4.83 page 175

Let $\bar{X}$ be the mean of a random sample of size 5 from a normal distribution with $\mu = 0$ and $\sigma^2 = 125$

Determine c so that $pr(\bar{X} < c) = 0.90$

Sol^n:

since $X_1, X_2, \dots, X_n \sim n(0, 125) \Rightarrow$

$\bar{X} \sim n(0, \frac{125}{5}) \Rightarrow \bar{X} \sim n(0, 25)$

$pr(\bar{X} < c) = 0.90$

$pr\left(\frac{\bar{X} - 0}{5} < \frac{c - 0}{5}\right) = 0.90$

$pr\left(Z < \frac{c}{5}\right) = 0.90$ from table

$\frac{c}{5} = 1.28 \Rightarrow$

$c = 5 * 1.28$

$c = 6.4$

4.84 page 175

If $\bar{X}$ is the mean of a random sample of size n from a normal distribution with mean μ and variance 100, find n so that

$pr(\mu-5 < \bar{X} < \mu+5) = 0.954$

Sol.: $X_1, X_2, \ldots, X_n \sim n(\mu, 100) \Rightarrow$

$$\bar{X} \sim n\left(\mu, \frac{100}{n}\right)$$

$$pr(\mu-5 < \bar{X} < \mu+5) = 0.954$$

$$pr\left(\frac{\mu-5-\mu}{\frac{10}{\sqrt{n}}} < \frac{\bar{X}-\mu}{\frac{10}{\sqrt{n}}} < \frac{\mu+5-\mu}{\frac{10}{\sqrt{n}}}\right) = 0.954$$

$$pr\left(\frac{-5}{\frac{10}{\sqrt{n}}} < Z < \frac{5}{\frac{10}{\sqrt{n}}}\right) = 0.954$$

$$pr\left(-\frac{\sqrt{n}}{2} < Z < \frac{\sqrt{n}}{2}\right) = 0.954$$

$$pr\left(Z < \frac{\sqrt{n}}{2}\right) - p\left(Z < -\frac{\sqrt{n}}{2}\right) = 0.954$$

$$pr\left(Z < \frac{\sqrt{n}}{2}\right) - 1 + p\left(Z < \frac{\sqrt{n}}{2}\right) = 0.954$$

$$2\,pr\left(Z < \frac{\sqrt{n}}{2}\right) = 1 + 0.954 = 1.954 \Rightarrow$$

$$pr\left(Z < \frac{\sqrt{n}}{2}\right) = \frac{1.954}{2} = 0.977 \text{ from table} \Rightarrow$$

$$\frac{\sqrt{n}}{2} = 2 \Rightarrow \sqrt{n} = 4 \Rightarrow n = 16$$

4.85 page 175

Let $X_1, X_2, \ldots, X_{25}$ and $Y_1, Y_2, \ldots, Y_{25}$ be two random sample from two independent normal distributions $n(0, 16)$ and $n(1, 9)$, respectively. Let $\bar{X}$ and $\bar{Y}$ denote the corresponding sample means. compute $pr(\bar{X} > \bar{Y})$.

Sol.: $X_1, X_2, \ldots, X_{25} \sim n(0,16) \Rightarrow \bar{X} \sim n(0, \frac{16}{25})$

$Y_1, Y_2, \ldots, Y_{25} \sim n(1,9) \Rightarrow \bar{Y} \sim n(1, \frac{9}{25})$

$pr(\bar{X} > \bar{Y}) = pr(0 > \bar{Y} - \bar{X}) = pr(\bar{Y} - \bar{X} < 0)$

$\mu_{\bar{Y}-\bar{X}} = E(\bar{Y} - \bar{X}) = E(\bar{Y}) - E(\bar{X}) = 1 - 0 = 1$

$var(\bar{Y} - \bar{X}) = var(\bar{Y}) + var(\bar{X}) = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$

$\therefore \bar{Y} - \bar{X} \sim n(1, 1)$

$pr(\bar{Y} - \bar{X} < 0) = pr\left(\frac{\bar{Y} - \bar{X} - 1}{1} < \frac{0-1}{1}\right) =$

$pr(Z < -1) = 1 - pr(Z < 1)$ from table

$= 1 - 0.84 = 0.16$

4.86 Page 175

Find the mean and variance of $S^2 = \frac{\sum_1^n (X_i - \bar{X})^2}{n}$ where $X_1, X_2, \ldots, X_n$ is a random sample from $n(\mu, \sigma^2)$

Hint. Find the mean and variance of $\frac{nS^2}{\sigma^2}$.

Sol:- $\frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$, If $x \sim \chi^2(r) \Rightarrow$

mean of chi-square $= r$

Variance of chi-square $= 2r$

$$\Rightarrow E(S^2) = E\left(\frac{\sigma^2}{n}\frac{nS^2}{\sigma^2}\right) = \frac{\sigma^2}{n} E\left(\frac{nS^2}{\sigma^2}\right) = \frac{\sigma^2}{n}\, n-1$$

$$= \frac{n-1}{n}\sigma^2$$

$$V(S^2) = V\left(\frac{\sigma^2}{n}\frac{nS^2}{\sigma^2}\right) = \frac{\sigma^4}{n^2} V\left(\frac{nS^2}{\sigma^2}\right)$$

$$= \frac{\sigma^4}{n^2}\, 2(n-1) = \frac{2(n-1)}{n^2}\sigma^2$$

4.87 page175

Let S^2 be the variance of a random sample of size 6 from the normal distribution $n(\mu, 12)$. Find

$pr(2.30 < S^2 < 22.2)$

Sol: $n=6$, $\sigma^2=12$

$$pr(2.30 < S^2 < 22.2)$$

$$= pr\left(\frac{6 * 2.30}{12} < \frac{nS^2}{\sigma^2} < \frac{6 * 22.2}{12}\right)$$

$$= pr\left(1.15 < \chi^2_{(n-1)} < 11.1\right)$$

$$= pr(\chi^2_{(5)} < 11.1) - pr(\chi^2_{(5)} < 1.15)$$ from table

$$= 0.95 - 0.050 = 0.90$$

4.89 page 176

Let $\bar{X}$ and S^2 be the mean and the variance of a random sample of size 25 from a distribution which is $n(3, 100)$

Evaluate $pr(0 < \bar{X} < 6, 55.2 < S^2 < 145.6)$.

Sol:- $X_1, X_2, \dots, X_{25} \sim n(3, 100) \Rightarrow \bar{X} \sim n(3, \frac{100}{25} = 4)$

$$\frac{nS^2}{6^2} = \frac{25S^2}{100} = \frac{S^2}{4} \sim \chi^2_{(n-1=25-1=24)}$$

$$pr(0 < \bar{X} < 6, 55.2 < S^2 < 145.6)$$

$$= pr(0 < \bar{X} < 6) * pr(55.2 < S^2 < 145.6) \quad \} \bar{x} \,\&\, s^2 \text{ indep.}$$

$$= pr\left(\frac{0-3}{2} < \frac{\bar{X}-3}{2} < \frac{6-3}{2}\right) * pr\left(\frac{55.2}{4} < \frac{S^2}{4} < \frac{145.6}{4}\right)$$

$$= pr\left(\frac{-3}{2} < Z < \frac{3}{2}\right) * pr\left(\frac{55.2}{4} < \chi^2_{(24)} < \frac{145.6}{4}\right)$$

$$= pr(-1.5 < Z < 1.5) * pr(13.8 < \chi^2_{(24)} < 36.4)$$

$$= \{pr(Z < 1.5) - pr(Z < -1.5)\} * \{pr(\chi^2_{(24)} < 36.4) - pr(\chi^2_{(24)} < 13.8)\}$$

$$= \{pr(Z < 1.5) - \{1 - pr(Z < 1.5)\}\} * \{0.95 - 0.05\}$$

$$= \{pr(Z < 1.5) - 1 + pr(Z < 1.5)\} * \{0.90\}$$

$$= \{2\,pr(Z < 1.5) - 1\} * \{0.90\}$$

$$= \{2 * 0.93 - 1\} * \{0.90\}$$

$$= 0.86 * 0.90 = 0.78$$