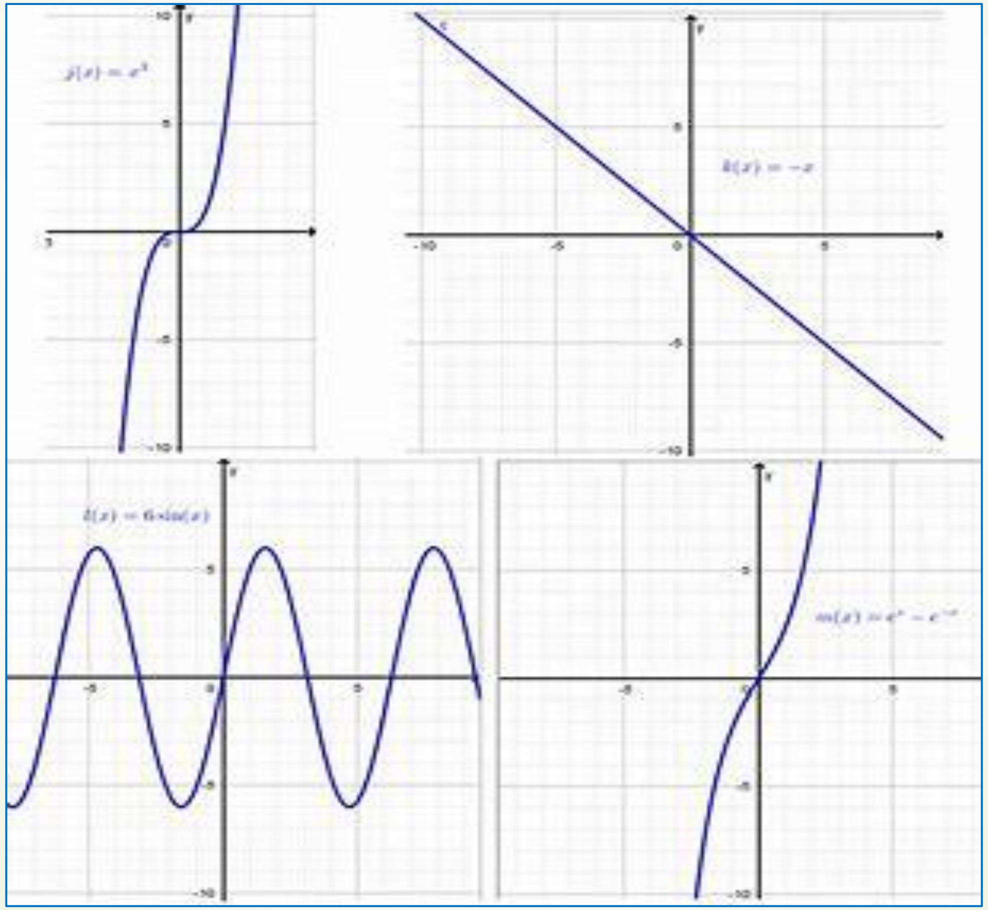


2. The function f(t) is said to be an even function if $f(t) = f(-t)$. It is always symmetrical about y-axis. Sometimes its dc value is zero.

- Note: the dc value is a horizontal line that divide the function area into two equal areas.

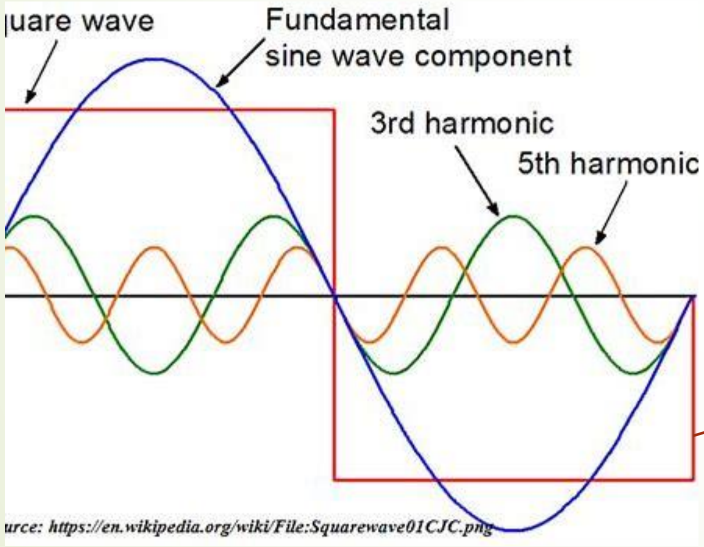


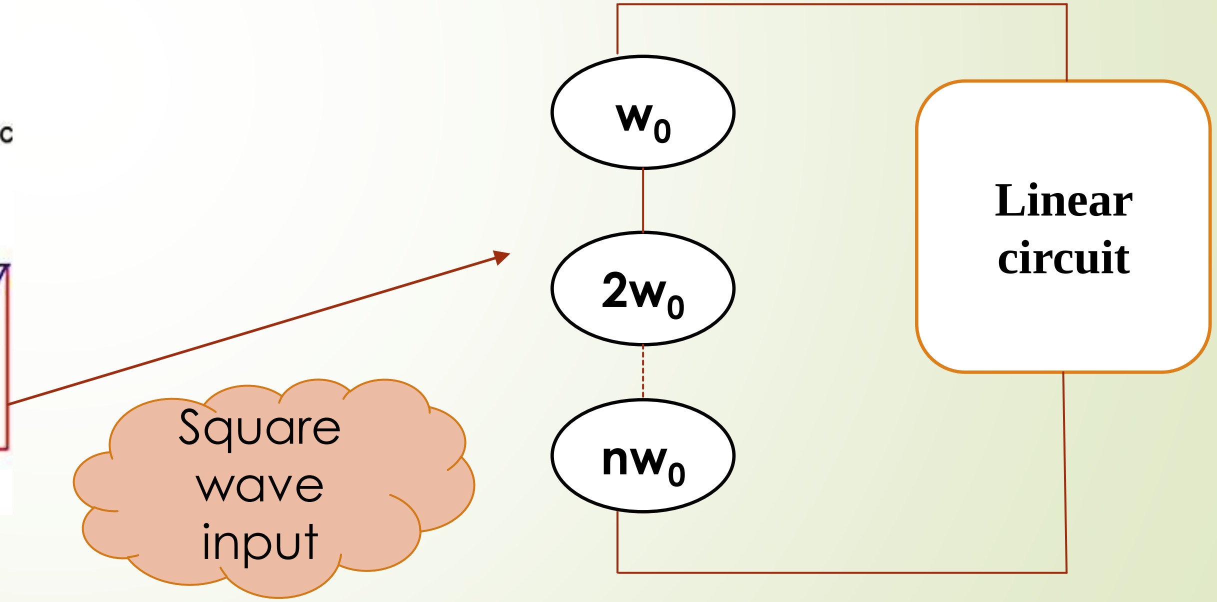
- 3. The function f(t) is said to be an odd function if $f(-t) = -f(t)$. It is always symmetrical about x-axis. its dc value is always zero.
-



Fourier Series (FS): Trigonometric form

- Any periodic function other than sinusoidal function (e.g., square wave) applied as an input to a linear circuit can be analyzed and represented by several sinusoidal sources considered as harmonics.





- **e.g.,= for example**

- $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nw_0 t + b_n \, sin \, nw_0 t\ )$, n =1,2,3,….(harmonics) …(1)
- Dc value = $\frac{a_0}{2}$, rad frequency = $w_0 = \frac{2\pi}{T}$, T = one period
- $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$ …….(2)
- $a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\, w_0 t\ dt$ ……(3)
- $b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\, w_0 t\ dt$ ……(4)

- Notes:

- $$\int_{-T/2}^{T/2} \cos m w_0 t \cos n\, w_0 t\; dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n, \quad m \neq 0 \\ T & m = n = 0 \end{cases}$$

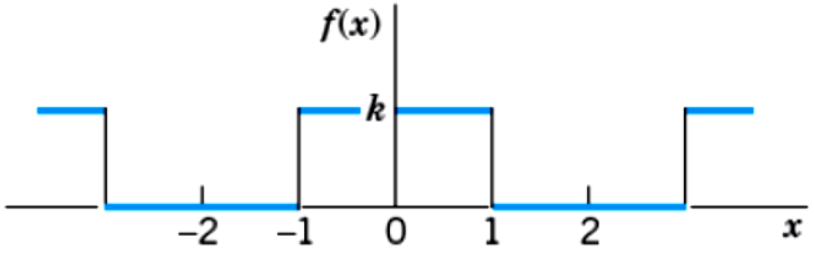
- $$\int_{-T/2}^{T/2} \sin m w_0 t\; sin\, n\, w_0 t\; dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

- $$\int_{-T/2}^{T/2} \cos m w_0 t \sin n\, w_0 t\; dt = 0 \quad for\, all\, m\, and\, n.$$

Ex: Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases}$$

Wo= 2π/4 = π/2
T= 4, $t = x$



$$a_0 = \frac{1}{4}\int_{-2}^{2} k\, dt = \frac{1}{4}\int_{-1}^{1} k\, dt = \frac{1}{4}\, k\big(1-(-1)\big) = \frac{k}{2}$$

$$a_n = \frac{2}{4}\int_{-2}^{2} \frac{k}{2} \cos(n\frac{\pi}{2}t)\, dt = \frac{1}{2}\int_{-1}^{1} \frac{k}{2}\cos\left(\frac{n\pi}{2}t\right) dt = \frac{2k}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

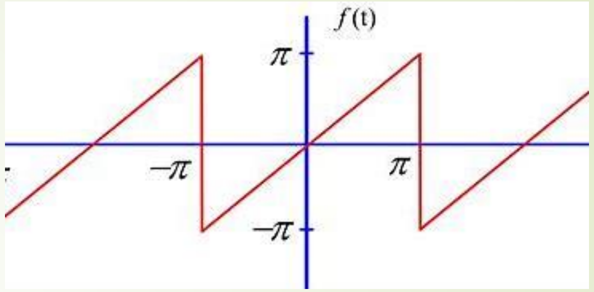
If n *is even then* $a_n = 0$, then:

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots$$
$$a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots$$

$$b_n = \frac{2}{4}\int_{-2}^{2} \frac{k}{2}\sin(n\frac{\pi}{2}t)\, dt = \frac{1}{2}\int_{-1}^{1} \frac{k}{2} \sin\left(\frac{n\pi}{2}t\right) dt = 0\,, \quad why?$$

$$f(t) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - \cdots + \cdots .\right)$$

- H.W1: Find FS for the function shown using the trigonometric form of F.S:
-



- H.W2: Find FS for the function shown using the trigonometric form of F.S:
$f(t) = t^2, \quad -2 \leq t \leq 2$

- Definitions:

- Full range expansion is a series consisting of both cosine and sine terms.

- Half-Range expansion is a series consisting of cosine terms only or series consisting of sine terms only

Exponential (Complex) form of Fourier Series

- By substituting $\cos nw_0 = \frac{e^{jnw_0} + e^{-jnw_0}}{2}$ and $\sin nw_0 = \frac{e^{jnw_0} - e^{-jnw_0}}{2j}$

Into equation (1) and rearrange values, then:

- $f(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{j\,n\,w_0 t}$ .....(5)

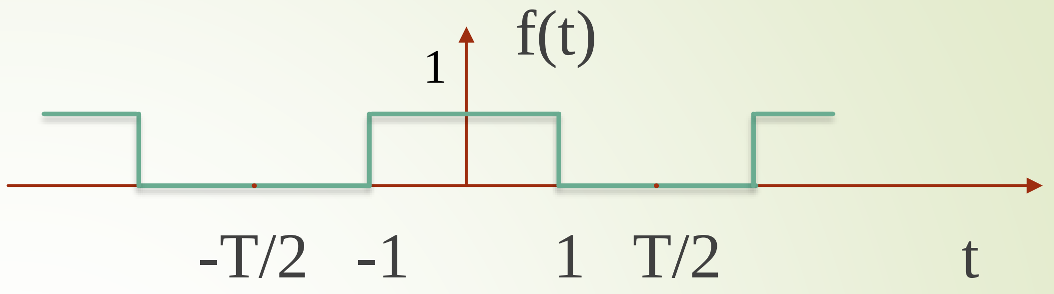
- $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jnw_0 t}\, dt$ ....(6)

- Were,

$c_0 = a_0 \qquad c_n = \frac{1}{2}(a_n - jb_n) \qquad c_{-n} = c_n^* = \frac{1}{2}(a_n + jb_n)$

- $|C_n| = \frac{\sqrt{a_n{}^2 + b_n{}^2}}{2}$

- Ex: Find and plot the complex form of FS for the function shown below for T = 4, 8, and 16.



- Using eq. (6)

$$C_n = \frac{1}{T}\int_{-1}^{1} 1\, e^{-jnw_0 t}\, dt$$

- $$= \frac{1}{T} \left. \frac{e^{-jnw_0 t}}{-jnw_0} \right|_{-1}^{1} = \frac{2}{T}\,\frac{e^{jnw_0} - e^{-jnw_0}}{2jnw_0} = \frac{2}{T}\,\frac{\sin nw_0}{nw_0}$$

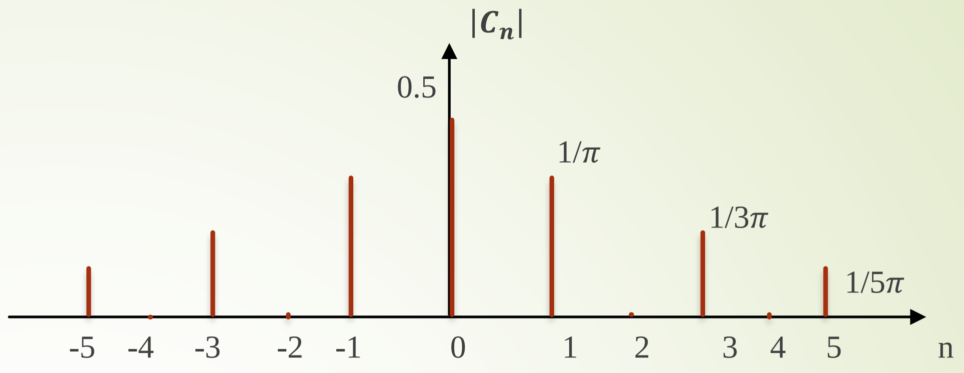
- For T = 4, $w_0 = \frac{2\pi}{T} = \pi/2$

Then $C_n = \frac{1}{2}\,\frac{\sin n\pi/2}{n\pi/2}$

- $C_n = 0$ when $\sin n\pi/2 = 0$, i.e., $\frac{n\pi}{2} = \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots\ldots\ldots$
- Then $n = \pm 2,\ \pm 4,\ \pm 6, \ldots\ldots$

i.e., means
That is

n	$\lvert C_n \rvert$
0	1/2
1	$1/\pi$
3	$1/3\pi$
5	$1/5\pi$



- H.W : solve for T=8, and 16 . As T is increased the spectrum ($|C_n|$) becomes denser and as T $\longrightarrow \infty$, the spectrum becomes continuous rather than discrete, i.e., the function becomes non periodic.