Chapter Three: The Laplace Transform- Part II

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Outlines:

Part I:

- ❑Laplace Transform definition
- ❑Laplace Transform of Some Functions
- ❑Properties of Laplace Transform
- ❑Initial and final value theorem
- ❑Laplace Transform of Periodic Function

Part II

- ❑Inverse of Laplace Transform
- ❑Application of Laplace Transform

Inverse of Laplace Transform

The inverse Laplace transform of $F(s)$ is $f(t)$, i.e:

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{ts}ds \qquad \ldots.(17)$$

Where $\mathcal{L}^{-1}$ is the inverse Laplace transform operator. It can be found by:

1. Using Properties.
2. Partial fraction method.
3. The convolution theorem.

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1. Inverse of LT using Properties

Ex14: Determine $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$ its

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \frac{1}{3}\sin 3t$$

Ex15: Determine $\mathcal{L}^{-1}\left\{\frac{3}{s^2-7}\right\}$ its

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2-7}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2-(\sqrt{7})^2}\right\}$$

$$= \frac{3}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2-(\sqrt{7})^2}\right\}$$

$$= \frac{3}{\sqrt{7}}\sinh\sqrt{7}t$$

See Table (1)

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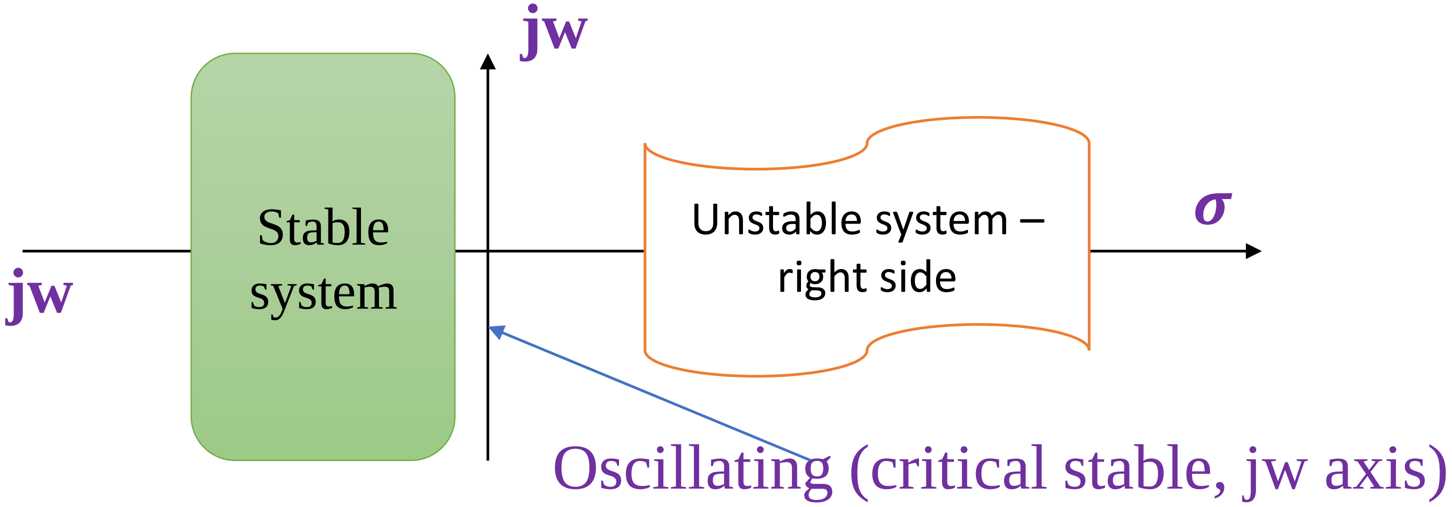
2. Inverse of LT using partial fraction method

F(s) is a rational function, i.e., :

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0} \quad ....(18)$$

Where, Zeros are the roots of the numerator, and Poles are the roots of the denominator. The system is stable if the poles lie on the left side of S-plane.

S-plane, s = **σ+ jw**



$$F(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s \pm a_1)(s \pm a_2) \ldots (s \pm a_n)(s \pm a)^r \, [(s+a)^2 + b^2]} \qquad \ldots.(19)$$

Distinct roots

Repeated roots

Complex conjugate roots

2.1 To find Distinct roots:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{(s - a_1)} + \frac{A_2}{(s - a_2)} + \cdots + \frac{A_n}{(s - a_n)} \qquad \ldots.(20)$$

$$A_i = \lim_{s \to a_i} (s - a_i) \, F(s), \qquad i = 1, 2, \ldots., n \qquad \ldots.(21)$$

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2.2 To find repeated roots:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_r}{(s-a)^r} + \cdots + \frac{A_2}{(s-a)^2} + \frac{A_1}{(s-a)} \qquad \ldots.(22)$$

$$A_{r-i} = \frac{1}{(i)\,!} \lim_{s \to a} \frac{d^i}{ds^i} (s-a)^r F(s)$$ *it is called residue theorem* …(23)

where r =highest order of the pole (degree), $i = 0,1,2,\ldots,r$-1

2.3. To find complex conjugate roots

$$F(s) = \frac{Es+D}{(s+a)^2+b^2} \qquad \ldots(24)$$

$$\mathcal{L}^{-1} \frac{Es+D}{(s+a)^2+b^2} = \frac{1}{b} e^{-at}\{G_I \cos bt + G_R \sin bt\} \qquad \ldots.(25)$$

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The real and imaginary values can be found by :

$$G_R + jG_I = \lim_{s \to -a+jb} \{(s+a)^2 + b^2\} F(s) \qquad \ldots(26)$$

Ex 16: Find y(t) if $Y(s) = \frac{s}{(s+2)^2\,(s^2+2s+10)}$

$$\mathcal{L}^{-1} \frac{s}{(s+2)^2\,(s^2+2s+10)} = \frac{A_2}{(s+2)^2} + \frac{A_1}{(s+2)} + \frac{Es+D}{(s+1)^2+9}$$

Using eq. (23) for repeated roots. The highest root has r =2 degree, i = 0,1

$$A_2 = A_{2-0} = \frac{1}{0\,!} \lim_{s \to -2} \frac{d^0}{ds^0}\,(s+2)^2 \frac{s}{(s+2)^2\,(s^2+2s+10)} = \frac{-1}{5}\,, \qquad i = 0$$

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$$A_1 = A_{2-1} = \frac{1}{1\,!} \lim_{s \to -2} \frac{d^1}{ds^1} (s+2)^2 \frac{s}{(s+2)^2\,(s^2+2s+10)} \quad , \qquad i = 1$$

$$A_1 = \lim_{s \to -2} \frac{(s^2+2s+10) - s(2s+2)}{(s^2+2s+10)^2} = \frac{3}{50}$$

Using equations (25) &(26) to find the complex conjugate roots:

$$\mathcal{L}^{-1} \frac{Es + D}{(s+1)^2 + 9} = \frac{1}{3} e^{-t} \{G_I \cos 3t + G_R \sin 3t\}$$

$$G_R + jG_I = \lim_{s \to -1+j3} \{(s+1)^2 + 9\} \frac{s}{(s+2)^2\,(s^2+2s+10)} = \frac{13}{50} - j\frac{9}{50}$$

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$$\mathcal{L}^{-1}\frac{s}{(s+2)^2\,(s^2+2s+10)} = \mathcal{L}^{-1}\{\frac{-1/5}{(s+2)^2}+\frac{3/50}{(s+2)}\} + \frac{1}{3}e^{-t}\left\{\frac{-9}{50}\cos 3t + \frac{13}{50}\sin 3t\right\}$$

$$\therefore y(t) = \frac{-1}{5}t\,e^{-2t} + \frac{3}{50}\,e^{-2t} + \frac{1}{3}e^{-t}\left\{\frac{-9}{50}\cos 3t + \frac{13}{50}\sin 3t\right\}$$

3. The convolution theorem.

If the Laplace Transform of two functions *F*(s) and *G*(s) exist, then the product *H*(s)=*F*(s). *G*(s) is the transform of h(t) given by:

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$, Note : * or ⊗ denote Convolution …(27)

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Ex.17: If $H(s) = \frac{1}{(s^2+w^2)^2}$. Find *h(t)*

$\mathcal{L}^{-1}\frac{1}{(s^2+w^2)} = \sin wt$, now using eq.(27) , then:

$$h(t) = \frac{\sin wt}{w} \otimes \frac{\sin wt}{w} = \frac{1}{w^2}\int_0^t \sin w\tau \sin w(t-\tau)d\tau$$

$$= \frac{1}{2w^2}\int_0^t [-\cos wt + \cos(2w\tau - wt)]\, d\tau$$

$$= \frac{1}{2w^2}\left[-\tau\, coswt + \frac{\sin w\tau}{w}\right]_{\tau=0}^{t} = \frac{1}{2w^2}\left[-t\cos wt + \frac{\sin wt}{w}\right]$$

Note:

Let $A = t - \tau$, then

$\sin w\tau \sin wA =$

$\frac{1}{2}\{\cos(w\tau - wA) - \cos(w\tau + wA)\}$

$= \frac{1}{2}\{\ \cos[w\tau - w(t-\tau)]$

$- \cos[w\tau + w(t-\tau)]\ \}$

$= \frac{1}{2}\{\cos(2w\tau - wt) - coswt\}$

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$$\mathcal{L}^{-1}H(s) = \mathcal{L}^{-1}\{F(s).G(s)\} = h(t) = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

multiplication in s-domain = Convolution in time

Application of Laplace Transform

1. LT applied to differential equation.

2. Simultaneous equations.

3. RLC circuit with initial condition

1. LT applied to differential equation (DE)

Ex18. Solve the following DE using LT: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 9, \quad y(0) = y'(0) = 0$

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$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dx}\right\} = \mathcal{L}\{9\}$$

Hence, $[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] - 3[s\mathcal{L}\{y\} - y(0)] = \dfrac{9}{s}$

$y(0) = 0$ and $y'(0) = 0$

Hence, $s^2\mathcal{L}\{y\} - 3s\mathcal{L}\{y\} = \dfrac{9}{s}$

Rearranging gives: $(s^2 - 3s)\mathcal{L}\{y\} = \dfrac{9}{s}$

i.e. $\mathcal{L}\{y\} = \dfrac{9}{s(s^2 - 3s)} = \dfrac{9}{s^2(s-3)}$

$$y = \mathcal{L}^{-1}\left\{\frac{9}{s^2(s-3)}\right\}$$

Let $\dfrac{9}{s^2(s-3)} \equiv \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-3} \equiv \dfrac{A(s)(s-3) + B(s-3) + Cs^2}{s^2(s-3)}$

Hence, $9 \equiv A(s)(s-3) + B(s-3) + Cs^2$

When $s = 0$, $9 = -3B$, from which, $B = -3$

When $s = 3$, $9 = 9C$, from which, $C = 1$

Equating s^2 terms gives: $0 = A + C$, from which, $A = -1$, since $C = 1$

Hence, $\mathscr{L}^{-1}\left\{\frac{9}{s^2(s-3)}\right\} = \mathscr{L}^{-1}\left\{-\frac{1}{s} - \frac{3}{s^2} + \frac{1}{s-3}\right\}$

$$\mathbf{y(t)} = -1 - 3x + e^{3x}$$

2. Simultaneous Differential equations.

Ex19. Find y(t) by solving the following Differential equations using LT:

$$y' + 2y + 6\int_0^t z\,dt = -2\,u(t)$$

$$y' + z' + z = 0\ , \qquad y(0) = y_0 = -5, \qquad z(0) = z_0 = 6$$

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Taking LT of both equations and substituting by I.C:

$$(s\,Y(s)+5)+2\,Y(s)+\frac{6}{s}\,Z(s)=-\frac{2}{s} \qquad (1)$$

$$(s\,Y(s)+5)+(s\,Z(s)-6)+Z(s)=0 \qquad (2)$$

Multiplying eq(1) by s and rearrange both equations:

$$(s^2+2s)\,Y(s)+6Z(s)=-2-5\,s \qquad (3)$$

$$s\,Y(s)+(s+1)Z(s)=1 \qquad (4)$$

Using Crammers rule (see Appendix A) then:

$$Y(s)=\frac{\begin{vmatrix}-2-5s & 6\\ 1 & s+1\end{vmatrix}}{\begin{vmatrix}s^2+2s & 6\\ s & s+1\end{vmatrix}}=\frac{-5s^2-7s-8}{s(s+4)(s-1)}=\frac{A}{s}+\frac{B}{s+4}+\frac{C}{s-1} \qquad \ldots.(5)$$

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Applying eq.(20) and eq.(21), the values of A= 2, B = -3, and C = $-$ 4. substituting them in eq.(5):

$$Y(s) = \frac{2}{s} + \frac{-3}{s+4} + \frac{-4}{s-1}$$

Then:

$$y(t) = 2\,u(t) - 3e^{-4t} - 4e^{t}$$

To check, substitute by t = 0 in $y(t)$ to find $y(0)$ which should be equal to -5

H.W: Find $z(t)$

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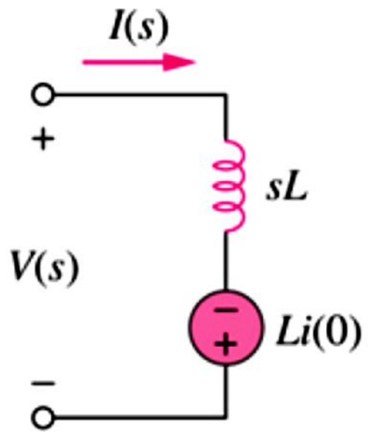
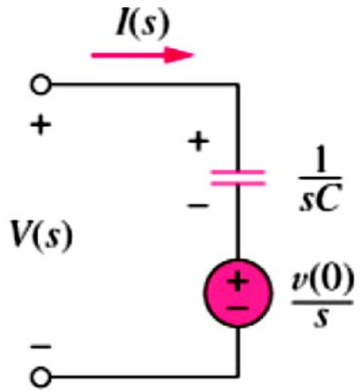
3. RLC circuit with initial condition



$$v_L(t) = L\frac{di_L(t)}{dt}$$

Taking the Laplace transform

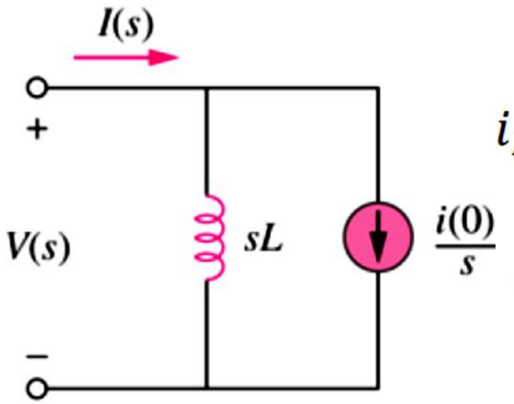
$$V_L(s) = (sL)I_L(s) - Li_L(0)$$



$$v_c(t) = \frac{1}{C}\int_0^t i_c(t)dt + v_c(0)$$

Taking the Laplace transform

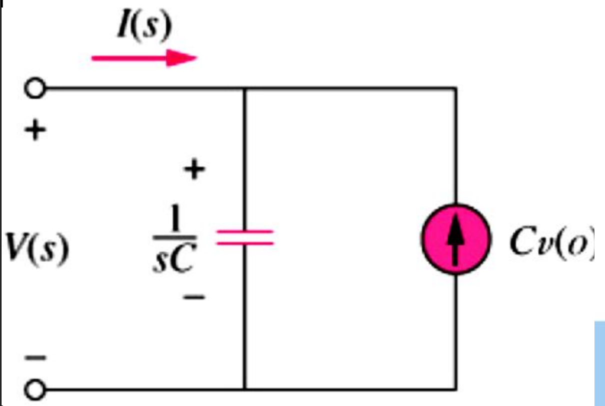
$$V_c(s) = \frac{1}{sC}I_c(s) + \frac{v_c(0)}{s}$$



$$i_L(t) = \frac{1}{L}\int_0^t v_L(t)dt + i_L(0)$$

Taking the Laplace transform

$$I_L(s) = \frac{V_L(s)}{sL} + \frac{i_L(0)}{s}$$



$$i_c(t) = C\frac{dv_c(t)}{dt}$$

Taking the Laplace transform

$$I_c(s) = \frac{V_c(s)}{1/sC} - Cv_c(0)$$

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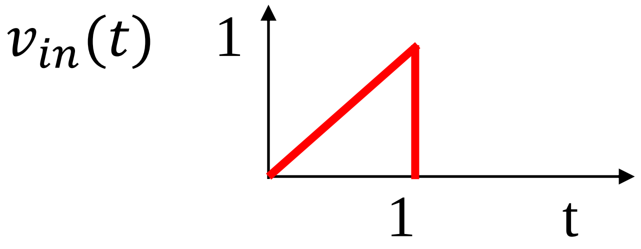
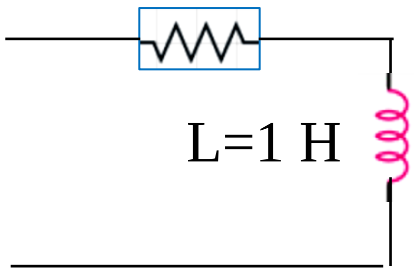
Ex20: Find *i(t)* for the circuit shown if $i_L(0) = 0\ A$.





$$R\,i + L\frac{di}{dt} = t\{u(t) - u(t-1)\}$$

Taking LT:

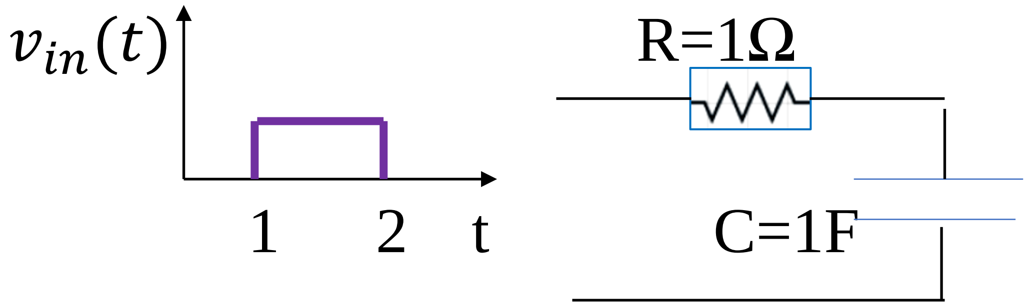
$$R\,I(s) + L\{sI(s) - i(0)\} = \frac{1}{s^2} - e^{-s}\mathcal{L}(t+1)$$

$$(s+1)I(s) = \frac{1}{s^2} - e^{-s}\left\{\frac{1}{s^2} + \frac{1}{s}\right\} = \frac{1}{s^2} - e^{-s}\left(\frac{1+s}{s}\right)$$

$\therefore\ I(s) = \frac{1}{s^2(S+1)} - \frac{e^{-s}}{s^2}$, using eq.(6) for first term, and eq.(11) for second term where *f(t)=t*

$\therefore\ i(t) = \int_0^t \int_0^t e^{-t}\,dt\,dt$ $-\{$(t-1) u(t-1)$\}$= $t + e^{-t} - 1 - \{(t-1)u(t-1)\}$

HW1: Find *i(t)* for the circuit shown if $v_c(0) = 2\ V$.



Ans:

$$i(t) = 4e^{-(t-1)}u(t-1) - 4e^{-(t-2)}u(t-2) - 2e^{-t}$$

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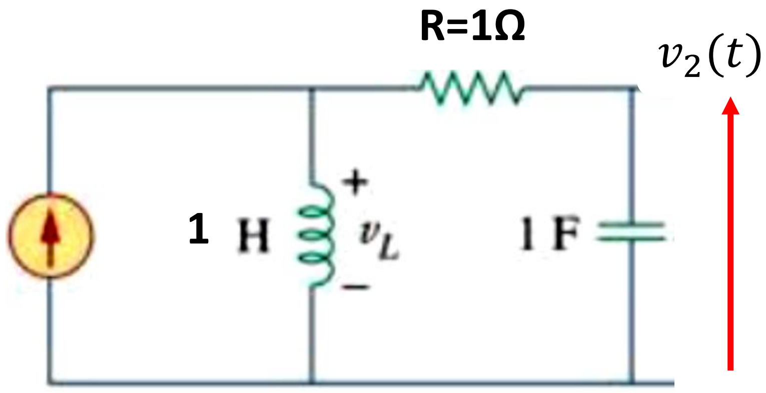
HW2: Find $v_2(t)$ *for the circuit shown if the current source is* $\delta(t), v_c(0) = 2V, i_L(0) = 1A.$



Ans:

$$v_2(t) = 2e^{-0.5t}cos\sqrt{0.75}\,t - \frac{2}{\sqrt{0.75}}e^{-0.5t}sin\sqrt{0.75}\,t$$

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Appendix A

Cramer's Rule for Two Equations

The unique solution to the equations:

$$ax + by = e$$

$$cx + dy = f$$

is given by:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}$$

in which

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \qquad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

If $\Delta = 0$ this method of solution cannot be used.