***Definition:*** **Linear Differential Equations** is a differential equation that can be written in the following form.

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x) \qquad (*)$$

The important thing to note about linear differential equations is that there are no products of the function, $y(x)$, and its derivatives and neither the function nor its derivatives occur to any power other than the first power.

The coefficients $a_0(x)$ ... $a_n(x)$and $g(x)$ can be zero or non-zero functions, constant or no constant functions, linear or non-linear functions. Only the function, $y(x)$, and its derivatives are used in determining if a differential equation is linear.
If a differential equation cannot be written in the form, (*) then it is called a ***non-linear*** differential equation.

***Definition:*** ***general solution***

If the solution of a differential equation contains as many arbitrary constants of integration as its order, then the solution is said to be the **general solution** of the differential equation.

***Definition:*** ***Particular solution***:

The solution obtained from the general solution by assigning particular values for the arbitrary constants, is said to be a **particular solution** of the differential equation.
For example,

| Differential equation | General solutuion | Particular solution |
| --- | --- | --- |
| (i) $\frac{dy}{dx} = \sec^2 x$ | $y = \tan x + c$ ($c$ is arbitrary constant) | $y = \tan x - 5$ |
| (ii) $\frac{dy}{dx} = x^2 + 2x$ | $y = \frac{x^3}{3} + x^2 + c$ | $y = \frac{x^3}{3} + x^2 + 8$ |
| (iii) $\frac{d^2y}{dx^2} - 9y = 0$ | $y = Ae^{3x} + Be^{-3x}$ | $y = 5e^{3x} - 7e^{-3x}$ |

## Methods of solving differential equations of the first order and first degree

**1. *Separation of variables*:**

If it is possible to re-arrange the terms of the first order and first degree differential equation in two groups, each containing only one variable, the variables are said to be separable.

When variables are separated, the differentail equation takes the form $f(x)\,dx + g(y)\,dy = 0$ in which $f(x)$ is a function of $x$ only and $g(y)$ is a function of $y$ only.

Then the general solution is

$$\int f(x)\,dx + \int g(y)\,dy = c \qquad (c \text{ is a constant of integration})$$

For example, consider $x\frac{dy}{dx} - y = 0$

$$x\frac{dy}{dx} = y \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{x} \qquad \text{(separating the variables)}$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x} + k \qquad \text{where } k \text{ is a constant of integration.}$$

$$\Rightarrow \quad \log y = \log x + k.$$

**Example**: Solve $\frac{dy}{dx} = 1 + e^{2x}$

**Solution.** $dy = (1 + e^{2x})dx$

Integrate both sides we get

$y = x + \frac{1}{2} e^{2x} + C$

**Example**: Solve $\frac{dy}{dx} = \sin x$

**Solution.** $dy = \sin x \, dx$

Integrate both sides we get

$y = -\cos x + C$ is a general solution of the given differential equation.

**Example** Solve $\frac{dy}{dx} = -\frac{x}{y}$ subject to $y(4) = 3$

**Solution.** $ydy = -xdx$

$\Rightarrow \int ydy = \int -xdx$

$x^2 + y^2 = C$ is general solution where $C = 2C_1$.

For $x = 4$ &$y = 3$

$4^2 + 3^2 = C \Rightarrow C = 25$

$\therefore x^2 + y^2 = 25$ ***Particular solution***

**Example** :Solve $(1 + x^2)\, dy - xy\, dx = 0$

**Solution.** Dividing by $y(1 + x^2)$ and transposing we get

$$\frac{dy}{y} = \frac{x\,dx}{1+x^2}$$

Integrating both sides, we get $\ln y = \frac{1}{2}\ln(1 + x^2) + \ln C$

Or

$\ln y = \ln C\,(1 + x^2)^{\frac{1}{2}}$

Taking exponentials $y = C(1+x^2)^{\frac{1}{2}}$

The arbitrary constant was added in the form "ln C" to facilitate the final representation.

**Example** :Solve $\frac{dy}{dx} = 2x(1+y^2)e^{x^2}$

**Solution.** $\frac{1}{1+y^2}\,dy = 2xe^{x^2}dx$

$$\int \frac{1}{1+y^2}\,dy = \int 2xe^{x^2}dx \qquad u = x^2 \;\&\; du = 2xdx$$

$$\int \frac{1}{1+y^2}\,dy = \int e^u du$$

$$\tan^{-1} y = e^u + C$$

$$\tan^{-1} y = e^{x^2} + C$$

$y = \tan(e^{x^2} + C)$ is a general solution of diff. eq.

**Example**: Solve $(x+1)ydx + (x-1)(y+1)dy = 0$

**Solution.** Divide the differential equation by $(x-1)y$

$\frac{x+1}{x-1}\,dx + \frac{y+1}{y}dy = 0$ is separable diff. eq.

$$\int \frac{x+1}{x-1}\,dx + \int \frac{y+1}{y}dy = 0$$

$$\int \frac{x+1+1-1}{x-1}\,dx + \int \frac{y+1}{y}dy = 0$$

$$\int (1+\frac{2}{x-1})\,dx + \int (1+\frac{1}{y})dy = 0$$

$$x + 2\ln(x-1) + y + \ln y = C$$

$$x + \ln(x-1)^2 + y + \ln y = C$$

$x + y + \ln(x-1)^2 y = C$ is a general solution.

**Example** :Solve $(xy+y)dx + (y^2x - y^2 - x + 1)dy = 0$

**Solution.** $(x+1)ydx + y^2(x-1) - (x-1)dy = 0$

$$(x+1)ydx + (y^2-1)(x-1)dy = 0$$

Divide the differential equation by $(x-1)y$

$$\frac{x+1}{x-1}\,dx + \frac{y^2-1}{y}\,dy = 0$$ is separable diff. eq.

$$\int \frac{x+1}{x-1}\,dx + \int \frac{y^2-1}{y}\,dy = 0$$

$$\int \frac{x+1+1-1}{x-1}\,dx + \int (y - \frac{1}{y})dy = 0$$

$$\int (1 + \frac{2}{x-1})\,dx + \int (y - \frac{1}{y})dy = 0$$

$$x + 2\ln(x-1) + \frac{1}{2}y^2 - \ln y = C$$

$$x + \ln(x-1)^2 + \frac{1}{2}y^2 - \ln y = C$$

$$x + \frac{1}{2}y^2 + \ln\frac{(x-1)^2}{y} = C$$ is a general solution.

Exercises:

1) $\frac{dy}{dx} = \sin 5x$ $\quad$ *Ans.* $y = -\frac{1}{5}\cos 5x + C$

2) $dx + e^{3x}dy = 0$ $\quad$ *Ans.* $y = \frac{1}{3}e^{-3x} + C$

3) $(1+x)\frac{dy}{dx} = x + 6$ $\quad$ *Ans.* $y = x + 5\ \ln|x+1| + C$

4) $xy' = 4y$ $\quad$ *Ans.* $y = Cx^4$

5) $\frac{dy}{dx} = \frac{y^3}{x^2}$ $\quad$ *Ans.* $y^{-2} = 2x^{-1} + C$

6) $\frac{dx}{dy} = \frac{x^2\ \ y^2}{1+x}$ $\quad$ *Ans.* $-3 + 3x\ln x = xy^3 + 3xC$

7) $\frac{dy}{dx} = e^{(3x+2y)}$ $\quad$ *Ans.* $-3e^{-2y} = 2e^{3x} + C\ \ where\ C = 6C1$

8) $(4y + y\,x^2)dy - (2x + xy^2)dx = 0$ $\quad$ *Ans.* $2 + y^2 = C(4 + x^2)\ \ where\ C = e^{2C1}$

9) $2y(x+1)dy = xdx$ $\quad$ *Ans.* $y^2 = x - \ln|x+1| + C$

10) $y\ \ln x\,\frac{dx}{dy} = (\frac{y+1}{x})^2$ $\quad$ *Ans.* $\frac{x^3}{3}\ln x - \frac{x^3}{9}$

$$= \frac{y^2}{2} + 2y + \ln|y| + C$$

11) $sec^2x\, dy + cscy\;\; dx = 0$ $\qquad$ $Ans.\;\; 4\cos y = 2x + \sin 2x + 4C$

12) $e^y\;\sin 2x\, dx + \cos x(\; e^{2y} - y)\, dy = 0$ $\;\; Ans. -2\;\cos x + e^y - ye^{-y} - e^{-y} = C$

13) $(e^y + 1)^2 e^{-y} dx + (e^x + 1)^3 e^{-x} dy = 0$ $\quad Ans.\;(e^x + 1)^{-2} + 2\;\;(e^y + 1)^{-1} = C$

**Definition: Homogenous equations**:

A function is called *homogeneous of degree* n if $f(tx, ty) = t^n f(x, y)$for all$x, y, t$.

**Example**:$f(x, y) = x - 3\sqrt{xy} + 5y$

**Solution**: $f(tx, ty) = tx - 3\sqrt{(tx)(ty)} + 5ty$

$$= tx - 3t\sqrt{xy} + 5ty$$

$$= t(x - 3\sqrt{xy} + 5y)$$

$$f(tx, ty) = tf(x, y)$$

Thus the given function is homogeneous With degree one.

**Example**:$f(x, y) = \sqrt{\;x^3 + y^3}$

**Solution**: $f(tx, ty) = \sqrt{\;t^3x^3 + t^3y^3}$

$$=\;t^{3/2}\sqrt{\;x^3 + y^3}$$

$$= t^{3/2} f(x, y)$$

Thus $f(x, y)$is homogeneous With degree $\;3/2$.

**Example**:$f(x, y) = x^2 + y^2 + 1$

**Solution**: $f(tx, ty) = (tx)^2 + (ty)^2 + 1$

$$= t^2x^2 + t^2y^2 + 1 \neq t^n f(x, y)$$

$f(x, y)$is not homogeneous

**Example**: $f(x, y) = \ln x^2 - 2\ln y = 2\ln x - 2\ln y$

**Solution**: $f(tx, ty) = 2\ln tx - 2\ln ty$

$$= 2\ln t + 2\ln x - 2\ln t - 2\ln y$$

$$= 2\ln x - 2\ln y$$