زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Game theory

Research Project

Submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of BSc. In Mathematics

By

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Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Dedication To

- *My father and mother*
- *My dear supervisor*
- *My brothers and sisters*
- *All who want to read it*

Kazhal Wale Hamad Ali

2024

Acknowledgment

Primarily, I would like to thanks my god for helping me to complete this research with success.

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Abstract

This research project delves into the fascinating realm of game theory, focusing on three key areas: two-person zero-sum games with saddle points, reducing games by dominance, and analyzing matrix strategies in 2x2 games. We explore the fundamental concepts and applications of these topics, highlighting their significance in understanding strategic decision-making in various real-world scenarios. Through theoretical analysis and practical examples, we demonstrate how game theory provides a powerful framework for analyzing competitive situations and predicting optimal strategies for players involved.

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Introduction

Game theory, a branch of applied mathematics, offers a systematic approach to analyzing strategic interactions between rational decision-makers. It provides a framework for understanding how individuals or entities make choices in situations where the outcome depends not only on their own actions but also on the actions of others. This project focuses on three core concepts within game theory:

Two-Person Zero-Sum Games with Saddle Points: These games involve two players with diametrically opposed interests, where one player's gain is the other's loss. The concept of a saddle point, representing an equilibrium where neither player can improve their outcome by unilaterally changing their strategy, is crucial in understanding the optimal strategies in such games.

Reducing Games by Dominance: This technique simplifies complex games by eliminating dominated strategies, which are strategies that are always inferior to other available options regardless of the opponent's actions. By reducing the game's complexity, we can more easily identify optimal strategies and analyze the game's equilibrium.

Matrix Strategies (2x2 Games): This section focuses on analyzing games represented by 2x2 matrices, where each player has two possible strategies. We explore various solution concepts, such as Nash equilibrium and maximin/minimax strategies, to understand the optimal choices for players in these simplified yet insightful games.

Through theoretical explanations and practical examples, we aim to demonstrate the applicability of these concepts in various real-world scenarios, ranging from economics and business to political science and even everyday life situations.

Chapter One

1 Background

This chapter will present some basic definitions related to our research project Rasmusen (2006) Tadelis (2013)

Definition 1.1 Player

each participant is called a player.

Definition 1.2 Strategy

the decision rule by which a player determines his of action is called strategy.

Definition 1.3 Pure Strategy

if a player decides to use only one particular course of action during every play, he is said to use a pure strategy.

Definition 1.4 Mixed Strategy

if a player decides, in advance, to use all or some of these available courses of action in some fixed proportion he is said to use mixed strategy.

Definition1.5 Payoff

Is the outcome of playing the game**.**

Definition 1.6 Payoff matrix

is a table showing the amounts received by player named at the left-hand side after all possible plays of the game.

The payment is made by the player named at the top of the table.

If a player A has m-courses of action and player B has n-courses of action, then the payoff matrix may be appear as follows:

$$
\begin{array}{ccc}
 & & \textbf{Player B} \\
 & & \begin{array}{ccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots & \mathbf{j} & \cdots & \mathbf{n} \end{array} \\
\textbf{Player A} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ . \\ \mathbf{i} \\ . \\ . \\ \mathbf{m} \end{array} & \left(\begin{array}{ccccccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn}
\end{array} \right)
\end{array}
$$

Playoff Matrix

Chapter Two

In this chapter we will present the steps and the procedure of some related topic with several examples Rasmusen (2006) Tadelis (2013)

2.1 Two-Person Zero-Sum Game with Saddle Point

The two-person zero-sum game could be easily solved because of the distribution of values within the game matrix, the basic rules employed in solving such games are described below:

Example (1)

Table below illustrates a game, where A and B are assumed to be equal in ability and intelligence.

Player A	Player B		Minimum of row
	Strategy 1	Strategy 2	
Strategy 1	+4	+6	4 (Max)
Strategy 2	+3	+5	3
Maximum of column	4 (Min)	6	

Both competitors know the payoff for every possible strategy. It should be noted that the game favors competitor A since al values are positive. Values that favor B would be negative. Based upon these conditions, game is against B. However, since B must play the game, he will play to minimize his losses.

A plays strategy 1and wins the highest game value, while B plays strategy 1 and minimize his losses. The game value must be 4 since A wins 4 points while B losses 4 points each time the game is played. This is called Two-Person Zero-Sum game.

Example (2)

In a certain game, player A has three possible choices L, M, and N while player B has two possible choices P and Q. Payments are to be made according to the choices made.

Choices	**Payment**
L, P	A pays B 3
L, Q	B pays A 3
M, P	A pays B 2
M, Q	B pays A 4
N, P	B pays A 2
N, Q	B pays A 3

What are the value of the game for A and B?

Player A Plans (Choices)	Player B Plans (Choices)		Minimum of Row
	P	Q	
L	-3	3	-3
M	-2	4	-2
N	2	3	2 (max)
Maximum of Column	2 (min)	4	

Player A selects plan (N) to maximizing his minimum gain. Player B wants to minimize his losses. He realizes that if he plays his first pure strategy P, he can lose no more than max = 2.

In the above example:

minimax value = maximin value = 2.

The game value(V) =2, and the game has a saddle point.

So when minimax value = maximin value, the corresponding pure strategies are called optimal strategies and the game is said to have a saddle point or equilibrium point.

2.1.1 Steps to detect a saddle point :-

a) At the right of each row, write the row minimum and ring the largest of them.

b) At the bottom of each column, write the column maximum and ring the smallest of them.

c) If these two elements are the same, the cell where the corresponding row and column met is a saddle point and the element in that cell is the value of the game.

d) if the two ringed elements are unequal, there is no saddle point, and the value of the game lies between these two values.

e) If there are more than one saddle points, then there will be more than one solution, each solution corresponding to each saddle point.

Example (3)

Find the best strategies for the following games.

1-

Player A Plans (Choices)	Player B Plans (Choices)		Minimum of Row
	1	2	
1	6	4	4 (max)
2	-4	-6	-6
3	-8	-10	-10
Maximum of Column	6	4 (min)	

Strategies for player A, row 1 and Player B, Column 2, Saddle point (1,2) Game value =4.

2-

Player A	Player B			Minimum of Row
	B_1	B_2	B_3	
A_1	6	18	16	6(max)
A_2	-14	10	-16	-16
A_3	5	0	18	0
Maximum of Column	6 (min)	18	18	

Strategies for player A, row 1 and for Player B, column 1, Saddle point (1,1), Game value=6.

2.2 Reduce Game by Dominance Osborne (1994) Myerson (1991)

If no pure strategies exist, the next step is to eliminate certain strategies (rows and / or columns) by dominance. The principle of dominance states that if one strategy of a player dominates over the other strategy in all conditions then the later strategy can be ignored.

The concept of dominance is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist. Generally, the dominance property is used to reduce the size of a large payoff matrix.

2.2.1 The rules as follows:

- If all the elements of a column (say i_{th} column) are greater than or equal to the corresponding elements of any other column (say j_{th} column), then the i_{th} column is dominated by the j_{th} column and can be deleted from the matrix.
- If all the elements of a row (say i_{th} row) are less than or equal to the corresponding elements of any other row (say j_{th} row), then the i_{th} row is dominated by the j_{th} row and can be deleted from the matrix.

Example (4)

Reduce the following game by dominance

Colors Chosen by Player P	Colors Chosen by Player Q		
	W	B	R
W	0	-2	7
B	2	5	6
R	3	-3	8

Colors Chosen by Player P	Colors Chosen by Player Q			Minimum of Row
	W	B	R	
W	0	-2	7	-2
B	2	5	6	$2_{(max)}$
R	3	-3	8	-3
Maximum of Column	$3_{(min)}$	5	8	

From the above table, minimax value ≠ min max value, this matrix has no saddle point.

The dominance rule for columns is: Every value in the dominating column(s) must be less than or equal to the corresponding value of the dominated column.

The values in column W and B are less than the values of column R, Column R deleted.

Player P	Player Q	
	W	B
W	0	-2

B	2	5
R	3	-3

From the Table Below:

Colors Chosen by Player P	Colors Chosen by Player Q		Minimum of Row
	W	B	
W	0	-2	-2
B	2	5	$2_{(max)}$
R	3	-3	-3
Maximum of Column	$3_{(min)}$	5	

minimax value ≠ min max value, this matrix has no saddle point.

The dominance rule for columns is: Every value in the dominating column(s) must be greater than or equal to the corresponding value of the dominated row.

The values in row B are greater than the values of the row W, row W deleted

Player P	Player Q	
	W	B
B	2	5
R	3	-3

Now we can solve this game as (2x2) game.

<u>Note:</u> that any strategy can be dominated if it is inferior to an average of two or more other pure strategies.

Example (5):

Reduce the following game by dominance.

Player A	Player B				Minimum of Row
	B_1	B_2	B_3	B_4	
A_1	3	2	4	0	0
A_2	3	4	2	4	$2_{(max)}$
A_3	4	2	4	0	0
A_4	0	4	0	8	0
Maximum of Column	$4_{(min)}$	4	4	8	

The matrix has no saddle point, by reducing the size of the matrix, for player A, row A_1, is dominated by row A_3, the reduced matrix will be:

Player A	Player B			
	B_1	B_2	B_3	B_4
A_2	3	4	2	4
A_3	4	2	4	0
A_4	0	4	0	8

The player B, column B_1 is dominated by column B_3, the reduced matrix will be

Player A	Player B		
	B_1	B_3	B_4
A_2	4	2	4
A_3	2	4	0

A_4	4	0	8

In the above matrix no single row (or column) dominates the other row (or column).

However, column B_2 is dominated by the average of columns B_3 and B_4 which equal to:

$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Hence column B_2 is deleted

Player A	Player B	
	B_3	B_4
A_2	2	4
A_3	4	0
A_4	0	8

Now row A_2 is dominated by the average of A_3 and A_4 rows which is equal to

$$\begin{pmatrix} 2 & 4 \end{pmatrix}$$

Hence row A_2 is deleted.

Now the matrix will be

Player A	Player B	
	B_3	B_4
A_3	4	0
A_4	0	8

The result will be 2x2 matrix, this matrix has no saddle point, to find optimum strategies and the game value, the matrix can be solved using arithmetic method.

2.3 Matrix Strategies (2x2 Games)

in case, where, there is no saddle point and dominance has been used to reduce the game matrix, arithmetic method are used for finding optimum strategies for a 2×2 game.

2.3.1 Arithmetic Method:-

It consists of the following steps:

a) Subtract the two digits in column 1and write them under column 2, ignoring. sign.
b) Subtract the two digits in column 2 and write them under column 1, ignoring sign.
c) Similarly proceed for the two rows.

These values are called oddments. They are the frequencies with which the players must use their courses of action in their optimum strategies. This method will described in the following example:

Example (6)

in a game of matching coins, player A wins 2 points if there are two heads, wins nothing if there are two tails and losses 1 point when there are one head and one tail. Determine the payoff matrix, and the best strategies for each player and the value of the game.

$$V_{BH}\ B_1 = \frac{A_{11}x + a_{21}v}{x+y}$$

$$V_{BT}\ B_2 = \frac{a_{12}x + a_{22}V}{x+y}$$

$$V_{AH}^{A_2} = \frac{a_{12}u + a_{22}v}{u+v}$$

a_{11}	a_{12}	X
a_{21}	a_{22}	Y

U	V

Solution

Player A	Player B		Minimum of Row
	B_1	B_2	
A_1	2	-1	-1
A_2	-1	0	-1
Maximum of Column	2	0	

since there is no saddle point, the optimal strategies will me mixed strategies.

Using the steps described before, we get:

Player A	Player B			
	B_1	B_1		
A_1	2	-1	1	¼=25%
A_2	-1	0	3	¾=75%
	1	3		
	¼=25%	¾=75%		

Player A should use strategy H for 25% of the time and strategy T for 75% of the time.

Player B should use strategy H for 25% of the time and strategy T for 75% of the time.

To obtain the value of the game,

Using A's oddments.

B plays H, the value of the game $V = \frac{(2)(1)-(1)(3)}{3+1} = -\frac{1}{4}$

B plays T, the value of the game $V = \frac{(-1)(1)+(0)(3)}{3+1} = -\frac{1}{4}$

Using B's oddments.

A plays H, the value of the game $V = \frac{(2)(1)-(1)(3)}{3+1} = -\frac{1}{4}$

A plays T, the value of the game $V = \frac{(-1)(1)+(0)(3)}{3+1} = -\frac{1}{4}$

Thus, the full solution of the game is

$$A(1,3), B(1,3), V = -\frac{1}{4}$$

A gains $-\frac{1}{4}$, $i.e$, he losses $\frac{1}{4}$ which B, in turn gets.

2.3.1.1 Mixed Strategies ($2 \times M$ and $M \times 2$)

Games where one player has only two courses of action while the other has more than two, are called 2XM or MX2 games.

1.

Player A	Player B	
	B_1	B_1
A_1	10	2
A_2	-3	7
A_3	2	12
A_4	4	-5

M * 2

4 * 2

2.

Player A	Player B		
	B_1	B_1	B_3
A_1	7	-1	3
A_2	8	0	-3

2 * M

2*3

If these games do not have a saddle point or are reducible by the dominance method, it can be still solved by algebraic method (sub-games method) and graphical method these methods are illustrated by the following examples.

Games with mixed strategies can be solved with algebraic method (sub-games method) as follows:

Example (7)

For the game below, determine the best strategies for player A, and B and the game value.

Player A	Player B				Minimum of Row
	B_1	B_1	B_3	B_4	
A_1	1	3	-3	7	-3
A_2	2	5	4	-6	$-6_{(max)}$
Maximum of Column	$2_{(min)}$	5	4	7	

There is no saddle point, the game value between $-3 < V > 2$.

1.

Player A	Player B		Minimum of Row
	B_1	B_2	
A_1	1	3	1
A_2	2	5	$2_{(max)}$
Maximum of Column	$2_{(min)}$	5	

The sub-games has a saddle point

2.

Player A	Player B		Minimum of Row
	B_1	B_2	
A_1	1	-3	-3
A_2	2	4	$2_{(max)}$
Maximum of Column	$2_{(min)}$	4	

The sub-games has a saddle point

3.

Player A	Player B		Minimum of Row
	B_1	B_2	
A_1	1	7	$1_{(max)}$
A_2	2	-6	-6
Maximum of Column	$2_{(min)}$	7	

The sub-game has no saddle point and the game value between $1 < V > 2$ using the arithmetic method for this sub-game, we have:

Player A	Player B		
	B_1	B_1	
A_1	1	7	8

A_2	2	-6	6
	13	1	

$$B_1 = \frac{(1)(8) + (2)(6)}{8 + 6} = \frac{10}{7}$$

$$B_4 = \frac{(7)(8) + (-6)(6)}{8 + 6} = -\frac{10}{7}$$

Comparing the result with the strategies B_2, B_3

$$B_2 = \frac{(3)(8) + (5)(6)}{8 + 6} = \frac{27}{7}$$

$$B_3 = \frac{(-3)(8) + (4)(6)}{8 + 6} = 0$$

The best strategy in this step is B_3

4.

Player A	Player B		Minimum of Row
	B_2	B_3	
A_1	3	-3	-3
A_2	5	4	$4_{(max)}$
Maximum of Column	5	$4_{(min)}$	

The sub-game has a saddle point

5.

Player A	Player B		Minimum of Row
	B_3	B_4	
A_1	-3	7	$-3_{(max)}$
A_2	4	-6	-6
Maximum of Column	$4_{(min)}$	7	

The sub-game has no saddle point and the game value between $-3 < V > 4$

Using the arithmetic method for this sub-game, we have:

Player A	Player B		
	B_3	B_4	
A_1	-3	7	10
A_2	4	-6	10
	13	7	

$$B_3 = \frac{(-3)(10) + (4)(10)}{10 + 10} = \frac{1}{2}$$

$$B_4 = \frac{(7)(10) + (-6)(10)}{10 + 10} = \frac{1}{2}$$

Comparing the result with the strategies B_1, B_2

$$B_1 = \frac{(1)(10) + (2)(10)}{10 + 10} = \frac{3}{2}$$

$$B_2 = \frac{(3)(10) + (5)(10)}{10 + 10} = 4$$

The best strategies for A is A_1 and A_2 and the best strategies for player B is B_3 and B_4, the number of playing the strategies for $A(10{,}10)$ and for $B(0{,}0{,}13{,}7)$.

Conclusion

This research project has provided a comprehensive exploration of key concepts within game theory, specifically focusing on two-person zero-sum games, dominance, and matrix strategies. By understanding these concepts, we gain valuable insights into strategic decision-making and the dynamics of competitive situations. Game theory offers a powerful framework for analyzing and predicting behavior in various fields, allowing us to make informed choices and develop optimal strategies in complex environments. Further research can explore more advanced game theory concepts and their applications in diverse fields, contributing to a deeper understanding of strategic interactions and decision-making processes.

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بوختە

ئەم پڕۆژە توێژینەوەیە قووڵ دەبێتەوە لە کایەی سەرنجڕاکێشی تیۆری یارییەکان، تیشک دەخاتە سەر سێ بوارە سەرەکییەکان: یارییەکانی دوو کەسی سفر-کۆمەڵ بە خاڵەکانی زین، کەمکردنەوەی یارییەکان بە باڵادەستی، و شیکردنەوەی ستراتیژییەکانی ماتریکس لە یارییەکانی 2x2. ئێمە بەدواداچوون بۆ چەمکە بنەڕەتییەکان و بەکارهێنانی ئەم بابەتانە دەکەین، تیشک دەخەینە سەر گرنگییەکانیان لە تێگەیشتن لە بڕیاردانی ستراتیژی لە سیناریۆ جیاوازەکانی جیهانی ڕاستەقینەدا. لە ڕێگەی شیکاری تیۆری و نموونەی پراکتیکییەوە، ئێمە نیشان دەدەین کە چۆن تیۆری یارییەکان چوارچێوەیەکی بەهێز بۆ شیکردنەوەی بارودۆخی کێبڕکێ و پێشبینیکردنی ستراتیژییە گونجاوەکان بۆ یاریزانە بەشداربووەکان دابین دەکات.