CHAPTER TWO : FOURIER TRANSFORMS

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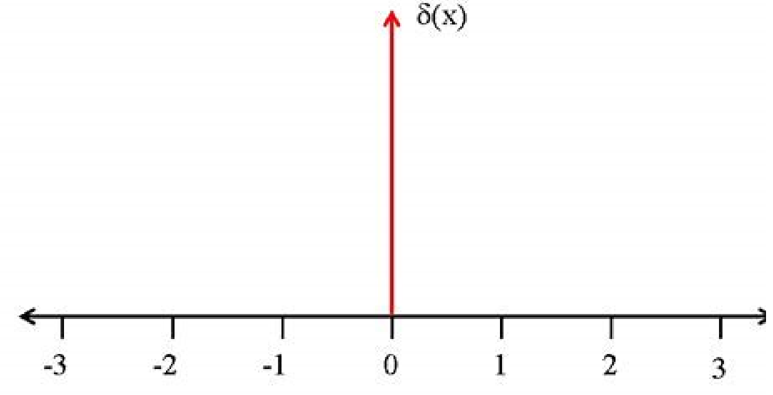
Some Special Functions

1. Impulse Function $\delta(t)$

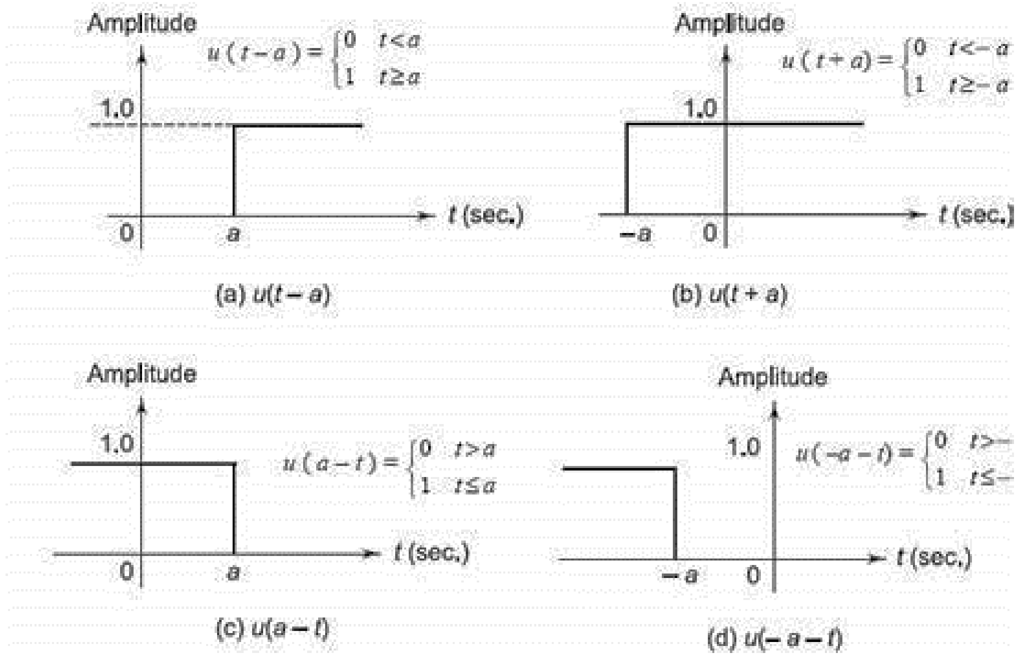
   or $\delta(x)$, if $t = x$.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

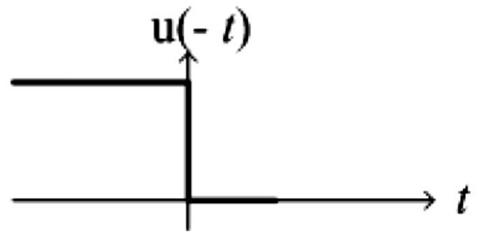


2. Unit step function u(t)

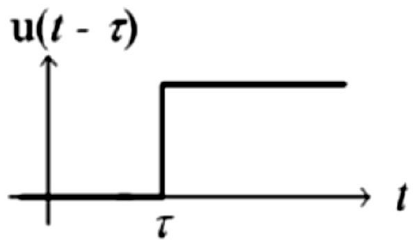
$$u(t) = 1 \qquad t \geq 0$$



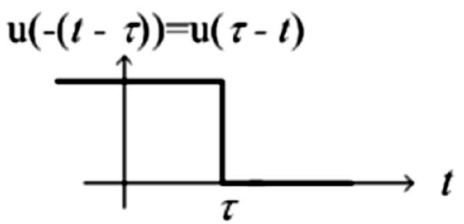
Reflection and shifting of unit step function



Reflected Unite Step Function



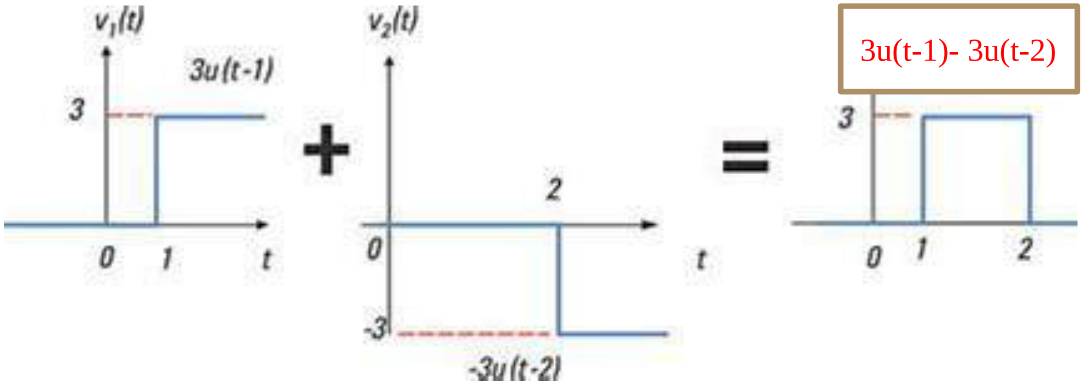
Shifted Unite Step Function



Reflected and Shifted Unite Step Function

EX1: Plot

3u(t-1)- 3u(t-2)



Continuous Fourier transform (C F T)

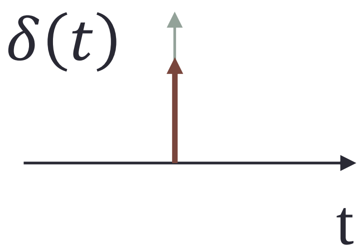
- Fourier transform (F.T) provides the link between the time-domain and frequency domain descriptions of a signal.
- Fourier transform can be used for both periodic and nonperiodic signals.

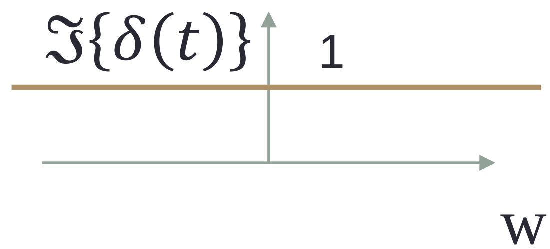
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$ Fourier Transform (1)

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$ Inverse Fourier Transform (2)

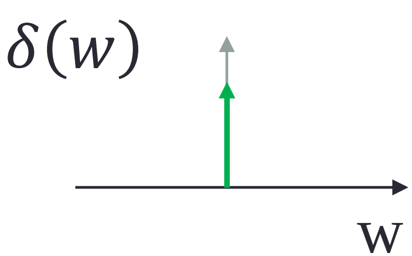
Fourier transform (FT) and inverse Fourier transform (IFT) of $\delta(t)$

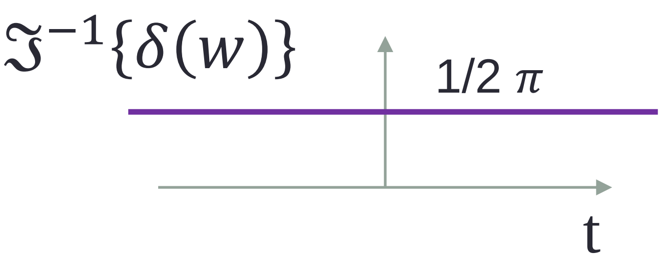
$$\Im\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-jwt} dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-jw(0)} dt = 1$$





$$\Im^{-1}\{\delta(w)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(w)\, e^{jwt} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(w)\, e^{j(0)t} dt = \frac{1}{2\pi}$$

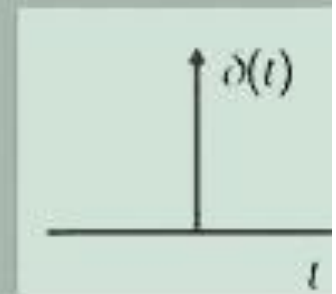




$f(t)$	$\Im\{f(t)\}$
$\delta(t)$	1
1	$2\pi\delta(w)=\delta(f)$
$\delta(t-t_0)$	e^{-jwt_0}
$e^{\pm jwt_0}$	$2\pi\delta(w \pm w_0) = \delta(f \pm f_0)$

Properties of delta- Dirac function

Dirac δ-function Properties



$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

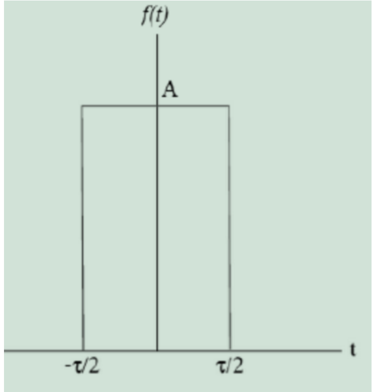
$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = \int_{-\infty}^{\infty} \delta(t-a)\, f(a)\, dt = f(a)$$

$$\int_{-\infty}^{\infty} \exp(\pm i\omega t)\, dt = 2\pi\, \delta(\omega)$$

$$\int_{-\infty}^{\infty} \exp[\pm i(\omega - \omega')t]\; dt = 2\pi\, \delta(\omega - \omega')$$

Fig.1 Properties of delta- Dirac function

Ex2: Find and plot the Fourier transform (F.T) for the rectangular function shown



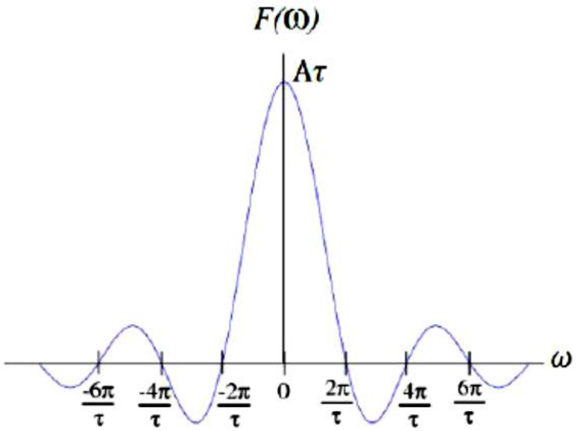
$$f(t) = A\,rect\left(\frac{t}{\tau}\right) = \begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & otherwise \end{cases}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} dt$$

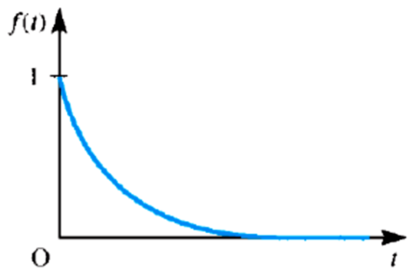
$$= \frac{A}{-j\omega}\left[e^{-j\omega t}\right]_{-\tau/2}^{\tau/2} = A\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} = A\tau\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j\omega\tau/2}$$

Since $\sin x = \frac{e^{jx} - e^{-jx}}{2j}$

$$F(\omega) = A\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2} = A\tau\,Sinc\left(\frac{\omega\tau}{2}\right)$$



Ex3: Find FT of $f(t) = e^{-at}\,u(t), \quad a > 0$



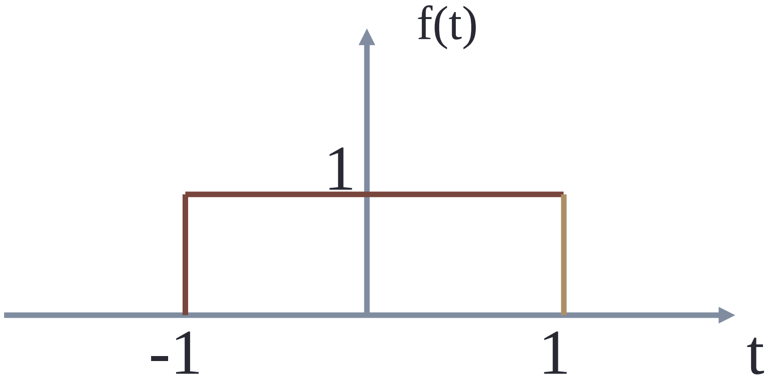
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

$$F(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$F(\omega) = \frac{1}{-(a+j\omega)}\left[e^{-(a+j\omega)t}\right]_0^{\infty} = \frac{1}{-(a+j\omega)}[0-1]$$

$$F(\omega) = \frac{1}{a+j\omega}$$

H.W : Find and plot FT of the function shown



Fourier Transform Pair (FTP) of even and odd non-periodic functions

- Fourier Transform Pair (FTP) of even non-periodic functions is:

$$f(t) = \frac{1}{\pi}\int_0^{\infty} F(w)\cos wt\, dw \qquad ....(3)$$

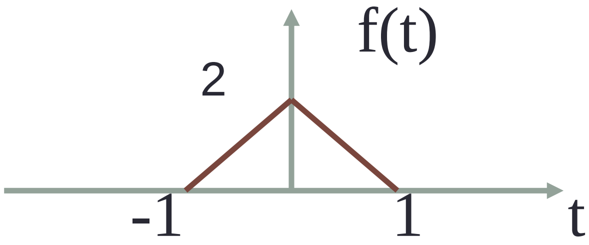
$$F(w) = 2\int_0^{\infty} f(t)\cos wt\, dt = \Im\{f(t)\} \qquad ....(4)$$

Fourier Transform Pair (FTP) of odd non-periodic functions is:

$$f(t) = \frac{1}{\pi}\int_0^{\infty} F(w)\sin wt\, dw \qquad ....(5)$$

$$F(w) = 2\int_0^{\infty} f(t)\sin wt\, dt = \Im\{f(t)\} \qquad ....(6)$$

Ex4: Find and Plot the spectrum of the function:



- Its even function
- $F(w) = 2\int_0^1 2(1-t)\cos wt\, dt = \frac{4}{w^2}\,(1-\cos w)\,\frac{2}{2}.$
- We have $sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, and assuming $2\ \theta = \text{w}$, then
- $\theta = \frac{w}{2}.$ Therefore:
- $F(w) = \frac{4}{w^2}\ sin^2\left(\frac{w}{2}\right) = 2\left(\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right)^2$
- H.W : Plot its spectrum.

Table 1: Fourier transform properties

	Property	Time domain $x(t)$	Fourier transform $X(j\omega)$
1)	Linearity	$x(t) = A\,x_1(t) + Bx_2(t)$	$X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega)$
2)	Time shifting	$x(t - t_0)$	$e^{-j\omega t_0}\,X(j\omega)$
3)	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4)	Differentiation in time	$\frac{d^n x(t)}{dt^n}$	$(j\omega)^n . X(j\omega)$
5)	Differentiation in frequency	$-jt\,x(t)$	$\frac{d\,X(j\omega)}{d\omega}$
6)	Time Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi . X(0) . \delta(\omega)$
7)	Time scaling	$x(at)$	$\frac{1}{\|a\|} X\left(j\frac{\omega}{a}\right)$
8)	Time reversal	$x(-t)$	$X(-j\omega)$
9)	Frequency shifting	$x(t) . e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$
10)	Duality	$X(t)$	$2\pi x(-j\omega)$
11)	Time convolution	$x(t) * h(t)$	$X(j\omega) . H(j\omega)$
12)	Parseval's Theorem	$E = \int_{-\infty}^{\infty} \|x(t)\|^2 dt$	$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|X(j\omega)\|^2 dt$
13)	Modulation	$z(t) = x(t) . y(t)$	$Z(\omega) = \frac{1}{2\pi} . X(j\omega) * Y(j\omega)$

Some examples of Fourier Transform properties:

1- Scaling property:

Time and frequency scaling:

$$x(at) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Example:

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a+j\omega} = \frac{1}{a}\frac{1}{1+j\left(\frac{\omega}{a}\right)}$$

2- Time shifting(delay):

Time shifting:

$$x(t-t_o) \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_o} X(\omega)$$

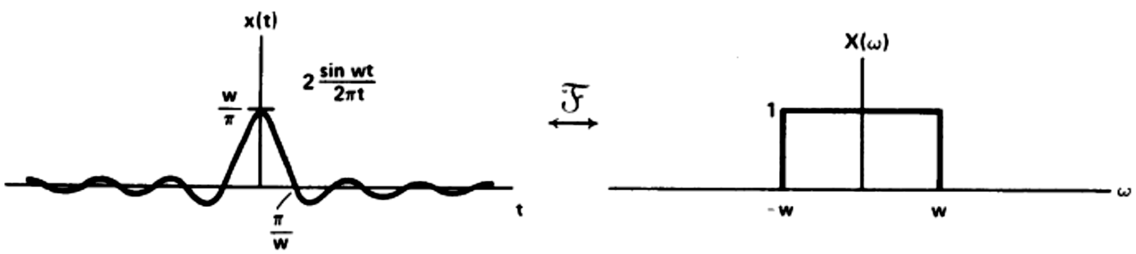
$$\Im\{\delta(t-3)\} = e^{-jw(3)}.1$$

3-Modulation (frequency shifting):

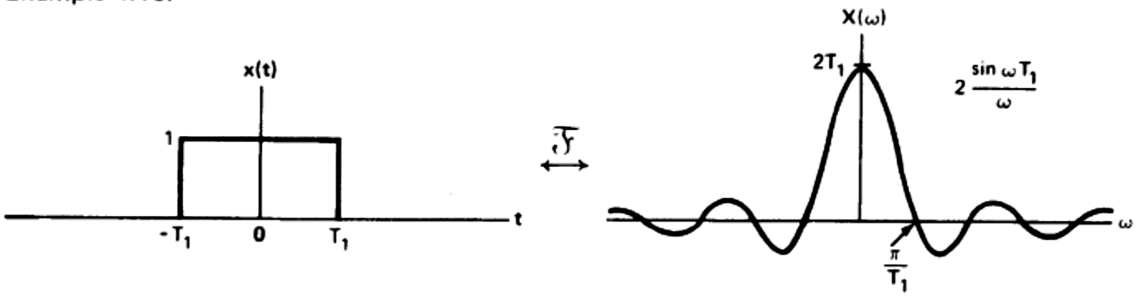
EX5: $\Im\{f(t)\cos w_0 t\} = \Im\left\{f(t)[\frac{e^{jnw_0} + e^{-jnw_0}}{2}]\right\} = \frac{1}{2}\{F(w - w_0) + F(w + w_0)\}$

Ex6: $\Im\left\{\frac{e^{jnw_0}+e^{-jnw_0}}{2}\right\} = \frac{1}{2}\{\delta((f-f_0)+\delta(f+f_0)\} =$
$\pi\{\,\delta((w-w_0)+\delta(w+w_0)\}$

4- Duality property:



Example 4.10:

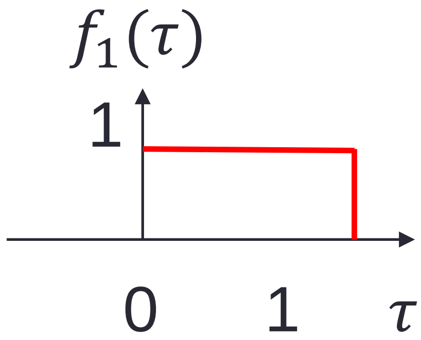


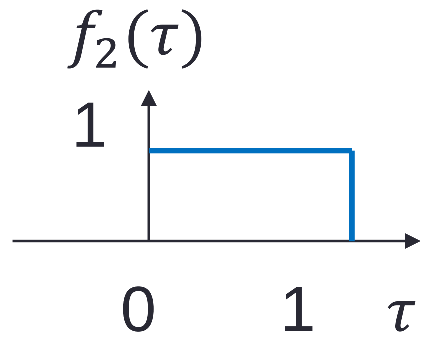
- Convolution

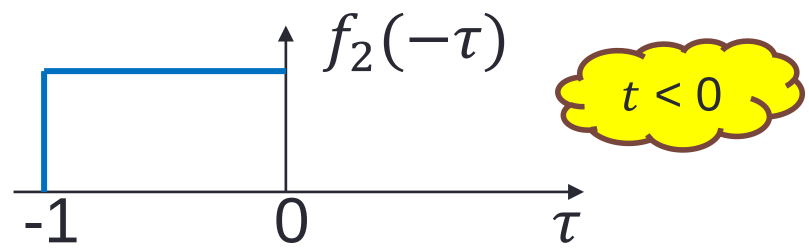
1. Change the axis to τ for both functions.
2. Let $f_1(\tau)$ to be fixed function.
3. Reflect (invert) the function $f_2(\tau)$ to be $f_2(-\tau)$.
4. Multiply and integrate both functions, the fixed function $f_1(\tau)$ with the reflected function $f_2(-\tau)$.
5. Now Shift $f_2(-\tau)$ by a value of t ,multiply and integrate with the fixed function $f_1(\tau)$.
6. The sifting of $f_2(-\tau)$ is stopped when there is no area (no relations) to be calculated.

Note: convolution in time = multiplication in frequency domain

- $\Im\{f_1(t)\otimes f_2(t)\} = \Im\{\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau\} = F_1(w).F_2(w)$







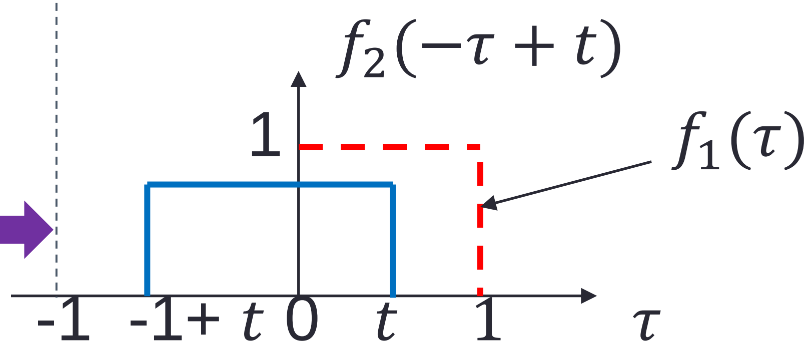
$$f(t) = \{f_1(t) \otimes f_2(t)\}$$

$$= \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau)\, d\tau$$

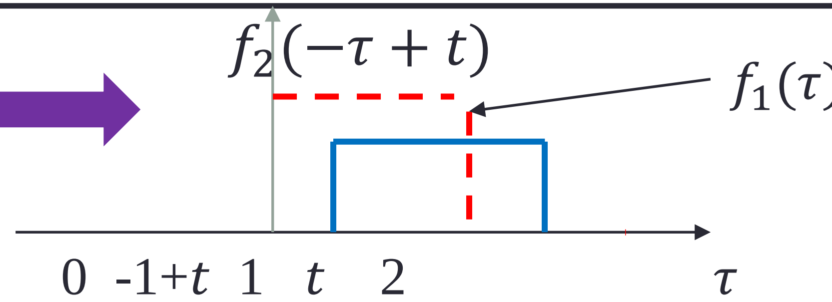
1. $0 < t < 1$

$$f(t) = \int_0^t 1.1\, d\tau = t$$



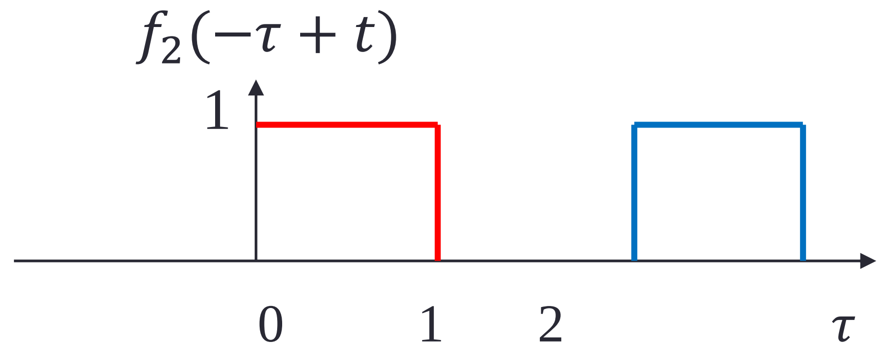
2. $1 < t < 2$

$$f(t) = \int_{t-1}^{1} 1.1\, d\tau = 2 - t$$

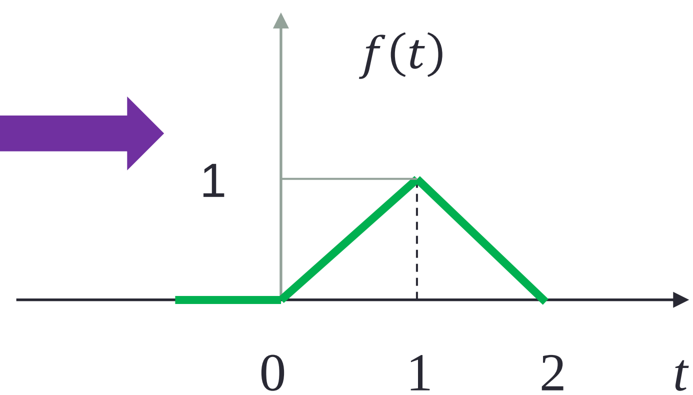


3. $t > 2$

$f(t) = 0$



$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$



Discrete Fourier Transform(DFT)

- Given a sequence x(n), 0 < n < N 1, its DFT is defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \text{ for } k = 0,\ 1, \ldots, N-1. \quad (7)$$

- where the factor W_N (called the twiddle factor) is defined as:

$$W_N = e^{-j2\pi/N} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right). \quad (8)$$

- The inverse DFT is given by:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \text{ for } n = 0, 1, \ldots, N-1. \quad (9)$$

MATLAB functions fft() and ifft() are used to compute the DFT coefficients

Ex7: Given a sequence x(n) for $0 \leq n \leq 3$, where x(0) = 1, x(1) = 2, x(2) =3, and x(3) = 4. Evaluate its DFT X(k).

Solution: Since N = 4 and $W_4 = e^{-j\pi/2}$, using Eq. (7):

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi k n}{2}}.$$

Thus, for $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 2 + 3 + 4 = 10$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$

$$= x(0) - jx(1) - x(2) + jx(3)$$

$$= 1 - j2 - 3 + j4 = -2 + j2$$

for $k = 2$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= x(0) - x(1) + x(2) - x(3)$$

$$= 1 - 2 + 3 - 4 = -2$$

and for $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$

$$= x(0) + jx(1) - x(2) - jx(3)$$

$$= 1 + j2 - 3 - j4 = -2 - j2$$

- Ex8:Using the DFT coefficients X(k) for $0 \le k \le 3$ computed in Example 7. Evaluate its inverse DFT to determine the time domain sequence x(n).
- Solution: Since N = 4 and $W_4 = e^{j\pi/2}$, using Eq. (9):

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k) W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} X(k) e^{j\frac{\pi kn}{2}}.$$

Then for $n = 0$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k) e^{j0} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}\right)$$

$$= \frac{1}{4}(10 + (-2 + j2) - 2 + (-2 - j2)) = 1$$

for $n = 1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}\right)$$

$$= \frac{1}{4}(X(0) + jX(1) - X(2) - jX(3))$$

$$= \frac{1}{4}(10 + j(-2 + j2) - (-2) - j(-2 - j2)) = 2$$

for $n = 2$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\pi} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right)$$

$$= \frac{1}{4}(X(0) - X(1) + X(2) - X(3))$$

$$= \frac{1}{4}(10 - (-2 + j2) + (-2) - (-2 - j2)) = 3$$

and for $n = 3$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi 3}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}\right)$$

$$= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3))$$

$$= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4$$

- using the MATLAB function fft( ) for Ex.7:
- X = fft([1 2 3 4])
- X = 10.0000 -2.0000+ 2.0000i -2.0000 -2.0000- 2.0000i

- Applying the MATLAB function ifft( ) for Ex.8 achieves:
- x = ifft([10 -2+ 2j -2 -2-2j])
- x =1 2 3 4

Fast Fourier Transform (FFT)

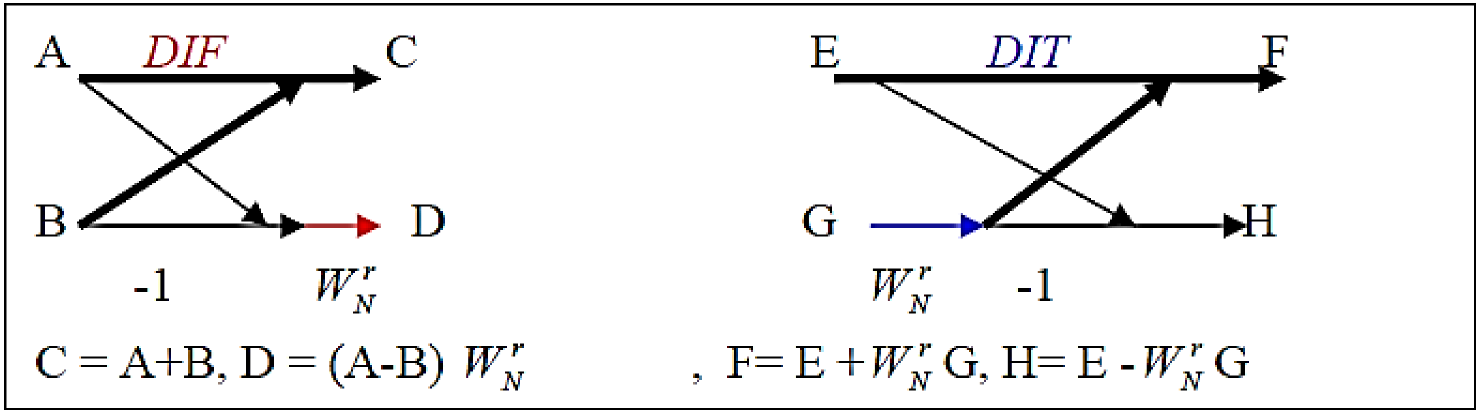
- $$\text{Complex multiplications of DFT} = N^2, \text{ and}$$ $$\text{Complex multiplications of FFT} = \frac{N}{2}\log_2(N). \quad \ldots..(10)$$

FFT-Reduced decimation in-frequency (Reduced DIF FFT)

- The definition of FFT is $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$ for $k = 0, 1, \ldots, N-1$ (11)
- where $W_N = e^{-j2\pi N}$ is the twiddle factor, and N = 2, 4, 8, 16, . .
- The inverse FFT (IFFT) is defined as below, where $\tilde{W}_N = W_N^{-1}$:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \tilde{W}_N^{kn}, \text{ for } k = 0, 1, \ldots, N-1. \quad (12)$$

- The Butterfly structure for DIF FFT and DIT FFT is shown below:



- Note the bit reversed order in Fig. 2 and Fig.3 in the *output*

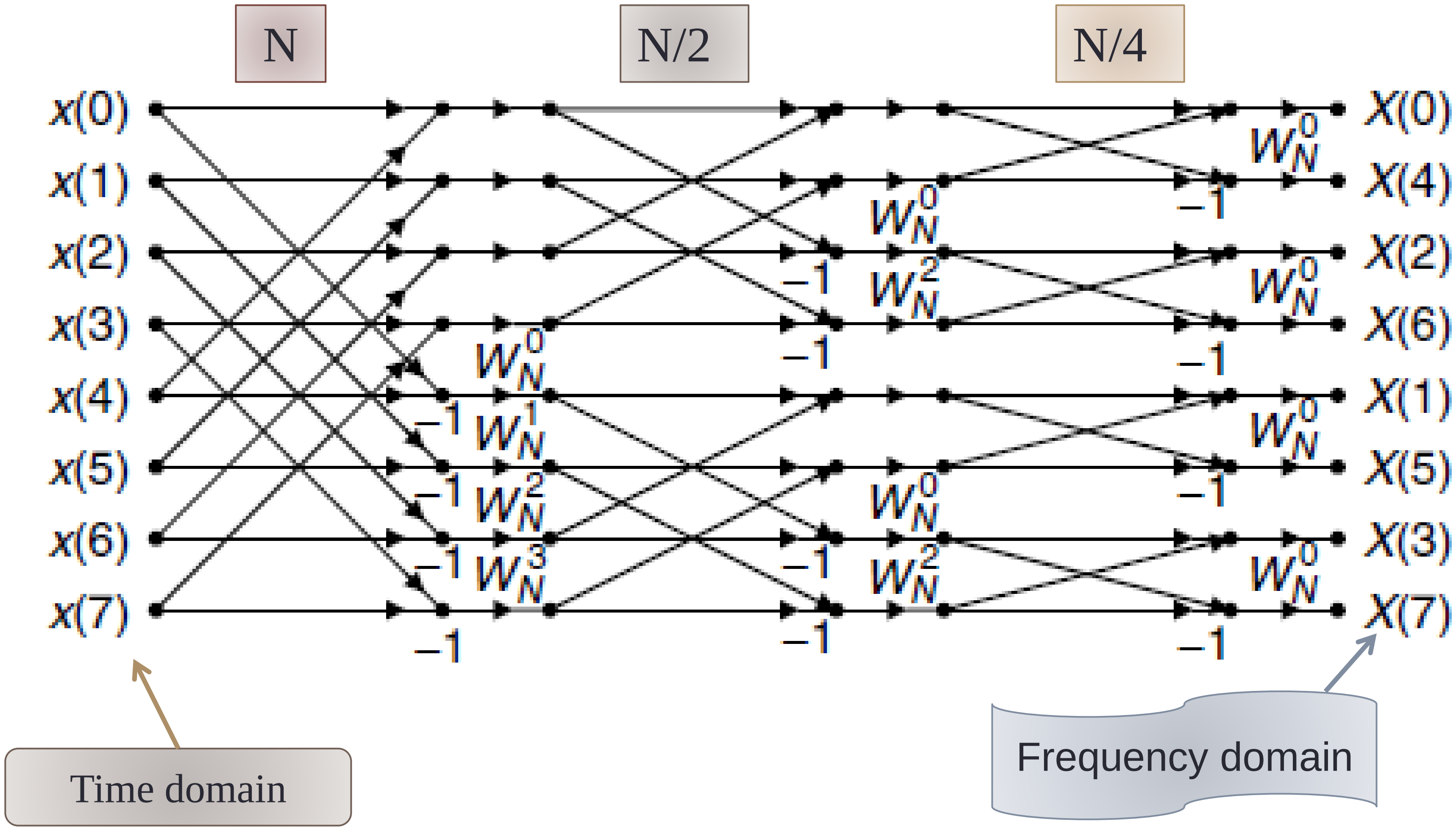


Fig.2 Block diagram for **reduced DIF FFT, N =8**

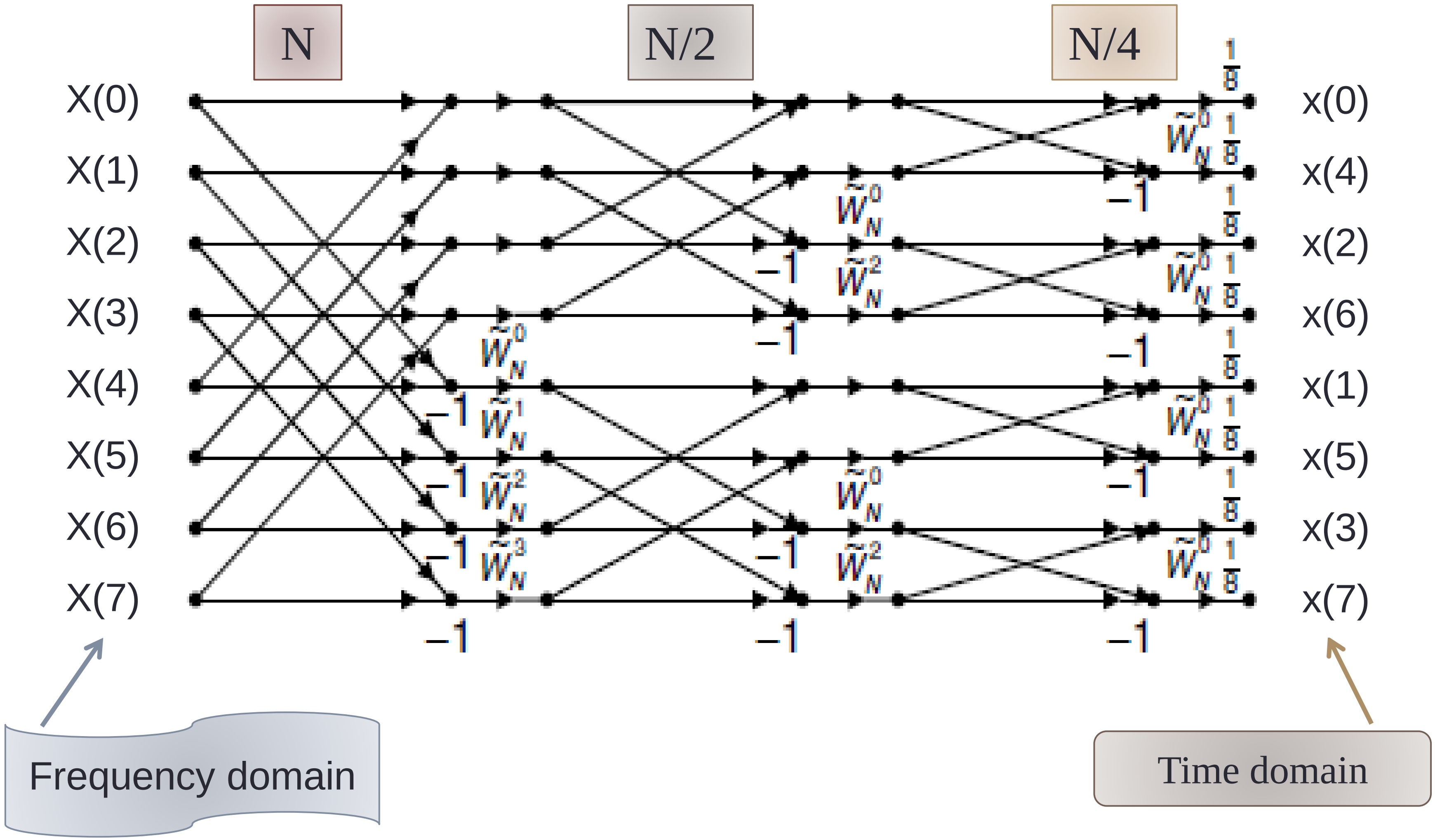
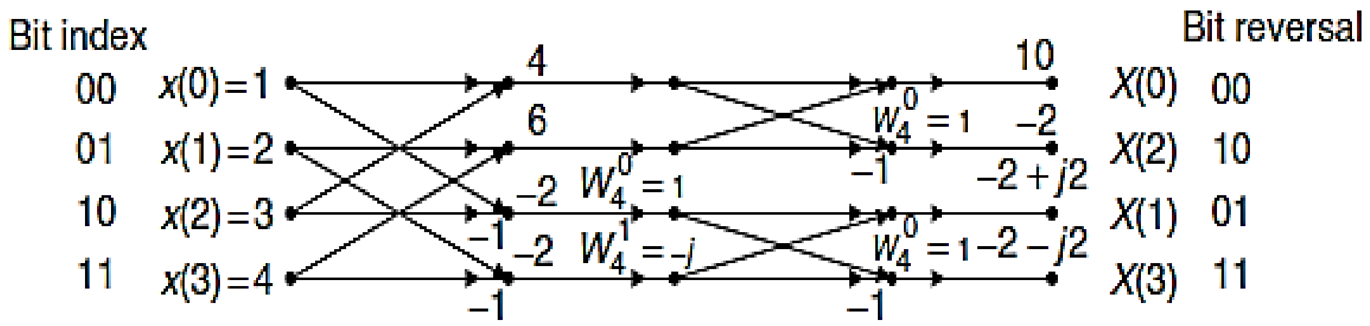


Fig.3 Block diagram for the **reduced DIF IFFT**, **N =8**

- Ex.9. Given a sequence x(n) for $0 \leq n \leq 3$, where x(0) = 1, x(1) = 2, x(2) = 3, and x(3) = 4,Evaluate its DFT X(k) using the decimation-in-frequency FFT method
- Solution:



- Ex.10. Given the DFT sequence X(k) for $0 \leq k \leq 3$ computed in Ex1. Evaluate its inverse DFT x(n) using the decimation-in-frequency FFT method.
- Solution: