



زانكۆی سه‌لاحه‌دین - هه‌ولێر  
Salahaddin University-Erbil

# Mathematical Model of Linear Programing Problem

Research Project

Submitted to the department of Mathematics in partial fulfillment of the requirements for the degree of BSc. In Mathematics

**By:**

Samyr Amir Irahim

**Supervised by:**

Dr. Maher A. Nawkhass

April 2024

## Certification of the Supervisors

I certify that this report was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

Signature:



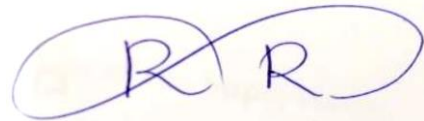
Supervisor: Dr. Maher A. Nawkhass

Scientific grade: Lecturer

Data: / 3/ 2024

In view of the available recommendations, I forward this report for debate by the examining committee.

Signature:



Name: Dr. Rashad Rashed Haji

Scientific grade: Assist. Professor

Chairman of the Mathematics Department

Date:

## ***Dedication To***

- *My father and mother*
- *My dear supervisor*
- *My brothers and sisters*
- *All who want to read it*

**Samyr Amir Irahim**

**2024**

## ***Acknowledgement***

I express my deep sense of gratitude and thanks to ALLAH the Almighty for providing me with strength, health, faith, patience, willing and self-confidence to accomplish this study.

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**Samyr Amir Irahim**

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## **Abstract**

This research explores the application of two established methods, Branch and Bound and Gomory's All-Integer Cutting Plane, for solving Integer Linear Programming (ILP) problems. ILP extends Linear Programming by introducing the constraint that all decision variables must be integers. Finding the optimal solution for ILP problems often proves more challenging than standard LP problems.

This study examines the theoretical foundations of both Branch and Bound and Gomory's Cutting Plane methods. We analyze their strengths and weaknesses in addressing ILP problems. Furthermore, the research aims to implement these methods and compare their performance on a set of benchmark ILP problems. The findings will provide valuable insights into the effectiveness of these methods in practical applications

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## **Introduction**

Linear Programming (LP) is a powerful mathematical optimization technique for solving problems with a linear objective function and a set of linear constraints. It finds extensive applications in various domains, including resource allocation, production planning, scheduling, and financial planning. However, in many real-world scenarios, the decision variables represent discrete choices, like the number of items produced or workers assigned. In such cases, the optimal solution obtained using standard LP methods might not be feasible with integer values for the variables.

This research project delves into exploring two prominent methods for tackling this challenge: the Branch and Bound method and Gomory's All-Integer Cutting Plane method. We aim to investigate the effectiveness of these methods in finding the optimal solution for Integer Linear Programming (ILP) problems, a specific subset of LP where all variables must take integer values.

## Chapter One

### Background

This chapter will present some basic definitions and theorems related to our research project

#### **Definition 1.1. Optimization Problem (OP) (Yang, 2008)**

Optimization is everywhere, from business to engineering design, from planning your holiday to your daily routine. Business organizations have to maximize their profit and minimize the cost. Engineering design has to maximize the performance of the designed product while of course minimizing the cost at the same time. Even when we plan holidays we want to maximize the enjoyment and minimize the cost. Therefore, the studies of optimization are of both scientific interest and practical implications and subsequently the methodology will have many applications .

The optimization problem is expressed as follows:

$$\text{Minimize (or maximize) } f(x) \tag{1.1}$$

$$\text{Subject to } x \in \Omega. \tag{1.2}$$

The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  that wish to minimize (or maximize) is a real-valued function, and is called the objective function, or cost function. The vector  $x$  is an  $n$ -vector of independent variables, that is  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ . The variables  $x_1, x_2, \dots, x_n$  are often referred to as decision variables. The set  $\Omega$  is a subset of  $\mathbb{R}^n$ , called the constraint set or feasible set.

The optimization problem above can be viewed as a decision problem that involves finding the “best” vector  $x$  of the decision variables over all possible vectors in  $\Omega$ . The “best” vector means the one that result in the smallest value of the objective function. This vector is called the minimizer of  $f$  over  $\Omega$ . It is



possible that there may be many minimizers. In this case, finding any of the minimizers will suffice.

There are also optimization problems that require maximization of the objective function. These problems, however, can be represented in above form because maximizing  $f$  is equivalent to minimizing  $-f$  (Chong and Zak, 2004).

The following diagram indicates the branch of optimization problems (Vankova, 2004)

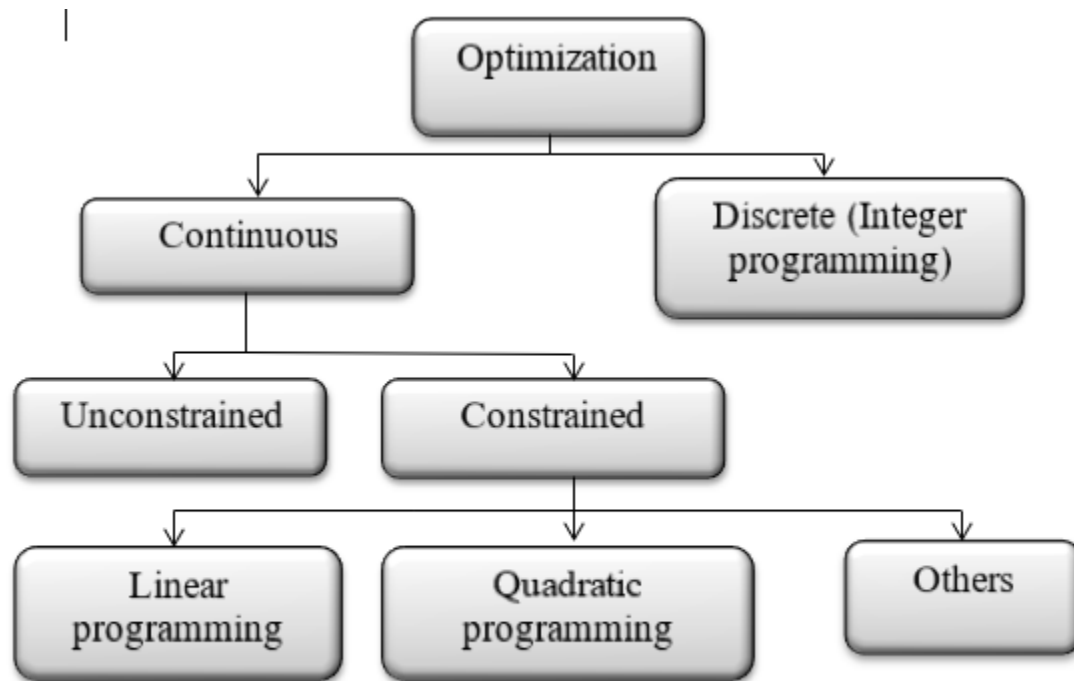


Figure 1.1: Classification of optimization

**Definition 1.2 (Vankova, 2004)**

A mathematical model in operation research is viewed generally as:

$$\text{Optimize } Z = f(x) \tag{1.3}$$

Subject to:

$$Ax \leq b \quad (1.4)$$

$$x \geq 0 \quad (1.5)$$

The function  $f$  is objective function while  $Ax \leq b$  represent the constraint, where  $b$  is known constants. The constricts  $x \geq 0$  are called the non-negative restrictions, which restrict the variables to zero or positive value only.

**Definition 1.3 (Chong and Zak, 2004)**

An objective function subject to constraint equations with nonnegative variables is called initial system.

**Definition 1.4 (STACHO, 2021)**

Slack variable is a variable that is added to the left-hand side of a less than or equal sign to type and convert the constraint into an equality.

**Definition 1.5 (STACHO, 2021)**

Surplus variable is a variable subtracted from the left-hand side of a greater than or equal to type constraint to convert the constraint into equality. It is also known as negative slack variable.

**Definition 1.6 (Taha, 2011)**

An artificial variable is a nonnegative variable added to the left-hand side of each of the equations corresponding to constraints of this types ( $\geq$ ) and ( $=$ ).

The artificial variables only provide a mathematical trick for obtaining a starting solution; the effect of these variables on the final solution is cancelled by high penalty in the objective function.

**Definition 1.7 (STACHO, 2021)**

The goal of linear programming is to determine the values of decision variables that maximize or minimize a linear objective function, where the decision variables are subject to linear constraints. The goal is to find a point that minimizes the objective function and at the same time satisfies the constraints. The point that satisfies the constraints is known as a feasible point. In a linear programming problem, the objective function is linear, and the set of feasible points is determined by a set of linear equations and / or inequalities.

A linear program is an optimization problem of the form

$$\text{Maximize } c^t x \quad (1.6)$$

Subject to

$$Ax = b \quad (1.7)$$

$$x \geq 0 \quad (1.8)$$

where  $c \in R^n$ ,  $b \in R^m$  and  $A \in R^{m \times n}$ .

**Definition 1.8 (Taha, 2011)**

Any set  $x = (x_1, x_2, \dots, x_n)$  of variables is called solution to LPP, if it satisfies the constraints only.

**Definition 1.9 (Winston, 2022)**

Any set  $x = (x_1, x_2, \dots, x_n)$  of variables is called feasible solution to LPP, if it satisfies the constraints and non-negative restrictions.

**Definition 1.10 (Taha, 2011)**

Basic solution is a solution obtained by setting any  $n$  variable (among  $m + n$  variables) equal to zero and solving remaining  $m$  variables provided the

determinant of the coefficients of these  $m$  variables is non-zero. Such  $m$  (any of them may be zero) are called basic variables and remaining  $n$  zero variables are called Non-basic variables.

**Definition 1.11(Winston, 2022)**

A basic feasible solution is a basic solution which also non-negative restrictions that is all basic variables are non-negative. Basic feasible solutions are two types

- a) Non-degenerate. A Non-degenerate basic feasible solution is a basic feasible solution which has exactly  $m$  positive  $x_i$  where  $i=1,2,\dots,m$ . In other words all  $m$  basic variables are positive, and remaining  $n$  variables will be zero
- b) Degenerate. A basic feasible solution is degenerate, if one or more basic variable zero. If an LPP in standard form has  $m$  constraints are  $n$  variables, then maximal number of basic solution is.

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}$$

**Definition 1.12 (Taha, 2011)**

A basic feasible solution is said to be optimum if it optimizes (Max or Min ) the objective function.

**Definition 1.13(Winston, 2022)**

If the value of the objective function can be increased or decreased indefinitely such solutions are called unbounded solution.

## Chapter Two

### 2.1 Branch and Bound Method

The Branch and Bound method developed first by A H Land and A G Doig is used to solve all-integer, mixed-integer and zero-one linear programming problems. The concept behind this method is to divide the feasible solution space of an LP problem into smaller parts called subproblems and then evaluate corner (extreme) points of each subproblem for an optimal solution. The branch and bound method starts by imposing bounds on the value of objective function that help to determine the subproblem to be eliminated from consideration when the optimal solution has been found. If the solution to a subproblem does not yield an optimal integer solution, a new subproblem is selected for branching. At a point where no more subproblem can be created, an optimal solution is arrived at. The branch and bound method for the profit-maximization integer LP problem can be summarized in the following steps:

#### 2.1.1 The Procedure

**Step 1: Initialization** Consider the following all integer programming problem

$$\text{Maximize } Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

subject to the constraints

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \dots$$

(LP-A) ..

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$$

And  $x_j \geq 0$  and non-negative integers.

Obtain the optimal solution of the given LP problem ignoring integer restriction on the variables.

(i) If the solution to this LP problem (say LP-A) is infeasible or unbounded, the solution to the given all-integer programming problem is also infeasible or unbounded, as the case may be.

(ii) If the solution satisfies the integer restrictions, the optimal integer solution has been obtained. If one or more basic variables do not satisfy integer requirement, then go to Step 2. Let the optimal value of objective function of LP-A be  $Z_1$ . This value provides an initial upper bound on objective function value and is denoted by  $Z_U$ .

(iii) Find a feasible solution by rounding off each variable value. The value of objective function so obtained is used as a lower bound and is denoted by  $Z_L$ .

**Step 2: Branching step**

(i) Let  $x_k$  be one basic variable which does not have an integer value and also has the largest fractional value.

(ii) Branch (or partition) the LP-A into two new LP subproblems (also called nodes) based on integer values of  $X_K$  that are immediately above and below its non-integer value. That is, it is partitioned by adding two mutually exclusive constraints.

$$X_K \leq [X_K] \text{ and } X_K \geq [X_K] + 1$$

to the original LP problem. Here  $[x_k]$  is the integer portion of the current non-integer value of the variable  $x_k$ . This is obviously done to exclude the non-integer value of the variable  $x_k$ . The two new LP subproblems are as follows:

<p>LP Subproblem B</p> <p>Max <math>Z \sum_{j=1}^n c_j x_j</math></p> <p>subject to <math>Z \sum_{j=1}^n a_{ij} x_j = b_i</math></p> <p><math>X_K \leq [X_K]</math></p> <p>and <math>X_j \geq 0</math> and <math>X_j \leq 0</math></p>		<p>LP Subproblem C</p> <p>Max <math>Z \sum_{j=1}^n c_j x_j</math></p> <p>subject to <math>Z \sum_{j=1}^n a_{ij} x_j = b_i</math></p> <p><math>X_K \leq [X_K] + 1</math></p>
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**Step 3: Bound step** Obtain the optimal solution of subproblems B and C. Let the optimal value of the objective function of LP-B be  $Z_2$  and that of LP-C be  $Z_3$ . The best integer solution value becomes the lower bound on the integer LP problem objective function value (Initially this is the rounded off value). Let the lower bound be denoted by  $Z_L$ .

**Step 4: Fathoming step** Examine the solution of both LP-B and LP-C

- (i) If a subproblem yields an infeasible solution, then terminate the branch.
- (ii) If a subproblem yields a feasible solution but not an integer solution, then return to Step 2.0
- (iii) If a subproblem yields a feasible integer solution, examine the value of the objective function. If this value is equal to the upper bound, an optimal solution has been reached. But if it is not equal to the upper bound but exceeds the lower bound, this value is considered as new upper bound and return to Step 2. Finally, if it is less than the lower bound, terminate this branch.

**Step 5: Termination** The procedure of branching and bounding continues until no further sub-problem remains to be examined. At this stage, the integer solution corresponding to the current lower bound is the optimal all-integer programming problem solution.

**Remark** The above algorithm can be represented by an enumeration tree. Each node in the tree represents a subproblem to be evaluated. Each branch of the tree creates a new constraint that is added to the original problem.

**Example 1.1.1:** Solve the following all integer programming problem using the branch and bound method.

Maximize  $Z = 2x_1 + 3x_2$

subject to the constraints

(i)  $6x_1 + 5x_2 \leq 25$ ,                      (ii)  $x_1 + 3x_2 \leq 10$

And  $x_1 + x_2 \geq 0$  and integers

[Jammu Univ., BE (Mach.) 2008]

**Solution:** Relaxing the integer conditions, the optimal non-integer solution to the given integer LP problem obtained by graphical method as shown in Fig. 7.4 is:  $x_1 = 1.92$ ,  $x_2 = 2.69$  and  $\max Z_1 = 11.91$ . The value of  $Z_1$  represents initial upper bound as:  $Z_L = 11.91$ . Since value of variable  $x_2$  is non-integer, therefore selecting it to decompose (branching) the given problem into two sub-problems by adding two new constraints  $x_2 \leq 2$  and  $x_2 \geq 3$  to the constraints of original LP problem as follows:

LP Sub-problem B

$$\text{Max } Z = 2x_1 + 3x_2$$

subject to (i)  $6x_1 + 5x_2 \leq 25$ ,

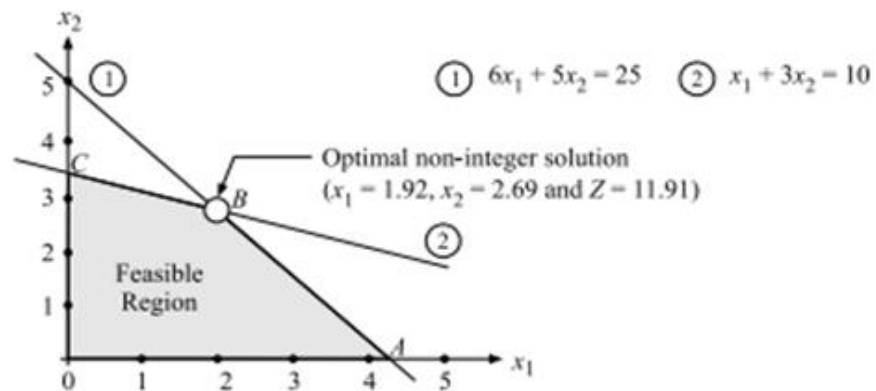
(iii)  $x_2 \leq 2$ ,

LP Sub-problem C

$$\text{Max } Z = 2x_1 + 3x_2$$

subject to (ii)  $x_1 + 3x_2 \geq 10$

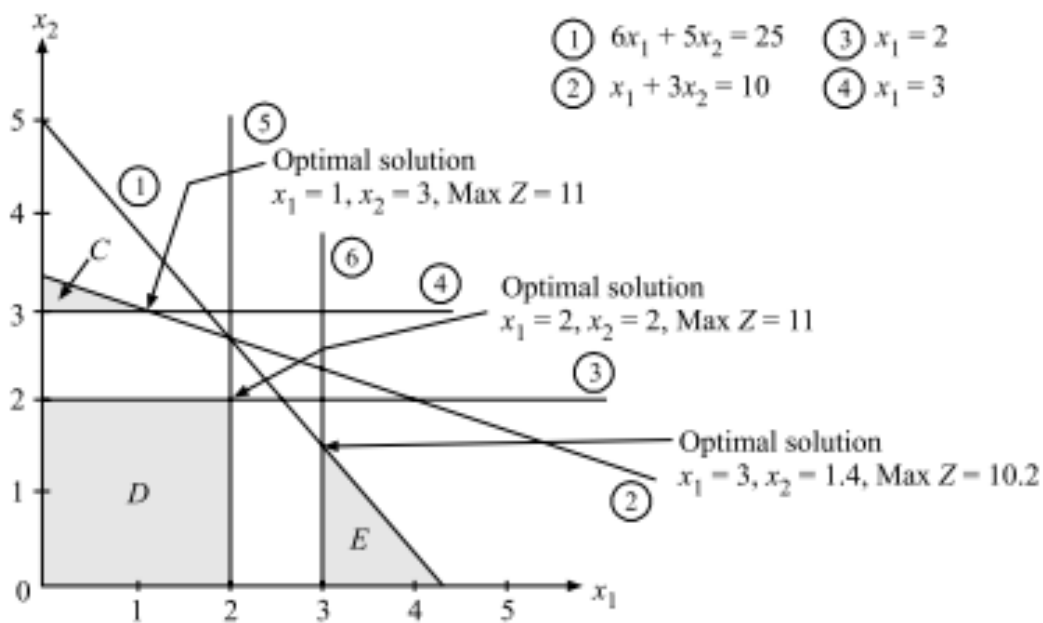
(iii)  $x_2 \geq 3$ ,



Sub-problem B and C are solved graphically as shown in Fig. . The feasible solutions are:

Sub-problem B :  $x_1 = 2.5, x_2 = 2$  and  $\text{Max } Z_2 = 11$

Sub-problem C :  $x_1 = 1, x_2 = 3$  and  $\text{Max } Z_3 = 11$





LP sub-problem *B* is further subdivided into two LP sub-problems *D* and *E* (shown in Fig. ) by taking variable  $x_1 = 2.5$ . Adding two new constraints  $x_1 \leq 2$  and  $x_1 \geq 3$  to sub-problem *B*. Also  $\text{Max } Z = 11$  is also not inferior to the  $Z_L = 11$ .

And  $x_1, x_2 \geq 0$  and integers  
integers

LP Sub-problem B

$$\text{Max } Z = 2x_1 + 3x_2$$

subject to (i)  $6x_1 + 5x_2 \leq 25$ , (ii)  $x_1 + 3x_2 \geq 10$   
 $25$ , (ii)  $x_1 + 3x_2 \leq 10$

(iii)  $x_2 \leq 2$ , (iv)  $x_1 \geq 2$

And  $x_1, x_2 \geq 0$  and integers  
integers

And  $x_1, x_2 \geq 0$  and

LP Sub-problem C

$$\text{Max } Z = 2x_1 + 3x_2$$

subject to (i)  $6x_1 + 5x_2 \leq$

(iii)  $x_2 \leq 2$ , (iv)  $x_1 \geq 3$

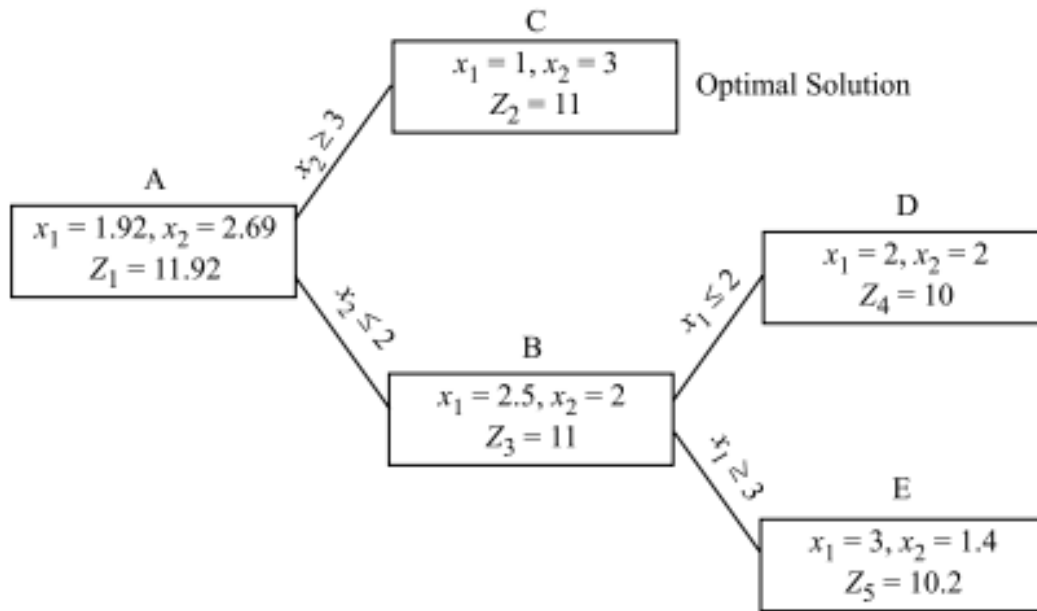
And  $x_1, x_2 \geq 0$  and

Sub-problems *D* and *E* are solved graphically as shown in Fig.(1.1.1) The feasible solutions are:

Sub-problem *D* :  $x_1 = 2, x_2 = 2$  and  $\text{max } Z_L = 10$

Sub-problem *E* :  $x_1 = 3, x_2 = 1.4$  and  $\text{max } Z_5 = 10.2$

The solution of LP sub-problem *D* is satisfying integer value requirement of variables but is inferior to the solution of LP sub-problem *E* in terms of value of objective function,  $Z_5 = 10.2$ . Hence the value of lower bound  $Z_L = 11$  remains unchanged and sub-problem *D* is not considered for further decomposition. Since the solution of sub-problem *E* is non-integer, it can be further decomposed into two sub-problems by considering variable,  $x_2$ . But the value of objective function ( $Z_5 = 10.2$ ) is inferior to the lower bound and hence this does not give a solution better than the one already obtained. The sub-problem *E* is also not considered for further branching. Hence, the best available solution corresponding to sub-problem *C* is the integer optimal solution:  $x_1 = 1, x_2 = 3$  and  $\text{Max } Z = 11$  of the given integer LP problem. The entire branch and bound procedure for the given Integer LP problem is shown in Fig(1.1.1).



**Ex9ample (1.1.2).**: Solve the following all-integer programming problem using the branch and bound method.

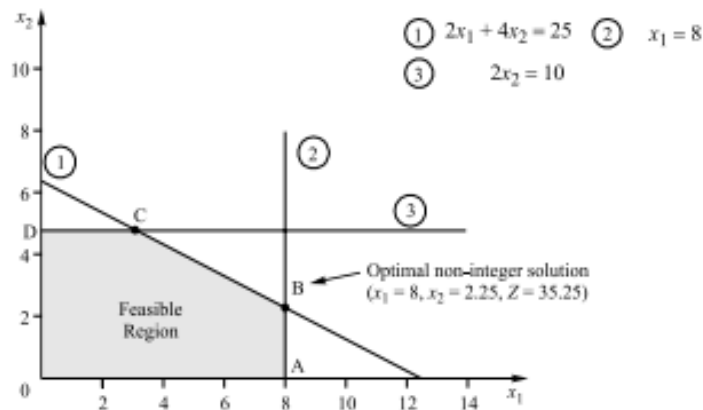
Maximize  $Z = 3x_1 + 5x_2$

subject to the constraints

(i)  $2x_1 + 4x_2 \leq 25$ , (ii)  $x_1 \leq 8$ , (iii)  $2x_2 \leq 10$

and  $x_1, x_2 \geq 0$  and integers

**Solution:** Relaxing the integer requirements, the optimal non-integer solution of the given Integer LP problem obtained by the graphical method, as shown in Fig. 7.8, is:  $X_1 = 8, X_2 = 2.25$  and  $Z_1 = 35.25$ . The value of  $Z_1$  represents the initial upper bound,  $Z_U = 35.25$  on the value of the objective function. This means that the value of the objective function in the subsequent steps should not exceed 35.25. The lower bound  $Z_L = 34$  is obtained by the rounded off solution values to  $X_1 = 8$  and  $X_2 = 2$ . The variable  $X_2 (= 2.25)$  is the non-integer solution value, therefore, it is selected for dividing the given LP-A problem into two subproblems LP-B and LP-C by adding two new constraints:  $X_2 \leq 2$  and  $X_3 \geq 3$  to the constraints of given LP problem as follows.

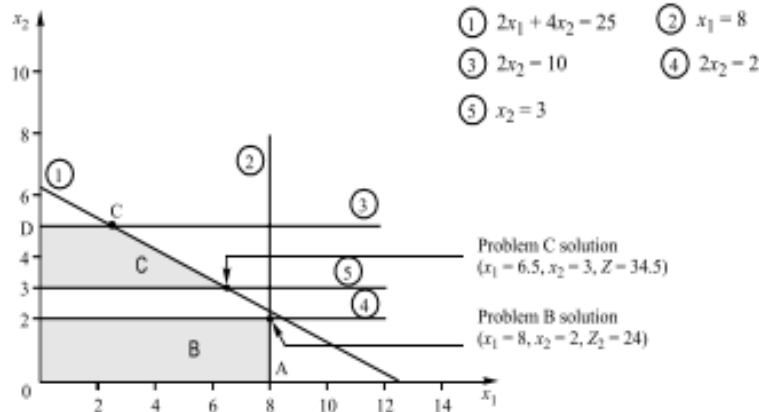


*LP Subproblem B*

Max  $Z = 3x_1 + 5x_2$   
 subject to (i)  $2x_1 + 4x_2 \leq 25$ ,    (ii)  $x_1 \leq 8$   
               (iii)  $2x_2 \leq 10$  (redundant),    (iv)  $x_2 \leq 2$   
 and  $x_1, x_2 \geq 0$  and integers.

*LP Subproblem C*

Max  $Z = 3x_1 + 5x_2$   
 subject to (i)  $2x_1 + 4x_2 \leq 25$ ,    (ii)  $x_1 \leq 8$   
               (iii)  $2x_2 \leq 10$ ,    (iv)  $x_2 \geq 3$   
 and  $x_1, x_2 \geq 0$  and integers.



Subproblems B and C are solved graphically as shown in Fig. (1.1.2). The feasible solutions are:

Subproblem B :  $x_1 = 8, x_2 = 2$  and max  $Z_2 = 34$

Subproblem C :  $x_1 = 6.5, x_2 = 3$ , and Max  $Z_2 = 34.5$

Since solution of the subproblem B is satisfying the integer value requirement of variables but value

of objective function  $Z_2 < Z_3$ , therefore this problem is not considered for further branching. However, if

$Z_3 \leq Z_2$ , then no further branching would have been possible for subproblem C.

The subproblem C is now branched into two new subproblems: D and E, by taking variable,  $x_1 = 6.5$ .

Adding two new constraints  $x_1 \leq 6$  and  $x_1 \geq 7$  to subproblem C. The two subproblems D and E are stated

as follows.

LP Sub-problem B

$$\text{Max } Z = 3x_1 + 5x_2$$

subject to (i)  $2x_1 + 4x_2 \leq 25$ , (ii)  $x_1 \leq 8$  (redundant)  
(ii)  $x_1 \leq 8$

(iii)  $x_2 \leq 10$ , (iv)  $x_2 \geq 2$  (v)  $x_1 \leq 6$

(v)  $x_1 \leq 6$

And  $x_1, x_2 \geq 0$  and integers

integers

LP Sub-problem C

$$\text{Max } Z = 3x_1 + 5x_2$$

subject to (i)  $2x_1 + 4x_2 \leq 25$ ,

(iii)  $x_2 \leq 10$ , (iv)  $x_2 \geq 2$

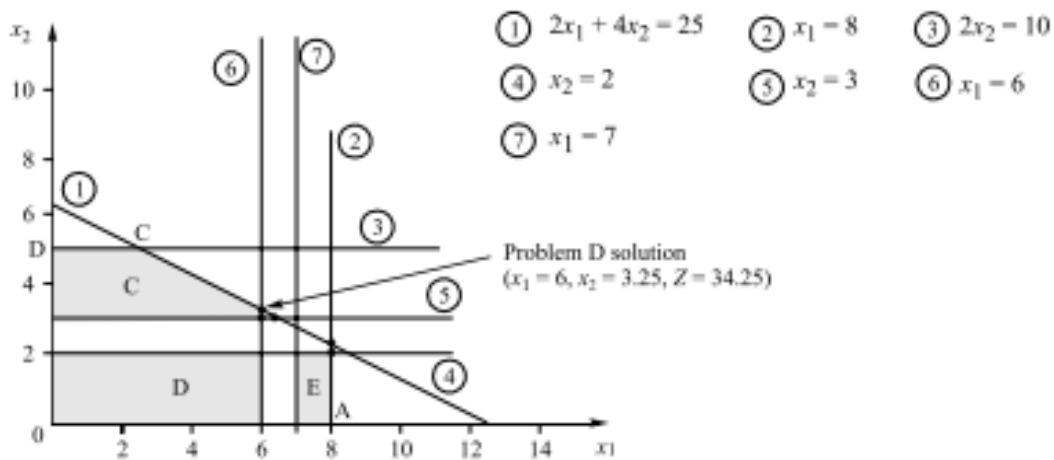
)  $x_1 \geq 6$

And  $x_1, x_2 \geq 0$  and

Subproblems D and E are solved graphically as shown in fig (1.1.2). the feasible solution are

Subproblem D :  $x_1 = 6, x_2 = 3.25$  and  $\text{Max } Z_4 = 34.25$ .

Subproblem E : No feasible solution exists because constraints  $x_1 \geq 7$  and  $x_2 \geq 3$  do not satisfy the first constraint. So this branch is terminated.



The non-integer solution obtained at sub-problem D yields an upper bound of 34.25 instead of 34.50 and also greater than  $Z_2$  (an upper bound for sub-problem B).

Once again we create sub-problems F and G from sub-problem D with two new constraints  $x_2 \leq 3$  and  $x_2 \geq 4$  as shown in fig (1.1.2).

LP Subproblem F

$$\text{Max } Z = 3x_1 + 5x_2$$

subject to (i)  $2x_1 + 4x_2 \leq 25$  (ii)  $x_1 \leq 8$   
 $4x_2 \leq 25$  (iii)  $2x_2$

(iii)  $2x_2 \leq 10$  (redundant) (iv)  $x_2 \geq 3$   
 (redundant)

(v)  $x_1 \leq 6$  (vi)  $x_2 \leq 3$

and  $x_1, x_2 \geq 0$  and integers  
 integers

LP Subproblem G

$$\text{Max } Z = 3x_1 + 5x_2$$

subject to (i)  $2x_1 +$

$\leq 10$ , (iv)  $x_2 \geq 3$

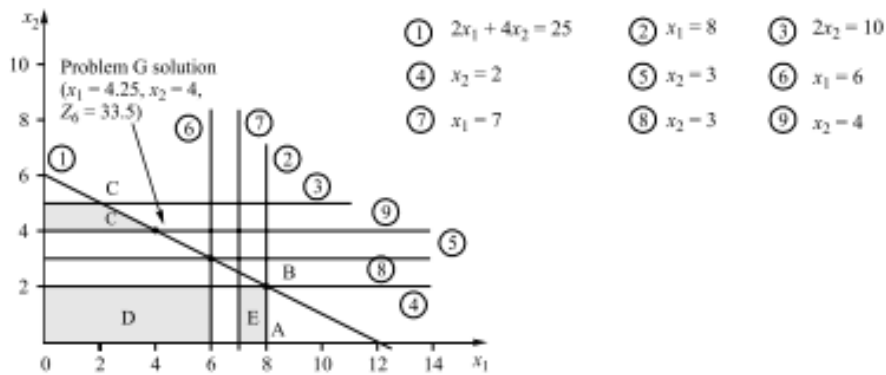
(v)  $x_1 \leq 6$ , (vi)  $x_2 \geq 4$

and  $x_1, x_2 \geq 0$  and

The graphical solution to subproblems F and G as shown in Fig. () is as follows:

Subproblem F :  $x_1 = 6, x_2 = 3$  and Max ,  $Z_5 = 33$ .

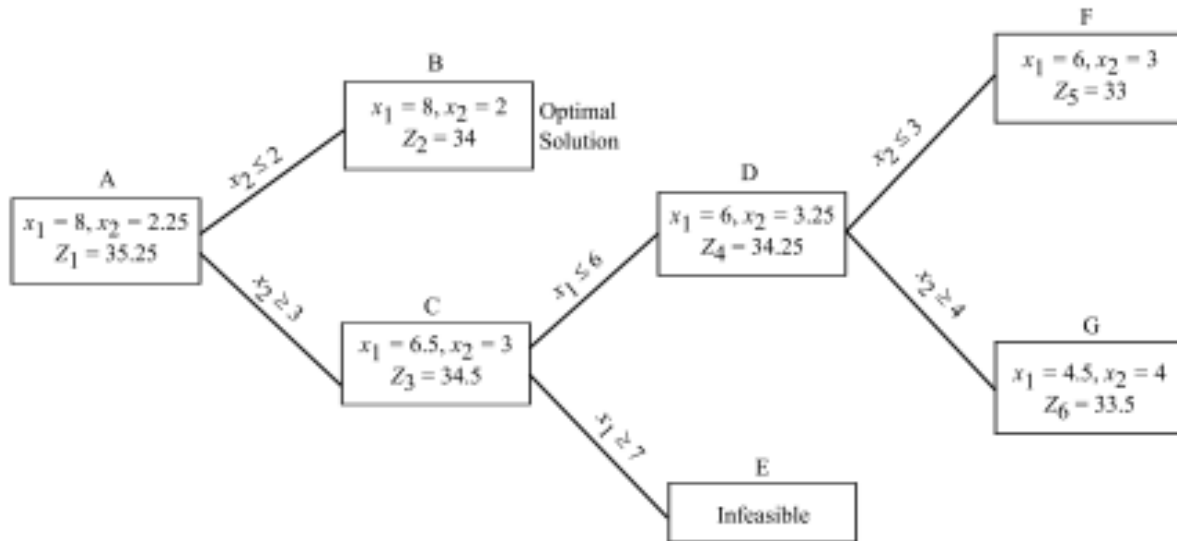
Subproblem G :  $x_1 = 4.25, x_2 = 4$  and Max,  $Z_6 = 33.5$



The solution at node G is non-integer, no additional branching is required from this node because  $Z_6 < Z_4$ . The branch and bound algorithm thus terminated and the optimal integer solution is:  $x_1 = 8, x_2 = 2$  and  $Z = 34$  yielded at node B.

The branch and bound procedure for the given Integer LP problem is shown in Fig. 7.12.

The branch and



**Example(1.1.3).** : Solve the following all-integer programming problem using the branch and bound method

Minimize  $Z = 3x_1 + 2.5x_2$

subject to the constraints

(i)  $x_1 + 2x_2 \geq 20$ , (ii)  $3x_1 + 2x_2 \geq 50$

and  $x_1, x_2 \geq 0$  and integers.

**Solution:** Relaxing the integer requirements, the optimal non-integer solution of the given integer LP

problem, obtained by the graphical method, is:  $x_1 = 15, x_2 = 2.5$  and  $Z_1 = 51.25$ . This value of  $Z_1$  represents

the initial lower bound,  $Z_L = 51.25$  on the value of the objective function, i.e. the value of the objective

function in the subsequent steps cannot be less than 51.25.

The variable  $x_2 (= 2.5)$  is the only non-integer solution value and is therefore is selected for dividing

the given problem into two subproblems: B and C. In order to eliminate the fractional part of  $x_2 = 2.5$ , two

new constraints  $x_2 \leq 2$  and  $x_2 \geq 3$  are created by adding in the given set of constraints as shown below:

LP Subproblem B

$$\text{Max } Z = 3x_1 + 2.5x_2$$

subject to (i)  $x_1 + 2x_2 \geq 20$ , (ii)  $3x_1 + 2x_2 \geq 50$   
 $20$ , (ii)  $3x_1 + 2x_2 \geq 50$

(iii)  $x_2 \leq 2$

and  $x_1, x_2 \geq 0$  and integers.  
 integers

LP Subproblem C

$$\text{Max } Z = 3x_1 + 2.5x_2$$

subject to (i)  $x_1 + 2x_2 \geq$

(iii)  $x_2 \geq 3$

and  $x_1, x_2 \geq 0$  and

Subproblems B and C are solved graphically. The feasible solutions are:

Subproblem B :  $x_1 = 16, x_2 = 2$  and  $\text{Min } Z_2 = 53$ .

Subproblem C :  $x_1 = 14.66, x_2 = 3$  and  $\text{Min } Z_3 = 51.5$ .

Since the solution of subproblem B is all-integer, therefore no further decomposition (branching) of this subproblem is required. The value of  $Z_2 = 53$  becomes the new lower bound. A non-integer solution of subproblem C and also  $Z_3 < Z_2$  indicates that further decomposition of this problem need to be done in order to search for a desired integer solution. However, if  $Z_3 \geq Z_2$ , then no further branching was needed from sub-problem C. The second lower bound takes on the value  $Z_L = 51.5$  instead of  $Z_L = 51.25$  at node A. Dividing subproblem C into two new subproblems: D and E by adding constraints  $x_1 \leq 14$  and  $x_1 \geq 15$ , as follows:

LP Subproblem D

$$\text{Max } Z = 3x_1 + 2.5x_2$$

subject to (i)  $x_1 + 2x_2 \geq 20$ , (ii)  $3x_1 + 2x_2 \geq 50$   
 $20$ , (ii)  $3x_1 + 2x_2 \geq 50$

(iii)  $x_2 \geq 3$ , (iv)  $x_1 \leq 14$   
 $15$

and  $x_1, x_2 \geq 0$  and integers  
 integers.

LP Subproblem E

$$\text{Max } Z = 3x_1 + 2.5x_2$$

subject to (i)  $x_1 + 2x_2 \geq$

(iv)  $x_2 \geq 3$ , (v)  $x_1 \geq$

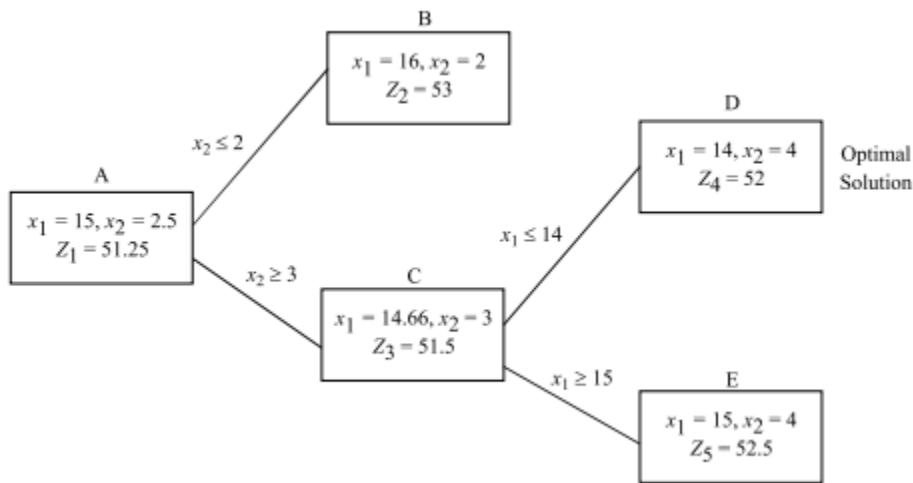
and  $x_1, x_2 \geq 0$  and

Subproblems D and E are solved graphically. The feasible solutions are:

Subproblem D :  $x_1 = 14, x_2 = 4$  and  $\text{Min } Z_4 = 52$ .

Subproblem E :  $x_1 = 15, x_2 = 3$  and  $\text{Min } Z_5 = 52.5$ .

The feasible solutions of both subproblems D and E are all-integer and therefore branch and bound procedure is terminated. The feasible solution of subproblem D is considered as optimal basic feasible solution because this solution is all-integer and the value of the objective function is the lowest amongst all such values. The branch and bound procedure for the given problem is shown in Fig.



## 2.2 GOMORY'S ALL INTEGER CUTTING PLANE METHOD

In this section, a procedure called Gomory's all-integer algorithm will be discussed for generating 'cuts' (additional linear constraints) so as to ensure an integer solution to the given LP problem in a finite number of steps. Gomory's algorithm has the following properties.

- (i) Additional linear constraints never cutoff that portion of the original feasible solution space that contains a feasible integer solution to the original problem.
- (ii) Each new additional constraint (or hyperplane) cuts off the current non-integer optimal solution to the linear programming problem.

### 2.2.1 Method for Constructing Additional Constraint (Cut)

Gomory's method begins by solving an LP problem ignoring the integer value requirement of the decision variables. If the solution so obtained is an



integer, i.e. all variables in the 'xB'-column (also called basis) of the simplex table assume non-negative integer values, the current solution is the optimal solution to the given ILP problem. However, if some of the basic variables do not have non-negative integer value, an additional linear constraint called the Gomory constraint (or cut) is generated. After having generated a linear constraint (or cutting plane), it is added to the bottom of the optimal simplex table. The new problem is then solved by using the dual simplex method. If the optimal solution, so obtained, is again a non-integer, then another cutting plane is generated. The procedure is repeated until all basic variables assume non-negative integer values.

### 2.2.2 Procedure

In the optimal solution simplex table, select a row called source row for which basic variable is non-integer. Then to develop a 'cut', consider only fractional part of the coefficients in source row. Such a cut is also referred to as fractional cut. Suppose the basic variable  $x_r$  has the largest fractional value among all basic variables required to assume integer value. Then the  $r$ th constraint equation (row) from the simplex table can be rewritten as:

$$x_{Br} (=b_r) = 1 \cdot x_r + (a_{r1} x_1 + a_{r2} x_2 + \dots) = x_r + \sum_{j \neq r} a_{rj} x_j \dots 1$$

where  $x_j$  ( $j = 1, 2, 3, \dots$ ) represents all the non-basic variables in the  $r$ th constraint (row), except the variables  $x_r$  and  $b_r (=x_{Br})$  is the non-integer value of variable  $x_r$ .

Decomposing the coefficients of variables  $x_j$  and  $x_r$  as well as  $x_{Br}$  into integer and non-negative fractional parts in Eq. (1) as shown below:

$$[x_{Br}] + f_r = (1 + 0) x_j + \sum_{j \neq r} \{ [a_{rj}] + f_{rj} \} x_j \dots 2$$

where  $[x_{Br}]$  and  $[a_{rj}]$  denote the largest integer value obtained by truncating the fractional part from  $x_{Br}$  and  $a_{rj}$  respectively

Rearranging Eq. (2) so that all the integer coefficients appear on the left-hand side, we get

$$f_r + \{ [x_{Br}] - x_r - \sum_{j \neq r} [a_{rj}] + x_j \} = \sum_{j \neq r} f_{rj} x_j \quad \dots 3$$

where  $f_r$  is strictly a positive fraction ( $0 < f_r < 1$ ) while  $f_{rj}$  is a non-negative fraction ( $0 \leq f_{rj} \leq 1$ ). Since all the variables (including slacks) are required to assume integer values, the terms in the bracket on the left-hand side as well as on the right-hand side must be non-negative numbers. Since the left-hand side in Eq. (3) is  $f_r$  plus a non-negative number, we may write it in the form of the following inequalities:

$$f_r + \sum_{j \neq r} f_{rj} x_j$$

$$\text{Or } \sum_{j \neq r} f_{rj} x_j \leq f_r + s_g \text{ or } -f_r = s_g - \sum_{j \neq r} f_{rj} x_j \quad \dots 4$$

where  $s_g$  is a non-negative slack variable and is also called Gomory slack variable. Equation (4) represents Gomory's cutting plane constraint. When this new constraint is added to the bottom of optimal solution simplex table, it would create an additional row in the table, along with a column for the new variable  $s_g$ .

### Steps of Gomory's All Integer Programming Algorithm

An iterative procedure for the solution of an all integer programming problem by Gomory's cutting plane method can be summarized in the following steps.

**Step 1: Initialization** Formulate the standard integer LP problem. If there are any non-integer coefficients in the constraint equations, convert them into integer coefficients. Solve the problem by the simplex method, ignoring the integer value requirement of the variables.

**Step 2: Test the optimality** (a) Examine the optimal solution. If all basic variables (i.e.  $x_{bi} = b_i \geq 0$ ) have integer values, then the integer optimal solution has been obtained and the procedure is terminated. (b) If one or more basic variables with integer value requirement have non-integer solution values, then go to Step 3.

**Step 3: Generate cutting plane** Choose a row  $r$  corresponding to a variable  $x_r$  that has the largest fractional value  $f_r$  and follow the procedure to develop a 'cut' (a Gomory constraint) as explained in Eqn. (5):

$$-f_r = s_g - \sum_{j \neq r} f_{rj} x_j \text{ where } 0 \leq f_{rj} < 1 \text{ and } 0 < f_r < 1 \quad \dots 5$$

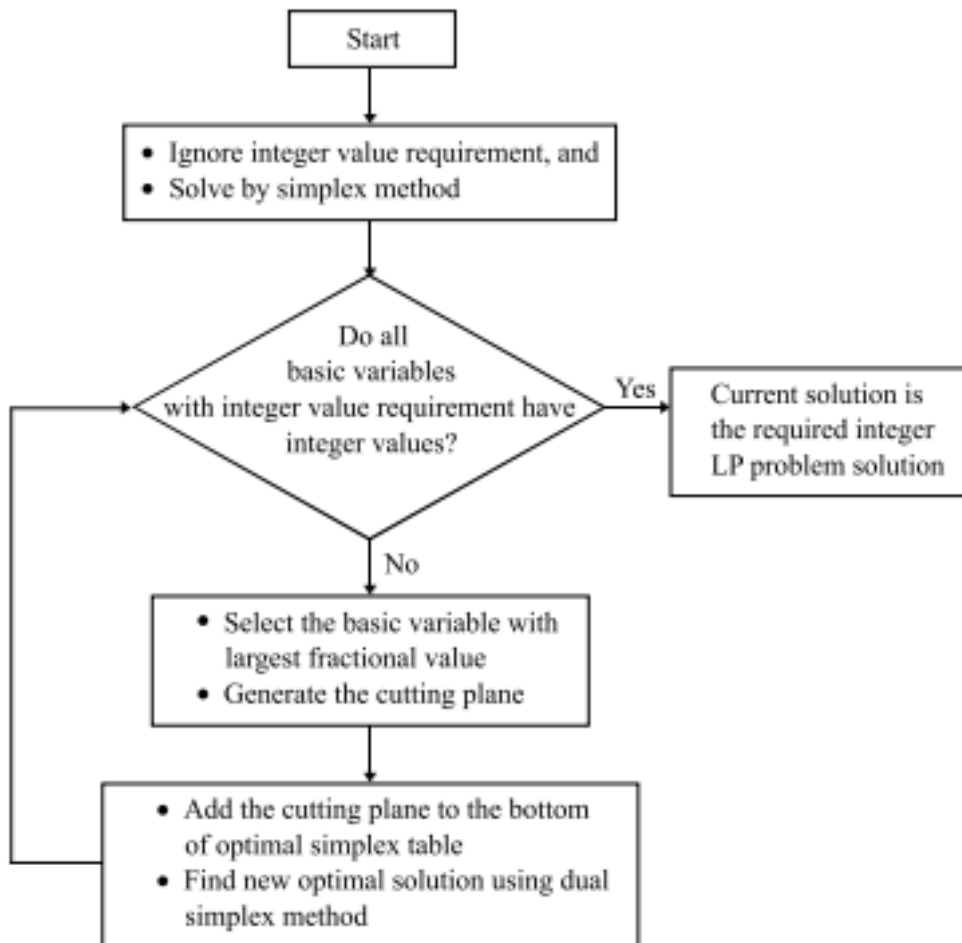
If there are more than one variables with the same largest fraction, then choose the one that has the smallest profit/unit coefficient in the objective function of

maximization LP problem or the largest cost/unit coefficient in the objective function of minimization LP problem.

**Step 4: Obtain the new solution** Add this additional constraint (cut) generated in Step 3 to the bottom of the optimal simplex table. Find a new optimal solution by using the dual simplex method, i.e. choose a variable that is to be entered into the new solution having the smallest ratio:  $\{(c_j - z_j) / y_{ij} ; < y_{ij} 0\}$  and return to Step 2. The process is repeated until all basic variables with integer value requirement assume non-negative integer values.

The procedure for solving an ILP problem is summarized in a flow chart shown in Fig

### Flow Chat for Solving Integer LP Problem



Example 2.2.1 Solve the following Integer LP problem using Gomory's cutting plane method.

Maximize  $Z = x_1 + x_2$

subject to the constraints

(i)  $3x_1 + 2x_2 \leq 5$ , (ii)  $x_2 \leq 2$

and  $x_1, x_2 \geq 0$  and are integers

**Solution Step 1:** Obtain the optimal solution to the LP problem ignoring the integer value restriction by the simplex method

		$c_j \rightarrow$	1	1	0	0
Basic Variables Coefficient $c_B$	Basic Variables $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$s_1$	$s_2$
1	$x_1$	1/3	1	0	1/3	-2/3
1	$x_2$	2	0	1	0	1
$Z = 7/2$		$c_j - z_j$	0	0	-1/3	-1/3

In Table 2.2.1, since all  $c_j - z_j \leq 0$ , the optimal solution of LP problem is:  $x_1 = 1/3$ ,  $x_2 = 2$  and  $\text{Max } Z = 7/2$ .

**Step 2:** In the current optimal solution, shown in Table 2.2.1 all basic variables in the basis ( $x_B$ -column) did not assume integer value. Thus solution is not desirable. To obtain an optimal solution satisfying integer value requirement, go to step 3.

**Step 3:** Since  $x_1$  is the only basic variable whose value is a non-negative fractional value, therefore consider first row ( $x_1$ -row) as source row in Table 2.2.1 to generate Gomory cut as follows:

$$\frac{1}{3} = x_1 + 0 \cdot x_2 + \frac{1}{3} s_1 - \frac{2}{3} s_2 \quad (x_1\text{-source row})$$

The factoring of numbers (integer plus fractional) in the  $x_1$ -source row gives

Each of the non-integer coefficients is factored into integer and fractional parts in such a manner that the fractional part is strictly positive.

$$\left(0 + \frac{1}{3}\right) = (1+0) x_1 + \left(0 + \frac{1}{3}\right) s_1 + \left(-1 + \frac{1}{3}\right) s_2$$

Rearranging all of the integer coefficients on the left-hand side, we get.

$$\frac{1}{3} + (s_1 - x_1) = \frac{1}{3} s_1 - \frac{2}{3} s_2$$

Since value of variables  $x_1$  and  $s_2$  is assumed to be non-negative integer, left-hand side must satisfy

$$\frac{1}{3} \leq \frac{1}{3} s_1 + \frac{1}{3} s_2 \quad (\text{Ref. Eq. 4})$$

$$\frac{1}{3} + s_{g1} = \frac{1}{3} s_1 + \frac{1}{3} s_2 \quad \text{or} \quad s_{g1} = \frac{1}{3} s_1 - \frac{1}{3} s_2 = -\frac{1}{3} \quad (\text{Cut I})$$

where  $s_{g1}$  is the new non-negative (integer) slack variable. Adding this equation (also called Gomory cut) at the bottom of Table 2.2.1, the new values so obtained is shown in Table

$c_j \rightarrow$			1	1	0	0	0
Basic Variables Coefficient $c_B$	Basic Variables $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$s_1$	$s_2$	$s_{g1}$
1	$x_1$	1/3	1	0	1/3	-2/3	0
1	$x_2$	2	0	1	0	10	0
0	$s_{g1}$	-1/3	0	0	-1/3	-1/3	1 →
$Z = 7/2$		$c_j - z_j$	0	0	-1/3	-1/3	0
		Ratio: $\text{Min } (c_j - z_j)/y_{3j} (< 0)$	-	-	1	1	-
					↑		

**Step 4:** Since the solution shown in Table 2.2.1 is infeasible, apply the dual simplex method to find a feasible as well as an optimal solution. The key row and key column are marked in Table 2.2.1. The new solution is obtained by applying the following row operations.

$R3(\text{new}) \rightarrow R3(\text{old}) \times -3$ ;  $R1(\text{new}) \rightarrow R1(\text{old}) - (1/3) R3(\text{new})$  The new solution is shown in Table

		$c_j \rightarrow$	1	1	0	0	0
Basic Variables Coefficient $c_B$	Basic Variables $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$s_1$	$s_2$	$s_{g1}$
1	$x_1$	0	1	0	0	-1	1
1	$x_2$	2	0	1	0	1	0
0	$s_{g1}$	1	0	0	1	1	-3
$Z = 2$		$c_j - z_j$	0	0	0	0	-1

Since all  $c_j$

–  $z_j \leq 0$  and value of basic variables shown in  $x_B$ -column of Table 2.2.1 is integer, the solution:

$x_1 = 0, x_2 = 2, s_{g1} = 1$  and  $\text{Max } Z = 2$ , is an optimal basic feasible solution of the given ILP problem.

**Example 2.2.2** Solve the following Integer LP problem using the cutting plane method.

Maximize  $Z = 2x_1 + 20x_2 - 10x_3$

subject to the constraints

(i)  $2x_1 + 20x_2 + 4x_3 \leq 15$ , (ii)  $6x_1 + 20x_2 + 4x_3 = 20$

and  $x_1, x_2, x_3 \geq 0$  and are integers.

Also show that it is not possible to obtain a feasible integer solution by simple rounding off method.

**Solution** Adding slack variable  $s_1$  in the first constraint and artificial variable in the second constraint,

the LP problem is stated in the standard form as:

Maximize  $Z = 2x_1 + 20x_2 - 10x_3 + 0s_1 - MA_1$

subject to the constraints

(i)  $2x_1 + 20x_2 + 4x_3 + s_1 = 15$ , (ii)  $6x_1 + 20x_2 + 4x_3 + A_1 = 20$

and  $x_1, x_2, x_3, s_1, A_1 \geq 0$  and are integers.

The optimal solution of the LP problem, ignoring the integer value requirement using the simplex

method is shown in Table 2.2.2

			$c_j \rightarrow$	2	20	-10	0
Basic Variables Coefficient $c_B$	Variables in Basis $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$x_3$	$s_1$	
20	$x_2$	5/8	0	1	1/5	3/40	
2	$x_1$	5/4	1	0	0	-1/4	
$Z = 15$		$c_j - z_j$	0	0	-14	-1	

The non-integer optimal solution shown in Table 2.2.2 is:  $x_1 = 5/4$ ,  $x_2 = 5/8$ ,  $x_3 = 0$  and  $\text{Max } Z = 15$ . To obtain an optimal solution satisfying integer value requirement, we proceed to construct Gomory's constraint. In this solution, the value of both basic variables  $x_1$  and  $x_2$  are non-integer. Since the fractional part of the value of basic variable  $x_2 = (0 + 5/8)$  is more than that of basic variable  $x_1 (= 1 + 1/4)$ , the  $x_2$ -row is selected for constructing Gomory cut as follows:

$$\frac{5}{8} = 0 \cdot x_1 + x_2 + \frac{1}{5} x_3 + \frac{3}{40} s_1 \quad (x_2\text{-source row})$$

The factoring of the  $x_2$ -source row yields

$$\left(0 + \frac{5}{8}\right) = (1 + 0)x_2 + \left(0 + \frac{1}{15}\right)x_3 + \left(0 + \frac{3}{40}\right)s_1$$

$$\frac{5}{8} - x_2 = \frac{1}{5}x_3 + \frac{3}{40}s_1 \quad \text{or} \quad \frac{5}{8} \leq \frac{1}{5}x_3 + \frac{3}{40}s_1$$

On adding a slack variable  $s_{g1}$ , the Gomory's fractional cut becomes:

$$\frac{5}{8} + s_{g1} = \frac{1}{5}x_3 + \frac{3}{40}s_1 \quad \text{or} \quad s_{g1} - \frac{1}{5}x_3 + \frac{3}{40}s_1 = \frac{5}{8} \quad (\text{Cut I})$$

Adding this additional constraint at the bottom of optimal simplex Table 2.2.2, the new values so obtained are shown in Table 2.2.2. Iteration 1: Remove the variable  $s_{g1}$  from the basis and enter variable  $s_1$  into the basis by applying the dual simplex method. The new solution is shown in Table 2.2.2

			$c_j \rightarrow$				
			2	20	-10	0	0
Basic Variables Coefficient $c_B$	Basic Variables $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$x_3$	$s_1$	$s_{g1}$
20	$x_2$	5/8	0	1	1/5	3/40	0
2	$x_1$	5/4	1	0	0	-1/4	0
0	$s_{g1}$	-5/8	0	0	-1/5	-3/40	1 $\rightarrow$
Z = 15		$c_j - z_j$	0	0	-14	-1	0
		Ratio: $\text{Min}(c_j - z_j)/y_{3j} (< 0)$	-	-	70	40/3	-
						$\uparrow$	

			$c_j \rightarrow$				
			2	20	-10	0	0
Basic Variables Coefficient $c_B$	Basic Variables $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$x_3$	$s_1$	$s_{g1}$
20	$x_2$	0	0	1	0	0	-1
2	$x_1$	10/3	1	0	2/3	0	-10/3
0	$s_1$	25/3	0	0	8/3	1	-40/3
Z = 20/3		$c_j - z_j$	0	0	-34/3	0	-40/3

The optimal solution shown in Table 7.6 is still non-integer. Therefore, one more fractional cut needs to be generated. Since  $x_1$  is the only basic variable whose value is a non-negative fractional value, consider the  $x_1$ -row (because of largest fractional part) for constructing the cut:

$$\frac{10}{3} = x_3 + \frac{2}{3}x_3 - \frac{10}{3}s_{g1} \quad (\text{x1-source row})$$

The factoring of the  $x_1$ -source row yields

$$\left(3 + \frac{1}{3}\right) = (1+0)x_1 + \left(0 + \frac{2}{3}\right)x_3 + \left(-4 + \frac{2}{3}\right)s_{g1}$$

On adding another Gomory slack variable  $s_{g1}$ , the second Gomory's fractional cut becomes:

$$\frac{1}{3} + (3 - x_1 + 4s_{g1}) = \frac{2}{3}x_3 + \frac{2}{3}s_{g1} \quad \text{or} \quad \frac{1}{3} \leq \frac{2}{3}x_3 + \frac{2}{3}s_{g1} \quad (\text{Cut II})$$

Adding this cut to the optimal simplex Table 2.2.2, the new table so obtained is shown in Table 2.2.2



			$c_j \rightarrow$					
			2	20	-10	0	0	0
Basic Variables Coefficient $c_B$	Basic Variables $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$x_3$	$s_1$	$s_{g1}$	$s_{g2}$
20	$x_2$	0	0	1	0	0	-1	0
2	$x_1$	10/3	1	0	2/3	0	-10/3	0
0	$s_1$	25/3	0	0	8/3	1	-40/3	0
0	$s_{g2}$	-1/3	0	0	-2/3	0	-2/3	1
$Z = 20/3$			0	0	-34/3	0	-40/3	0
Ratio: $\text{Min } (c_j - z_j)/y_{kj} (< 0)$			—	—	17	—	20	—
					↑			

**Iteration 2:** Enter non-basic variable  $x_3$  into the basis to replace basic variable  $s_{g2}$  by applying the dual simplex method. The new solution is shown in Table 7.8.

			$c_j \rightarrow$					
			2	20	-10	0	0	0
Basic Variables Coefficient $c_B$	Basic Variables $B$	Basic Variables Value $b (= x_B)$	$x_1$	$x_2$	$x_3$	$s_1$	$s_{g1}$	$s_{g2}$
20	$x_2$	0	0	1	0	0	1	0
2	$x_1$	3	1	0	0	0	-4	0
0	$s_1$	7	0	0	0	1	-16	4
-10	$x_3$	1/2	0	0	1	0	1	-3/2
$Z = 1$			0	0	0	0	-2	-17
$c_j - z_j$			0	0	0	0	-2	-17

The optimal solution shown in Table 2.2.2 is still non-integer because variable  $x_3$  does not assume integer value. Thus, a third fractional cut needs to be constructed with the help of the  $x_3$ -row:

$$\frac{2}{3} = x_3 + s_{g1} - \frac{3}{2}s_{g2} \quad (x_3\text{-source row})$$

$$\left(0 + \frac{1}{2}\right) = (1+0)x_3 + (1+0)s_{g1} + \left(-2 + \frac{1}{2}\right)s_{g2}$$

$$\frac{1}{2} + (2s_{g2} - x_3 - s_{g1}) = \frac{1}{2}s_{g2} \quad \text{or} \quad \frac{1}{2} \leq \frac{1}{2}s_{g2}$$

The required Gomory's fractional cut obtained by adding slack variable  $s_{g3}$  is:

$$\frac{1}{2} + s_{g3} = \frac{1}{2}s_{g2} \quad \text{or} \quad s_{g3} - \frac{1}{2}s_{g2} - \frac{1}{2} \text{ Cut III}$$

Adding this cut to the bottom of the optimal simplex Table 2.2.2 the new table so obtained is shown in Table 2.2.2.

			$c_j \rightarrow$						
			2	20	-10	0	0	0	0
Basic Variables	Basic	Basic Variables	$x_1$	$x_2$	$x_3$	$s_1$	$s_{g1}$	$s_{g2}$	$s_{g3}$
Coefficient	Variables	Value							
$c_B$	$B$	$b (= x_B)$							
20	$x_2$	0	0	1	0	0	1	0	0
2	$x_1$	3	1	0	0	0	-4	1	0
0	$s_1$	7	0	0	0	1	-16	4	0
-10	$x_3$	1/2	0	0	1	0	1	-3/2	0
0	$s_{g3}$	-1/2	0	0	0	0	0	-1/2	1 $\rightarrow$
$Z = 1$			0	0	0	0	-2	-17	0
Ratio: $\text{Min}(c_j - z_j)/y_{sj} (< 0)$			-	-	-	-	-	34	-
								$\uparrow$	

Iteration 3: Remove the variable  $sg_3$  from the basis and enter variable  $sg_2$  into the basis by applying the dual simplex method. The new solution is shown in Table 7.10.

			$c_j \rightarrow$						
			2	20	-10	0	0	0	0
Basic Variables	Basic	Basic Variables	$x_1$	$x_2$	$x_3$	$s_1$	$s_{g1}$	$s_{g2}$	$s_{g3}$
Coefficient	Variables	Value							
$c_B$	$B$	$b (= x_B)$							
20	$x_2$	0	0	1	0	0	1	0	0
2	$x_1$	2	1	0	0	0	-4	0	2
0	$s_1$	3	0	0	0	1	-16	0	8
-10	$x_3$	2	0	0	1	0	1	0	-3
0	$s_{g2}$	1	0	0	0	0	0	1	-2
$Z = -16$			0	0	0	0	-2	0	-34
$c_j - z_j$									

In Table 2.2.2, since value of all basic variables is an integer value and all  $c_j - z_j \leq 0$ , the current solution is an integer optimal solution:  $x_1 = 2$ ,  $x_2 = 0$ ,  $x_3 = 2$  and  $\text{Max } Z = -16$ .

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## بوخنه

ئەم توپزىنەھىيە بەدوۋادىچوون بۇ بەكارھىنئانى دوو شىۋازى دامەزراۋ دەكات، Branch and Bound ۋە Gomory's All-Integer Cutting Plane، بۇ چارەسەر كىشەكانى Integer Linear Programming (ILP). ILP بەرنامەسازى ھىلى درىژ دەكاتەھە بە ناساندنى ئەو سنووردار كىشەھىيە كە ھەموو گۇراۋە بىر يارىيەكان دەبىت ژمارەھى ئەواۋ بىن. دۆزىنەھىيە چارەسەرى گونجاۋ بۇ كىشەكانى ILP زۇر جار دەيسەلمىنىت كە چالاكتەرە لە كىشە ستانداردەكانى LP.

ئەم توپزىنەھىيە بىنەما تىۋرىيەكانى ھەردوۋ شىۋازى تەختەھى بىرىنى برانچ ۋە باوند ۋە گۇمورى دەخاتە بەر لىكۆلىنەھە. ئىمە خالە بەھىز ۋە لاوازەكانىيان لە چارەسەر كىشەكانى ILP شى دەكەپنەھە. جگە لەھەش، ئامانجى توپزىنەھە كە جىبەجىيە كىشەكانى ئەم شىۋازانە ۋە بەر اورد كىشەكانى ئەدەي كار كىشەكانى لەسەر كۆمەلىك كىشەھىيە پىۋەرەكانى ILP. دۆزىنەھەكان تىروانىنىكى بەنرخ لەبارەھى كارىگەرىيە ئەم شىۋازانە لە بەكارھىنئانى پراكتىكىدا دەدەن