Chapter_4 The Z-Transform

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The Z-Transform

The z transform of a discrete time signal $x(n)$is defined as the power series:

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (1)$$

- Z-transform inherently concerned with sampled data systems whereas Fourier transform, and Laplace transform were developed and used with analog systems.
- The one-sided Z-transform of a causal sequence
- x(n) is simply defined by:

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \ldots \qquad \ldots(2)$$

Note: $Z = e^{jW}$

Stable system if poles lie inside the unit circle

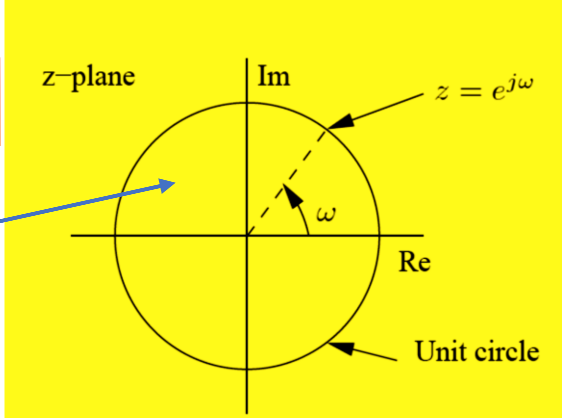


Fig.1 Unit circle Z-Transform

Note: A causal system is one in which the output y(n) at time n depends only on the current input x(n) at time n, and its past input sample values such as x(n -1), x(n- 2), ….

- Z-transform is an infinite power series, it exists only for those values of z for which this series converges. The Region Of Convergence ROC of $X(z)$is the set of all values of z for which $X(z)$ attains a finite value.

- Ex1: Find $Z\{u(n)\}$:

$$Z[u[n]] = \sum_{n=-\infty}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} 1.z^{-n} = 1 + z^{-1} + Z^{-2} + Z^{-3} + \cdots = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

- This series will converge if $\frac{1}{Z} < 1$, or $|Z| > 1$

$$Z[u(n)] \iff \frac{z}{z-1} \qquad \text{ROC } |z| > 1$$

Ex2: Find $Z\{a^n u(n)\}$

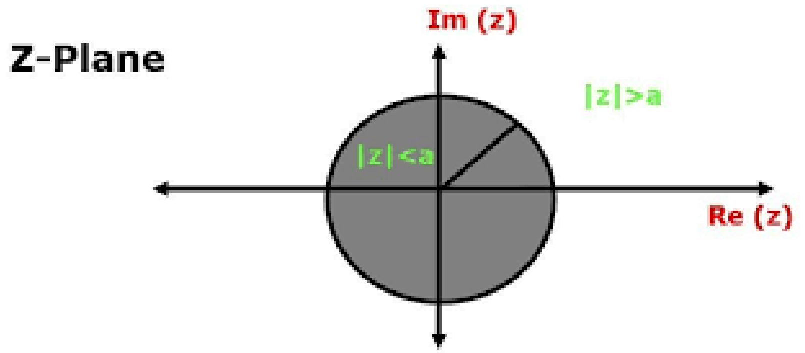
$$Z\{a^n u[n]\} = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

- This series will converge if $\frac{a}{z} < 1$, or $|Z| > a$

$$Z[a^n u[n]] \Longleftrightarrow \frac{z}{z-a}$$

ROC $|z| > |a|$



Ex.3: Find the ROC for the signal $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n}$$

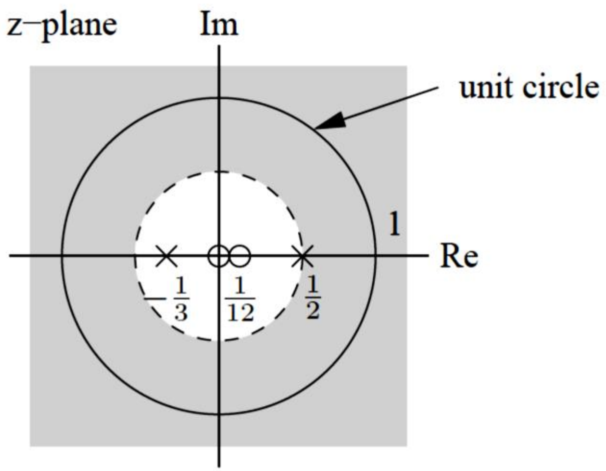
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n .$$

From the example for the right-handed exponential sequence, the first term in this sum converges for $|z| > 1/2$, and the second for $|z| > 1/3$. The combined transform $X(z)$ therefore converges in the intersection of these regions,

namely when $|z| > 1/2$. In this case

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}.$$

The pole-zero plot and region of convergence of the signal is



Z-Transform of some functions

$\boldsymbol{x(n)}$	$\boldsymbol{X(Z)}$	ROC
$\delta(n)$	1	All values of Z
$u(n)$	$\frac{Z}{Z-1}$	$\lvert Z\rvert > 1$
$a^n u(n)$	$\frac{Z}{Z-a}$	$\lvert Z\rvert > a$
$nu(n)$	$\frac{Z}{(Z-1)^2}$	$\lvert Z\rvert > 1$
$e^{-An}u(n)$	$\frac{Z}{Z-e^{-A}}$	$\lvert Z\rvert > e^{-A}$
$r^n \cos(\theta n)$	$\frac{Z^2 - Z\,r\cos(\theta)}{Z^2 - 2Z\,r\,cos(\theta) + r^2}$	$\lvert Z\rvert > r$
$r^n \sin(\theta n)$	$\frac{Z\,r\sin(\theta)}{Z^2 - 2Z\,r\,cos(\theta) + r^2}$	$\lvert Z\rvert > r$

Properties of Z-Transform

Linearity Property

If $x(n) \overset{Z.T}{\longleftrightarrow} X(Z)$

and $y(n) \overset{Z.T}{\longleftrightarrow} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \overset{Z.T}{\longleftrightarrow} a\,X(Z) + b\,Y(Z)$$

Time Shifting Property

If $x(n) \overset{Z.T}{\longleftrightarrow} X(Z)$

Then Time shifting property states that

$$x(n-m) \overset{Z.T}{\longleftrightarrow} z^{-m}X(Z)$$

Multiplication by Exponential Sequence Property

If $x(n) \overset{Z.T}{\longleftrightarrow} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n\,.\,x(n) \overset{Z.T}{\longleftrightarrow} X(Z/a)$$

Differentiation in Z-Domain OR Multiplication by n Property

If $x(n) \overset{Z.T}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \overset{Z.T}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

Convolution Property

If $x(n) \overset{Z.T}{\longleftrightarrow} X(Z)$
and $y(n) \overset{Z.T}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \overset{Z.T}{\longleftrightarrow} X(Z).Y(Z)$$

Initial Value Theorem

For a causal signal xn, the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal xn, the final value theorem states that

$$x(\infty) = \lim_{z \to 1}[z-1]X(z)$$

Ex.4: Find

- $Y(Z) = Z\{y(n)\} = Z\{4\,u(n) + 3\,\delta(n)\}$
- B) $G(Z) = Z\{g(n)\} = Z\{5\,u(n-3)\}$
- C) $R(Z) = Z\{\delta(n) \otimes \delta(n)\}$

Solution:

- A) $Y(Z) = 4\left(\frac{Z}{Z-1}\right) + 3\,(1)$
- B) $G(Z) = Z^{-3}\left\{\left(\frac{Z}{Z-1}\right)\right\}$
- C) $R(Z) = (1)\,.\,(1)$

Ex5.: Find $Y(Z) = Z\{ 2^n u(n) \}$

Solution: $Y(Z) = \left[\frac{Z}{Z-1}\right]_{\frac{Z}{2}} = \frac{Z/2}{\left(\frac{Z}{2}\right)-1}$

Ex.6: Find $Z\{(n-2)\, a^{(n-2)} \cos[w(n-2)]\, u(n-2).$

$$= Z^{-2}\, Z\{ n\, a^n \cos wn\, u(n) \}$$

$$= Z^{-2}\, (-Z) \frac{d}{dZ} Z\{ a^n \cos wn\, u(n) \}$$

$$= -Z^{-1} \frac{d}{dZ} \frac{Z^2 - Z\cos w}{Z^2 - 2Z\cos w + 1} \Big|_{Z \to \frac{Z}{a}}$$

Inverse of Z-Transform

$$x(n) = Z^{-1}\{X(Z)\} \quad \text{...(3)}$$

The inverse z-transform may be obtained using:

1. Inspection method
2. Partial fraction expansion method.
3. The Residue theorem method.

1. Inspection method

Ex.7: Find x(n), if $X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2},$

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad \text{for } |z| > |a|.$$

then $x[n] = \left(\frac{1}{2}\right)^n u[n].$

2. Partial fraction expansion method.

$$X(Z) = \frac{P(Z)}{Q(Z)} = \frac{P(Z)}{(Z-a)(Z-b)(Z-c)\ldots}$$

Put $\frac{X(z)}{z} = \frac{A}{(Z-a)} + \frac{B}{(Z-b)} + \frac{C}{(Z-c)} + \cdots.$

The values of A,B, C,.. Can be found by using the limit of each term. Then:

$$X(Z) = \frac{ZA}{(Z-a)} + \frac{ZB}{(Z-b)} + \frac{ZC}{(Z-c)} + \cdots.$$

EX.8: Find *x*(n) using partial fraction method if:

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})}.$$

$$= \frac{z^2}{(z-1)(z-0.5)}$$

- Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} \quad = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}.$$

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1.$$

- Substituting for A, B

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

$$x(n) = 2u(n) - (0.5)^n u(n).$$

3- The Residue theorem method.

$x(n) = \sum$ residues of $X(Z)\, Z^{n-1}$ at the poles of $X(Z)\, Z^{n-1} = a_{-1} + b_{-1} + c_{-1} + \ldots$

$$a_{-1} = \frac{1}{(m-1)!} \lim_{Z \to a} \frac{d^{m-1}}{dZ^{m-1}} \{(Z-a)^m Z^{n-1} X(Z)\}, \quad m \text{ is the order of the pole}$$

Ex.9 :Find x(n) using the residue theorem method if $X(Z) = \dfrac{2Z}{(Z-1)^2 (Z-2)(Z-3)}$

$$a_{-1} = \frac{1}{1!} \lim_{Z \to 1} \frac{d}{dZ} \frac{2Z\; Z^{n-1}}{(z-2)(Z-3)} = n + \frac{3}{2}$$

$$b_{-1} = \frac{1}{0!} \lim_{Z \to 2} \frac{2\,Z^n}{(Z-1)^2 (Z-3)} = -2\,(2)^n$$

$$c_{-1} = \frac{1}{0!} \lim_{Z \to 3} \frac{2\,Z^n}{(Z-1)^2 (Z-2)} = \frac{1}{2}(3)^n$$

$$x(n) = a_{-1} + b_{-1} + c_{-1} = n + \frac{3}{2} - 2\,(2)^n + \frac{1}{2}(3)^n$$

Solution of Difference Equations (DE) Using the Z-Transform

To solve a difference equation with initial conditions, we must deal with time shifted sequences such as y(n -1), y(n - 2), . . . , y(n - m), and the time shifting property becomes:

$$Z\{x(n-m)\} = Z^{-m}\{X(Z) + \sum_{k=-m}^{-1} x(k)\, Z^{-k}\} \qquad (4)$$

Ex.10: Solve $y(n) - (3/2)\, y(n-1) + (1/2)\, y(n-2) = (1/4)^n$, if $y(-1) = 4$, $y(-2) = 10$ for $n \geq 0$

$$Y(Z) - \frac{3}{2}\{Y(Z) . Z^{-1} + y(-1)\} + \frac{1}{2}\{Z^{-2}\, Y(Z) + Z^{-1}\, y(-1) + y(-2)\} = \frac{Z}{Z - \frac{1}{4}}$$

$$Y(Z)\{1 - \frac{3}{2} Z^{-1} + \frac{1}{2} Z^{-2}\} = \frac{Z}{Z - \frac{1}{4}} + 1 - 2Z^{-1}$$

$$Y(Z) = \frac{Z\,(2Z^2 - \frac{9}{4}Z + \frac{1}{2})}{(Z - \frac{1}{4})\,(Z - \frac{1}{2})\,(Z - 1)}$$

$$Y(Z) = \frac{(1/3)\,Z}{(Z - \frac{1}{4})} + \frac{Z}{Z - \frac{1}{2}} + \frac{(2/3)\,Z}{Z - 1}$$

$$y(n) = \{\frac{1}{3}\,(\frac{1}{4})^n + (\frac{1}{2})^n + \frac{2}{3}\}\,u(n)$$

Relations between systems representation