Diversity

Main story

- Communication over a flat fading channel has poor performance due to significant probability that channel is in deep fading.
- Reliability is increased by provide more signal paths that fade independently.
- Diversity can be provided across time, frequency and space.
- Name of the game is how to expoit the added diversity in an efficient manner.

Baseline: AWGN Channel

y = x+ w

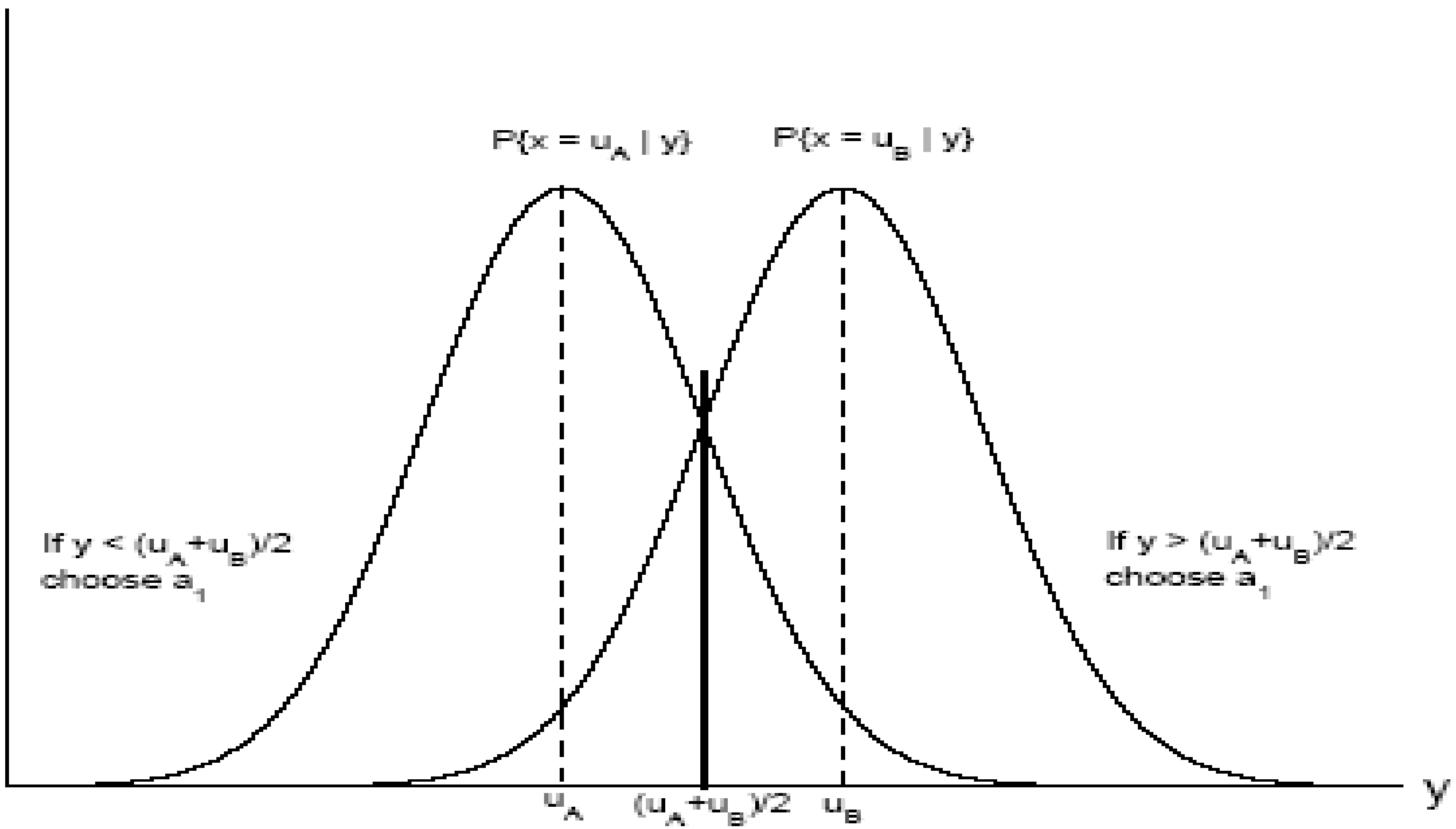
BPSK modulation x = § a

$$p_e = Q\left(\frac{a}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2\mathsf{SNR}}\right)$$

$$\mathsf{SNR} := \frac{a^2}{N_0}$$

Error probability decays exponentially with SNR.

Gaussian Detection



Rayleigh Flat Fading Channel

$$y = hx + w$$

$$h \sim \mathcal{CN}(0, 1)$$

BPSK: $x = \pm a$.

Conditional on h,

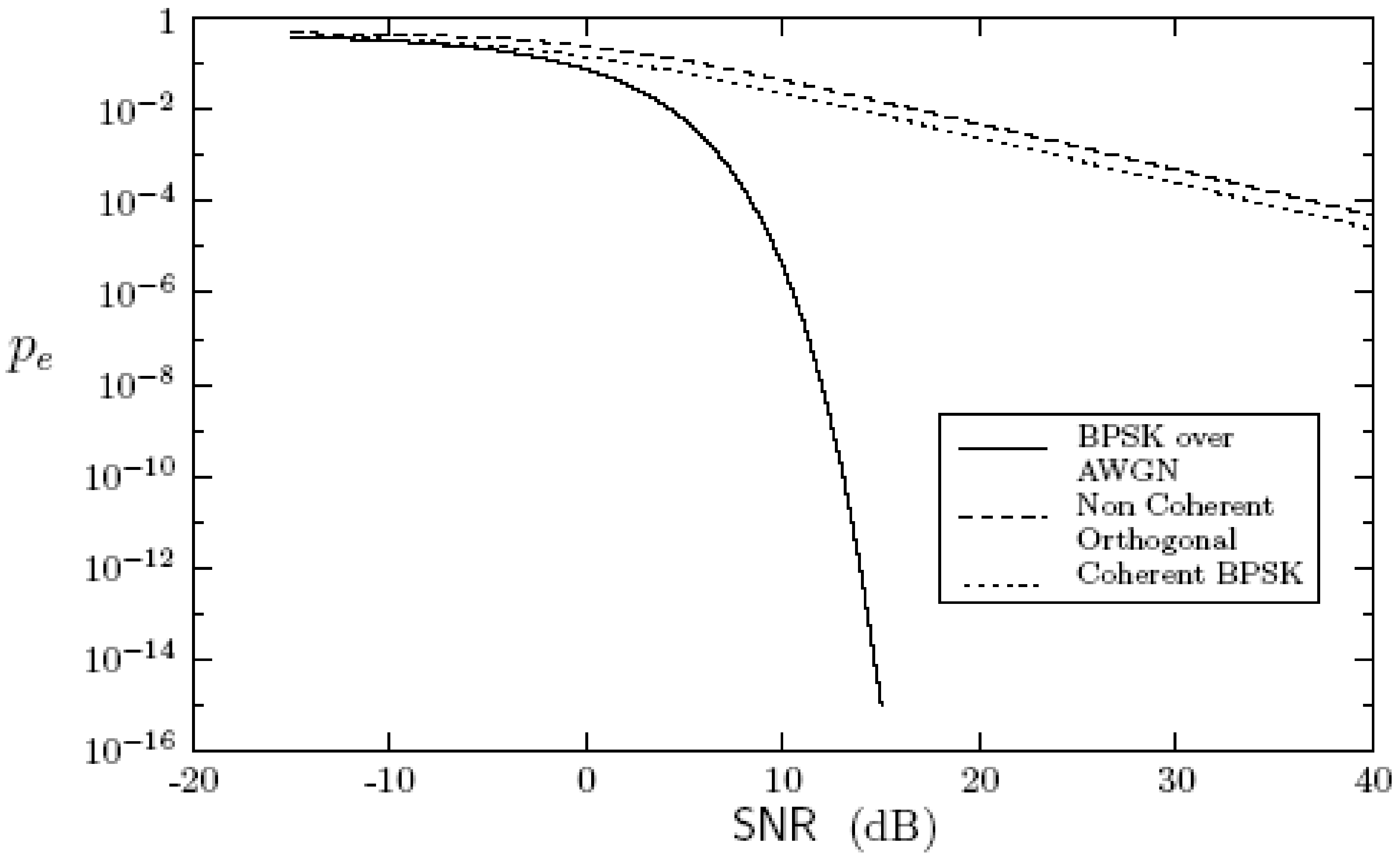
$$p_e = Q\left(\sqrt{2|h|^2 \mathsf{SNR}}\right)$$

Averaged over h,

$$p_e = \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{SNR}}{1 + \mathsf{SNR}}}\right) \approx \frac{1}{4\mathsf{SNR}}$$

at high SNR.

Rayleigh vs AWGN



Typical Error Event

Conditional on h,

$$p_e = Q\left(\sqrt{2|h|^2\mathsf{SNR}}\right)$$

When $|h|^2 >> \frac{1}{\mathsf{SNR}}$, error probability is very small.

When $|h|^2 < \frac{1}{\mathsf{SNR}}$, error probability is very significant.

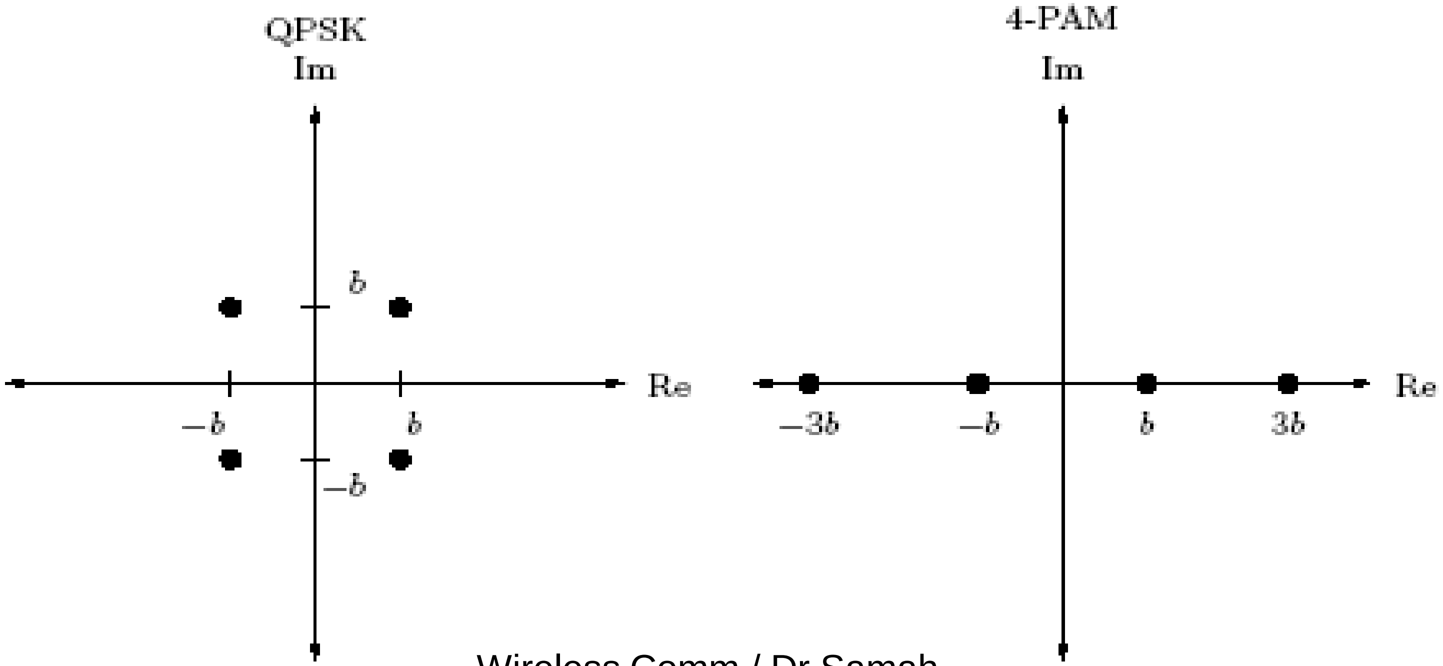
$$p_e \approx P\left(|h|^2 < \frac{1}{\mathsf{SNR}}\right) \approx \frac{1}{\mathsf{SNR}}$$

$$|h|^2 \sim \exp(1).$$

Typical error event is due to channel being in deep fade rather than noise being large.

BPSK, QPSK and 4-PAM

- BPSK uses only the I-phase.The Q-phase is wasted.
- QPSK delivers 2 bits per complex symbol.
- To deliver the same 2 bits, 4-PAM requires 4 dB more transmit power.
- QPSK exploits the available degrees of freedom in the channel better.

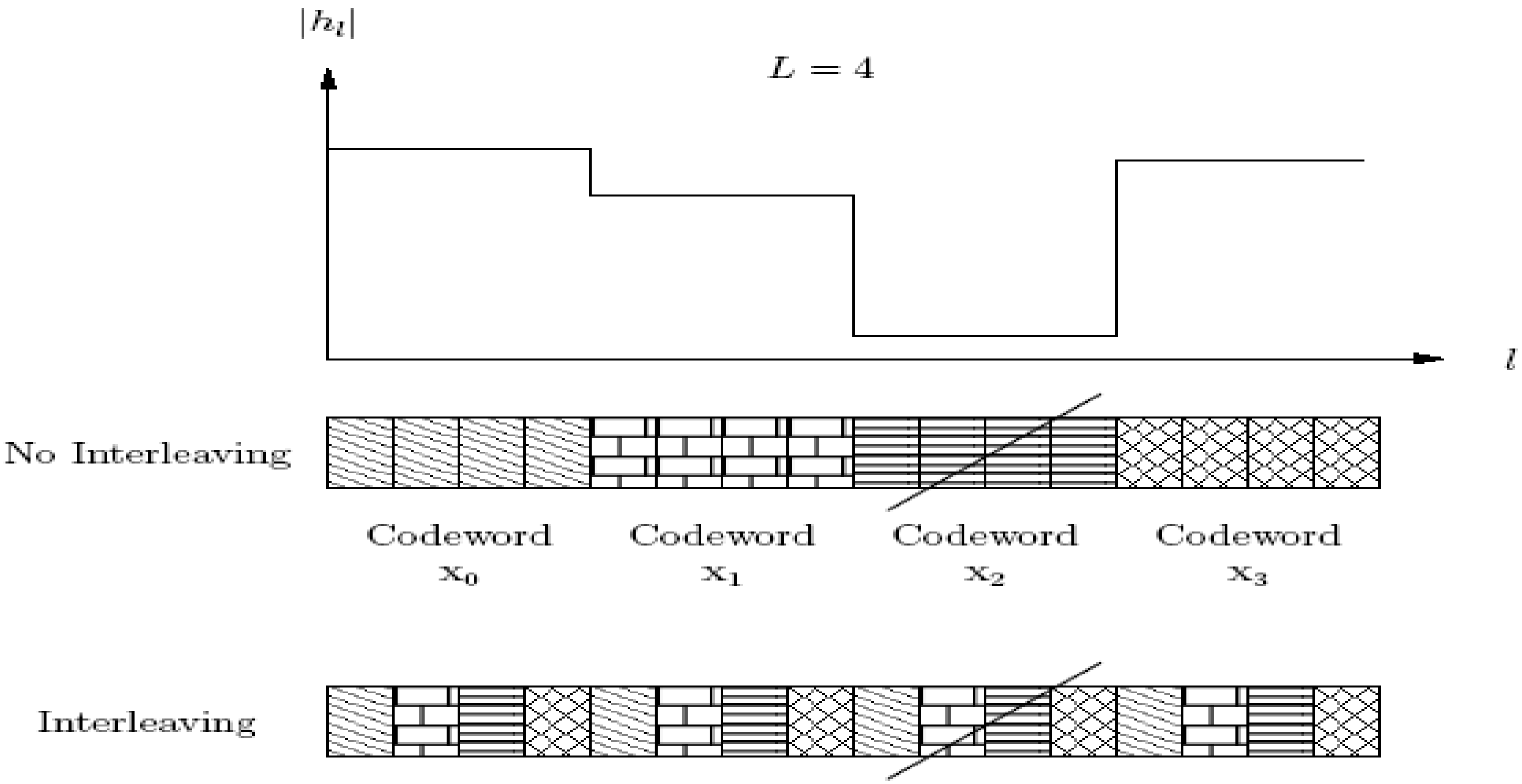


Diversity Techniques

- In time or frequency or space
- In each , a simple scheme based on repetition coding can be considered:
- Repetition coding achieves the maximal diversity gain.
- However, it is usually wasteful of the degrees of freedom of the channel.
- More sophisticated schemes to increase data rate and achieve a coding gain along with the diversity gain.

Time Diversity

- Typically Tc ≈ (10-100) Ts i.e the channel is highly correlated across consecutive symbol
- Time diversity can be obtained by interleaving and coding over symbols across different coherent time periods.



Repetition Coding

After interleaving over L coherence time periods, and transmitting L symbols

$$y_\ell = h_\ell x_\ell + w_\ell, \qquad \ell = 1, \ldots, L$$

Repetition coding: $x_\ell = x$ for all ℓ.

$$\mathbf{y} = \mathbf{h}x + \mathbf{w}$$

where $\mathbf{y} = [y_1, \ldots, y_L]^t$, $\mathbf{h} = [h_1, \ldots, h_L]^t$ and $\mathbf{w} = [w_1, \ldots, w_L]^t$.

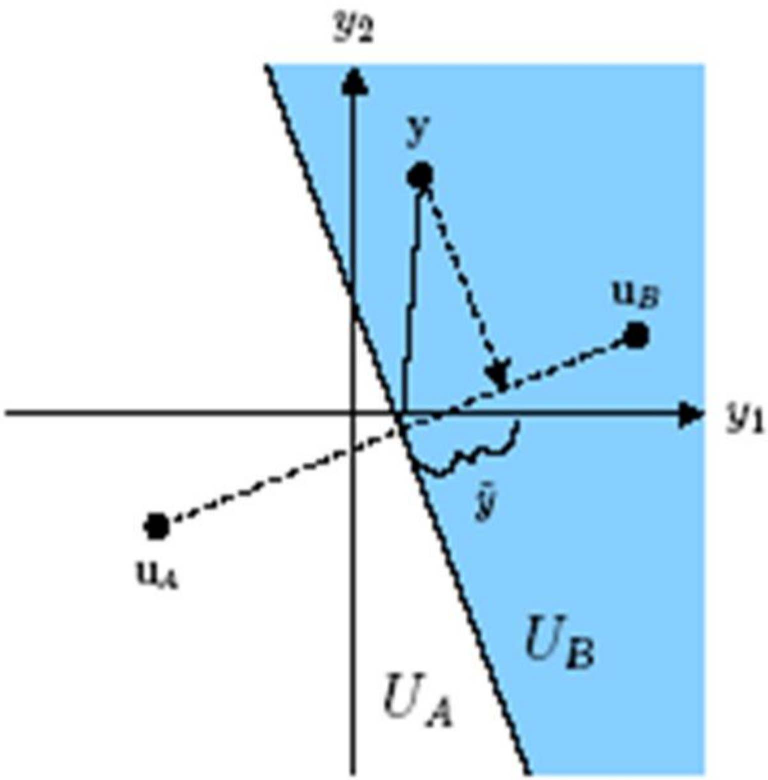
This is classic vector detection in white Gaussian noise.

- With coherent detection

$$\frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{y} = \|\mathbf{h}\| x_1 + \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{w}$$

- Using matched filter or MRC to weight received signal in each of L branches and align the signal phase in the summation to maximize the output SNR

Geometry



For BPSK $x = \pm a$,

$$\mathbf{u}_A = +a\mathbf{h}, \mathbf{u}_B = -a\mathbf{h}.$$

$$\tilde{y} = \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{y}$$

is a sufficient statistic (match filtering.)

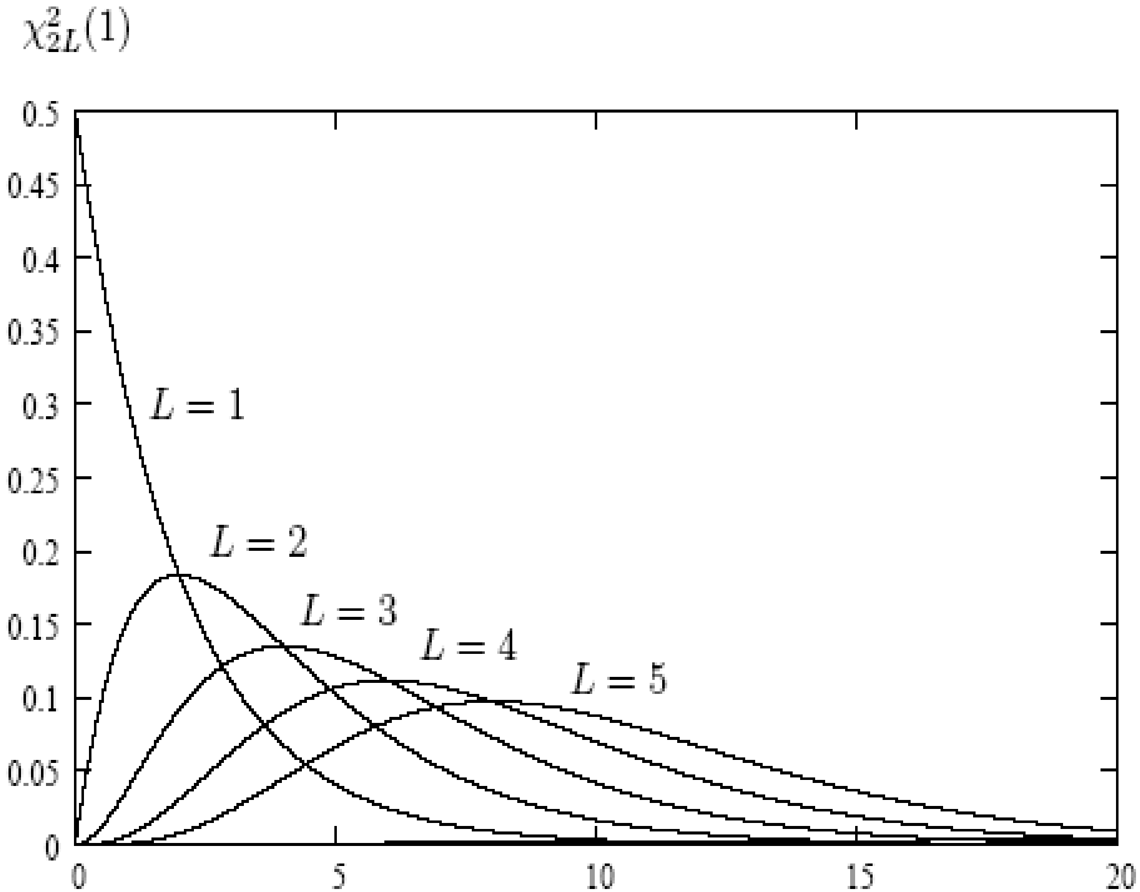
Reduces to scalar detection problem:

$$\tilde{y} = \|\mathbf{h}\|x + \tilde{w}$$

Error probability $Q\left(\sqrt{2\|\mathbf{h}\|^2 \mathsf{SNR}}\right)$

SNR=a^2/N_o & $\|\mathbf{h}\|^2 = \sum_{\ell=1}^{L} |h_\ell|^2$

Deep Fades Become Rarer



Prob. density function of $||h||^2$

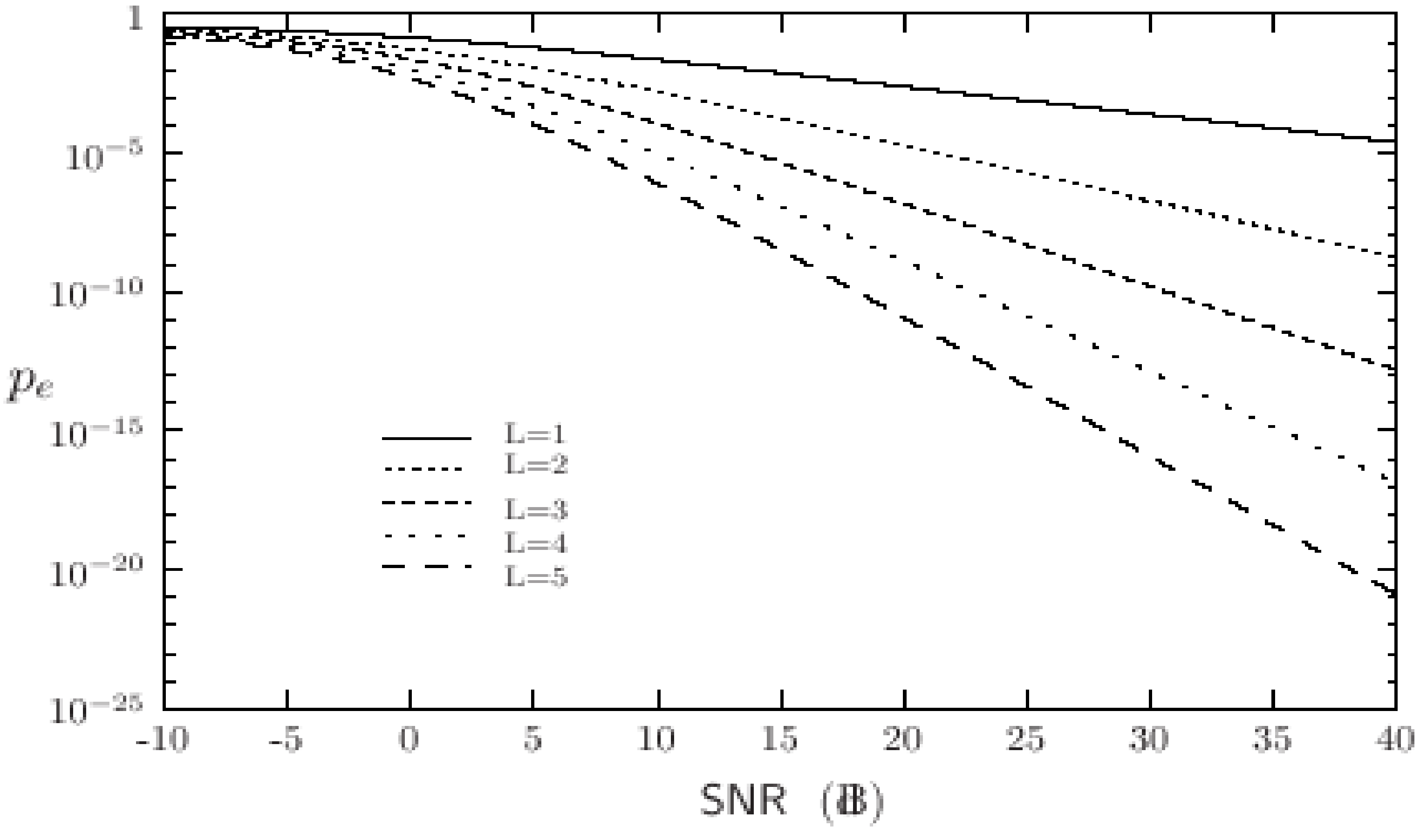
<u>Pe depends on the overall channel gain</u>

$$P\left(\|\mathbf{h}\|^2 < \epsilon\right) \approx \frac{1}{L!}\epsilon^L$$

$$p_e \approx P\left(\|\mathbf{h}\|^2 < \frac{1}{\mathsf{SNR}}\right) \approx \frac{1}{L!}\frac{1}{\mathsf{SNR}^L}$$

L is the diversity gain

Performance



Example: Hybrid ARQ in EV-DO

- Evolution-data Optimized is a telecommunication standard
- wireless transmission of data through radio signal
- Typically for broadband Internet access
- It is Evolution of CDMA 2000 standard
- Packets are retransmitted via ARQ protocol
- Decoding is done by combining the current and the previous received copies
- No. of repetitions is variable

Beyond Repetition Coding

- Repetition coding gets full diversity, but sends only one symbol every L symbol times: does not exploit fully the degrees of freedom in the channel.
- How to do better? code design for fading channels
  - a coding gain beyond the diversity gain

Example: Rotation code (L=2)

- Consider a transmitter scheme where rotation matrix R is used $\theta \in 0\text{-}2\pi$

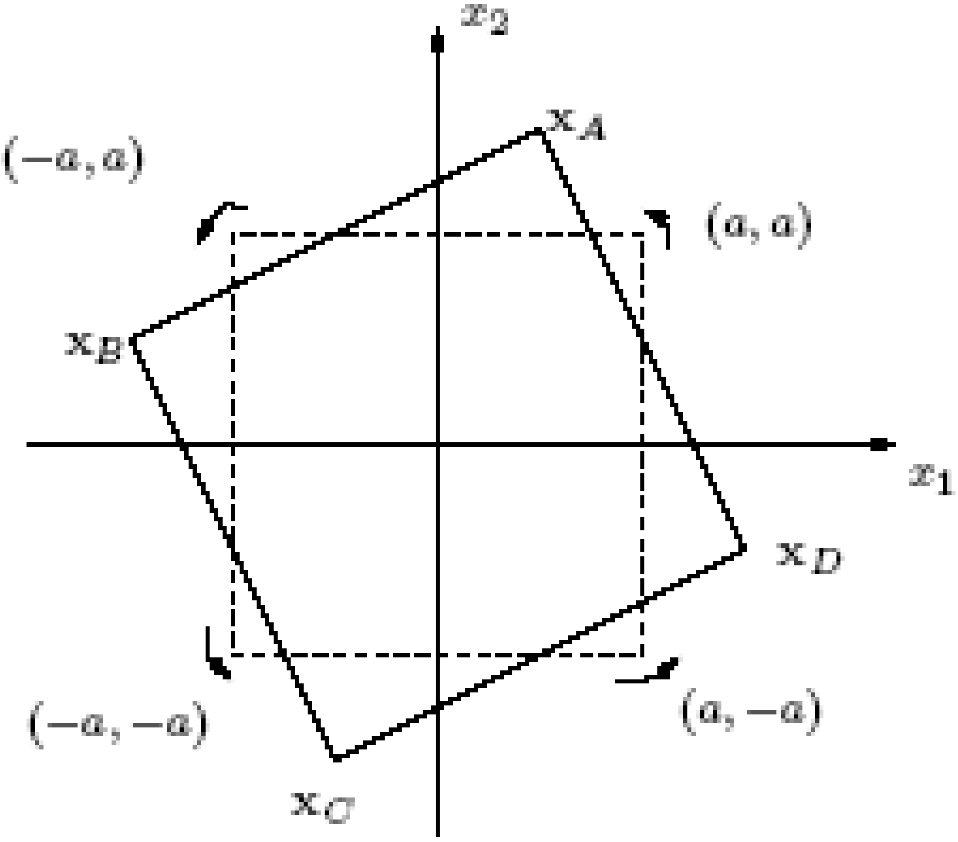
$$\mathbf{x} = \mathbf{R}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

over the two symbol times, where

$$\mathbf{R} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The four codewords of rotation code X_A, X_B, X_C & X_D

$$\mathbf{x}_A = \mathrm{R}\begin{bmatrix} a \\ a \end{bmatrix}, \quad \mathbf{x}_B = \mathbf{R}\begin{bmatrix} -a \\ a \end{bmatrix}, \quad \mathbf{x}_C = \mathbf{R}\begin{bmatrix} -a \\ -a \end{bmatrix}, \quad \mathbf{x}_D = \mathbf{R}\begin{bmatrix} a \\ -a \end{bmatrix}$$



- Error probability
  - Union bound rather than exact prob.
- If X_A is transmitted then

$$p_e \le \mathbb{P}\{\mathrm{x}_A \to \mathrm{x}_B\} + \mathbb{P}\{\mathrm{x}_A \to \mathrm{x}_C\} + \mathbb{P}\{\mathrm{x}_A \to \mathrm{x}_D\}$$

where

$$\mathbb{P}\{\mathrm{x}_A \to \mathrm{x}_B | h_1, h_2\} = Q\left(\frac{\|\mathbf{u}_A - \mathbf{u}_B\|}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\mathsf{SNR}/2 \cdot (|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}\right)$$

$$\mathbf{u}_A = \begin{bmatrix} h_1 x_{A1} \\ h_2 x_{A2} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_B = \begin{bmatrix} h_1 x_{B1} \\ h_2 x_{B2} \end{bmatrix}$$

$\mathsf{SNR} = a^2/N_0$ and

$$\mathbf{d} := \frac{1}{a}(\mathbf{x}_A - \mathbf{x}_B) = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \end{bmatrix}$$

d is the normalized difference between codewords

Product Distance

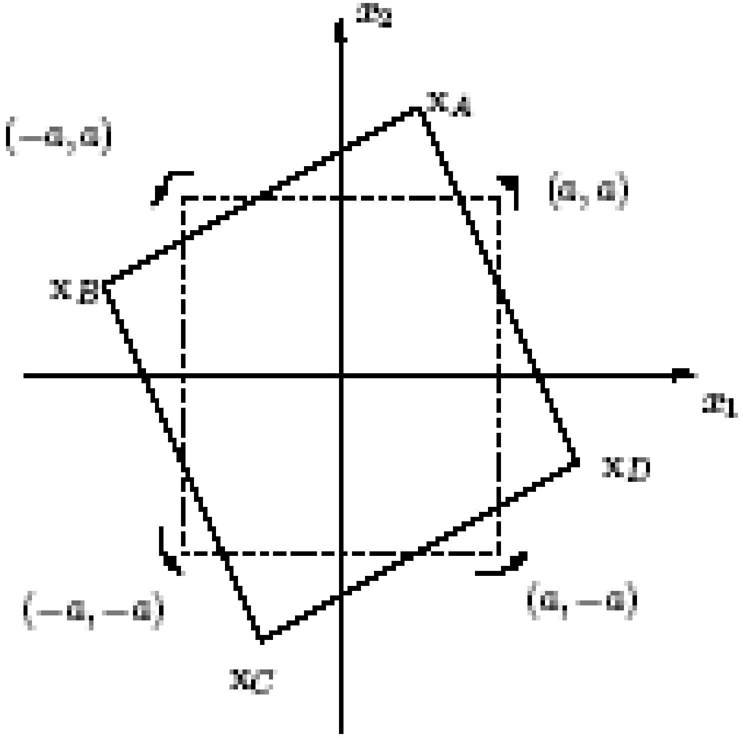
$$\mathcal{P}\{\mathbf{x}_A \to \mathbf{x}_B | h_1, h_2\} = Q\left(\frac{\mathsf{SNR}}{2}\left[|d_1|^2|h_1|^2 + |d_2|^2|h_2|^2\right]\right)$$

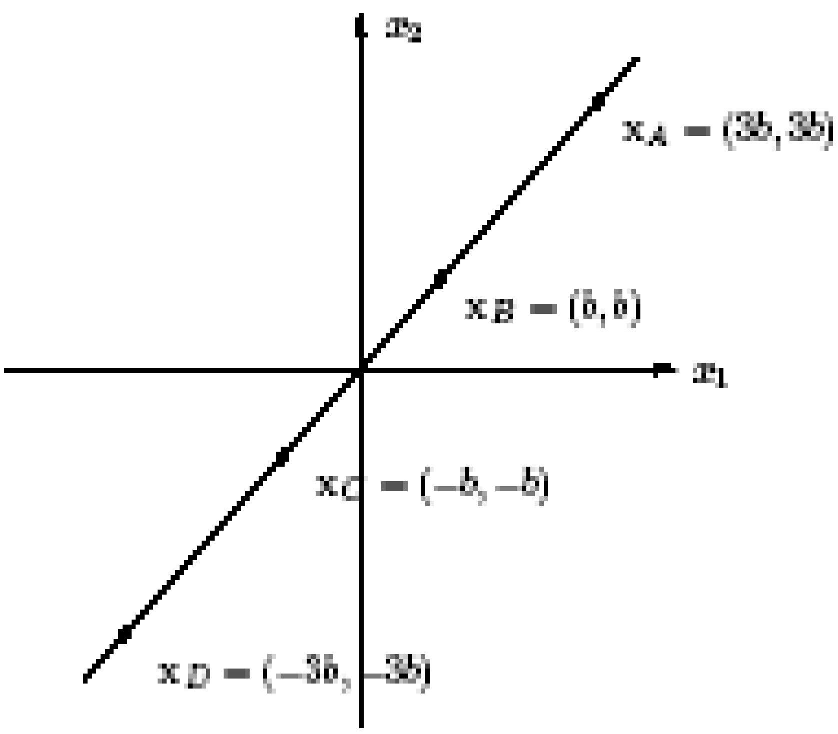
$$\begin{aligned} & \mathcal{P}\{\mathbf{x}_A \to \mathbf{x}_B\} \\ \approx \; & \mathcal{P}\left\{|d_1|^2|h_1|^2 < \frac{1}{\mathsf{SNR}} \quad \& \quad |d_2|^2|h_2|^2 < \frac{1}{\mathsf{SNR}}\right\} \\ \approx \; & \frac{1}{|d_1|^2|d_2|^2}\mathsf{SNR}^{-2} \end{aligned}$$

$$\text{product distance } = |d_1||d_2|.$$

Choose the rotation angle to maximize the worst-case product distance to all the other codewords:

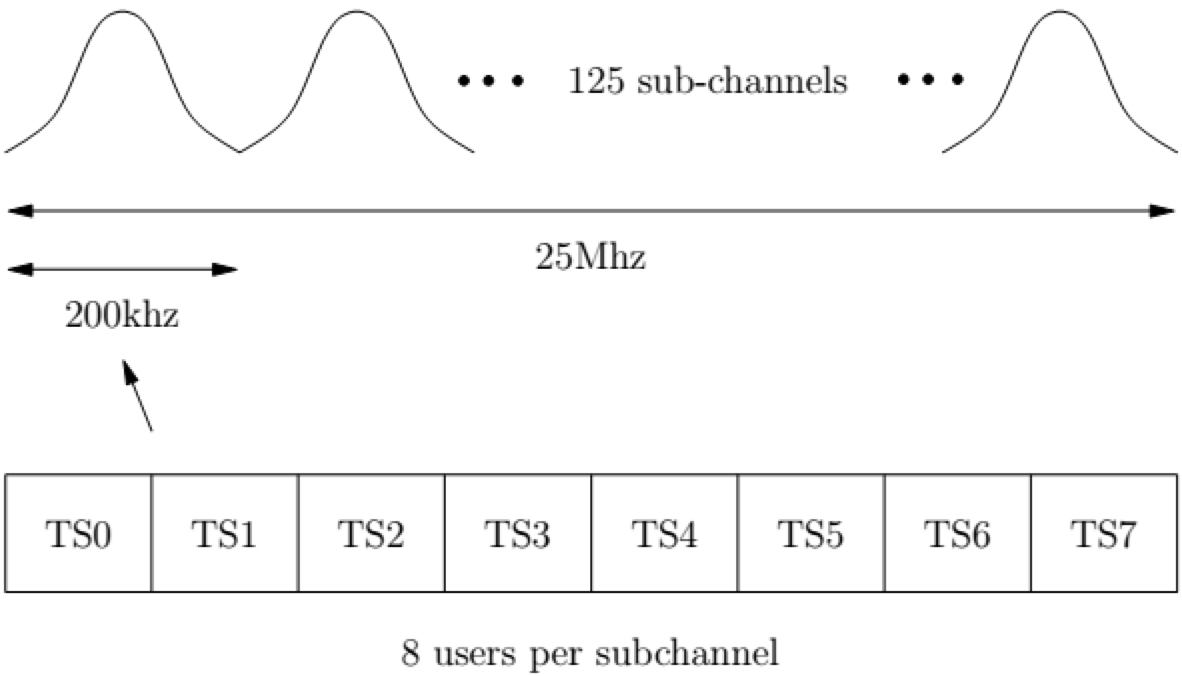
Rotation vs Repetition Coding





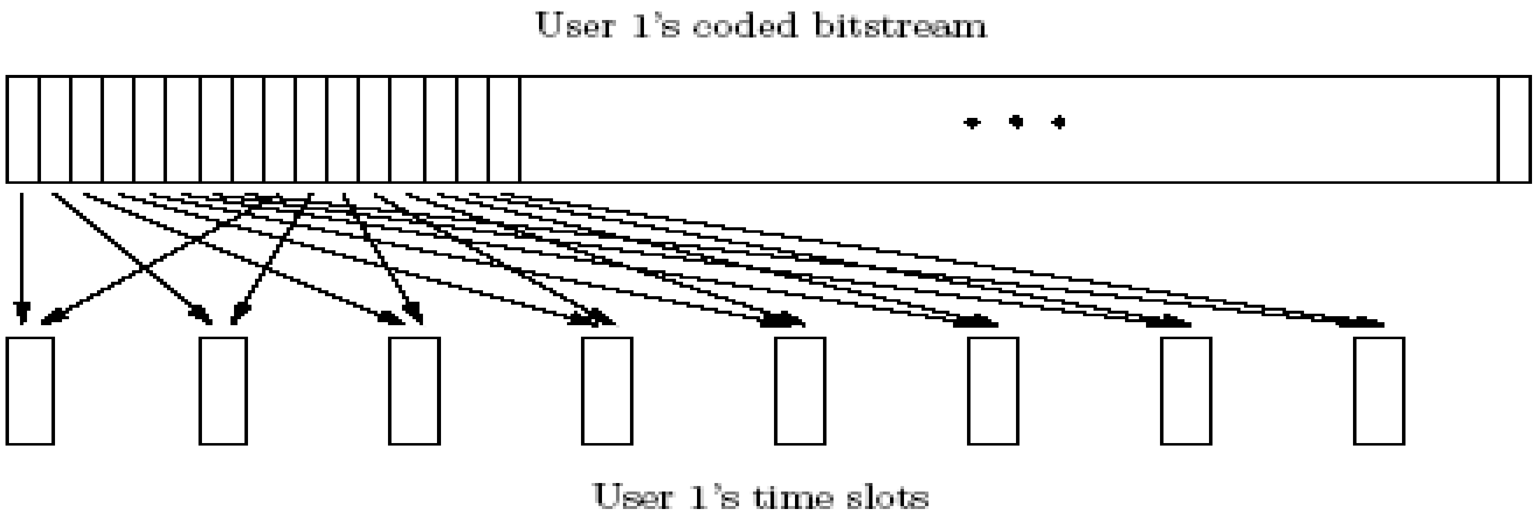
Example: GSM

- Global System for Mobile (GSM) is a digital cellular standard
- GSM is a frequency division duplex (FDD) system and uses two 25 MHz bands



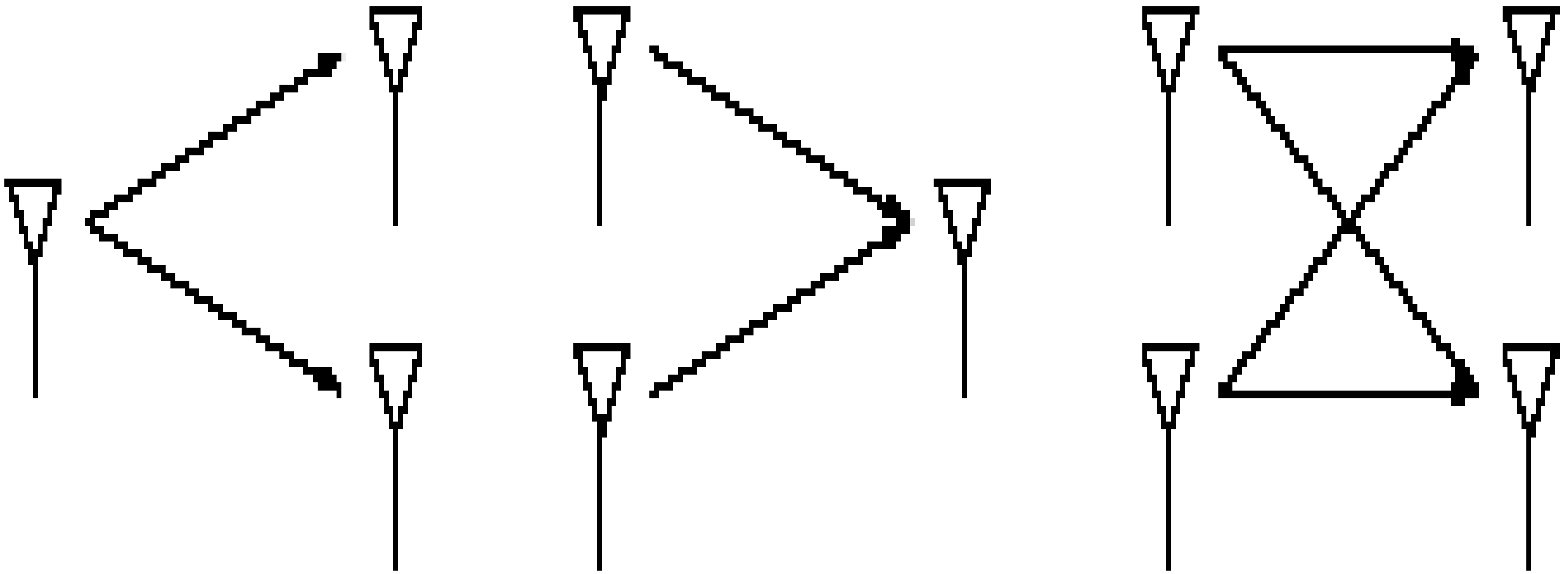
- each user has time slots 577 μs
- 8 users together form a frame of 4.615 ms

Example: GSM



- Amount of diversity limited by delay constraint and how fast channel varies.
- In GSM, delay constraint is 40ms (voice).
- To get full diversity of 8, needs v > 30 km/hr at f_c = 900Mhz.

Antenna Diversity



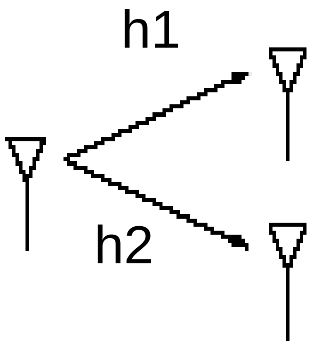
Receive

Transmit

Both

Receive Diversity

$$\mathbf{y} = x\mathbf{h} + \mathbf{w}$$



Same as repetition coding in time diversity, except that there is a further power gain.

Optimal reception is via match filtering (receive beamforming).

<u>With sufficient separation between Rx's antennas, i.e. channel gains are likely independent, we get a diversity gain of L</u>

Receive Diversity

- Two types of gain: power and diversity gains
- For BPSK, P_e

$$Q\left(\sqrt{2\|\mathbf{h}\|^2 \mathsf{SNR}}\right)$$

$$\|\mathbf{h}\|^2 \mathsf{SNR} = L\mathsf{SNR} \cdot \frac{1}{L}\|\mathbf{h}\|^2$$

- Power gain: coherent combining at the receiver, the effective total received power increases linearly with L
- Diversity gain: probability that overall gain is small decreases by averaging over multiple independent paths.
  - However, a diminishing marginal return as L increases

$$\frac{1}{L}\|\mathbf{h}\|^2 = \frac{1}{L}\sum_{\ell=1}^{L} |h_\ell[1]|^2$$

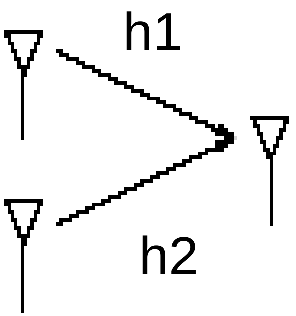
Transmit Diversity

- common in the downlink of a cellular system

$$y = \mathbf{h}^* \mathbf{x} + w$$

If transmitter knows the channel, send:

$$\mathbf{x} = x \frac{\mathbf{h}}{\|\mathbf{h}\|}.$$



maximizes the received SNR by in-phase addition of signals at the receiver (transmit beamforming).

Reduce to scalar channel:

$$y = \|\mathbf{h}\| x + w,$$

same as receive beamforming.

What happens if transmitter does not know the channel?

Space-time Codes

- Transmitting the same symbol simultaneously at the antennas doesn’t work.
- Using the antennas one at a time and sending the same symbol over the different antennas is like repetition coding.
  - But these codes are quite wasteful of degrees of freedom
- More generally, can use any time-diversity code by turning on one antenna at a time.
  - transmit coded symbols successively over L different antennas: one antenna at a time → code gain
- Space-time coding for transmit diversity systems

Alamouti Scheme

- Over flat fading and two symbol times, transmit [u_1 u_2] in T_1 and [$-u_2^*$ u_1^*] in T_2
- Assume the channel is constant over two symbol times

$$\begin{bmatrix} y[1] & y[2] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} + \begin{bmatrix} w[1] & w[2] \end{bmatrix}.$$

$$\begin{bmatrix} y[1] \\ y[2]^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w[1] \\ w[2]^* \end{bmatrix}$$

<u>Since we want to detect u_i</u>

- The columns are orthogonal

Projecting onto the two columns of the H matrix yields:

$$r_i = ||\mathbf{h}||u_i + w_i, \qquad i = 1, 2.$$

- double the symbol rate of repetition coding.
- 3dB loss of received SNR compared to transmit beamforming.

Space-time Code Design

A space-time code is a set of matrices $\{\mathbf{X}_i\}$.

Complex codeword Xi (L by N matrix).. L is transmit antennas & N is code block length

Assuming that the channel remains constant for N symbol times

$$\mathbf{y}^t = \mathbf{h}^*\mathbf{X} + \mathbf{w}^t,$$

where

$$\mathbf{y} := \begin{bmatrix} y[1] \\ \cdot \\ \cdot \\ y[N] \end{bmatrix}, \quad \mathbf{h} := \begin{bmatrix} h_1^* \\ \cdot \\ \cdot \\ h_L^* \end{bmatrix}, \quad \mathbf{w} := \begin{bmatrix} w[1] \\ \cdot \\ \cdot \\ w[N] \end{bmatrix}$$

$$\mathbb{P}\{\mathbf{X}_A \to \mathbf{X}_B \mid \mathbf{h}\} = Q\left(\frac{\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|}{2\sqrt{N_0/2}}\right)$$

$$\mathbb{P}\{\mathbf{X}_A \to \mathbf{X}_B \mid \mathbf{h}\} = Q\left(\frac{\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|}{2\sqrt{N_0/2}}\right)$$

$$\mathbb{P}\{\mathbf{X}_A \to \mathbf{X}_B\} \leq \frac{4^L}{\mathsf{SNR}^L \det[(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*]}$$

- diversity gain of L is achieved if all pairwise differences have full rank

- coding gain is determined by the minimum of $\det[(X_A - X_B)(X_A - X_B)^*]$ over all codeword pairs….. the determinant criterion.

Cooperative Diversity

- Different users can form a distributed antenna array to help each other in increasing diversity.
- Distributed versions of space-time codes may be applicable.
- Interesting characteristics:
  - Users have to exchange information and this consumes bandwidth.
  - Operation typically in half-duplex mode
  - Broadcast nature of the wireless medium can be exploited.

Frequency Diversity

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell x[m - \ell + w[m]$$

Resolution of multipaths provides diversity.

Full diversity is achieved by sending one symbol every L symbol times.

But this is inefficient (like repetition coding).

Sending symbols more frequently may result in intersymbol interference.

Challenge is how to mitigate the ISI while extracting the inherent diversity in the frequencyp-selective channel.

Approaches

- Time-domain equalization (eg. GSM)
- Direct-sequence spread spectrum (eg. IS-95 CDMA)
- Orthogonal frequency-division multiplexing OFDM (eg. 802.11a)

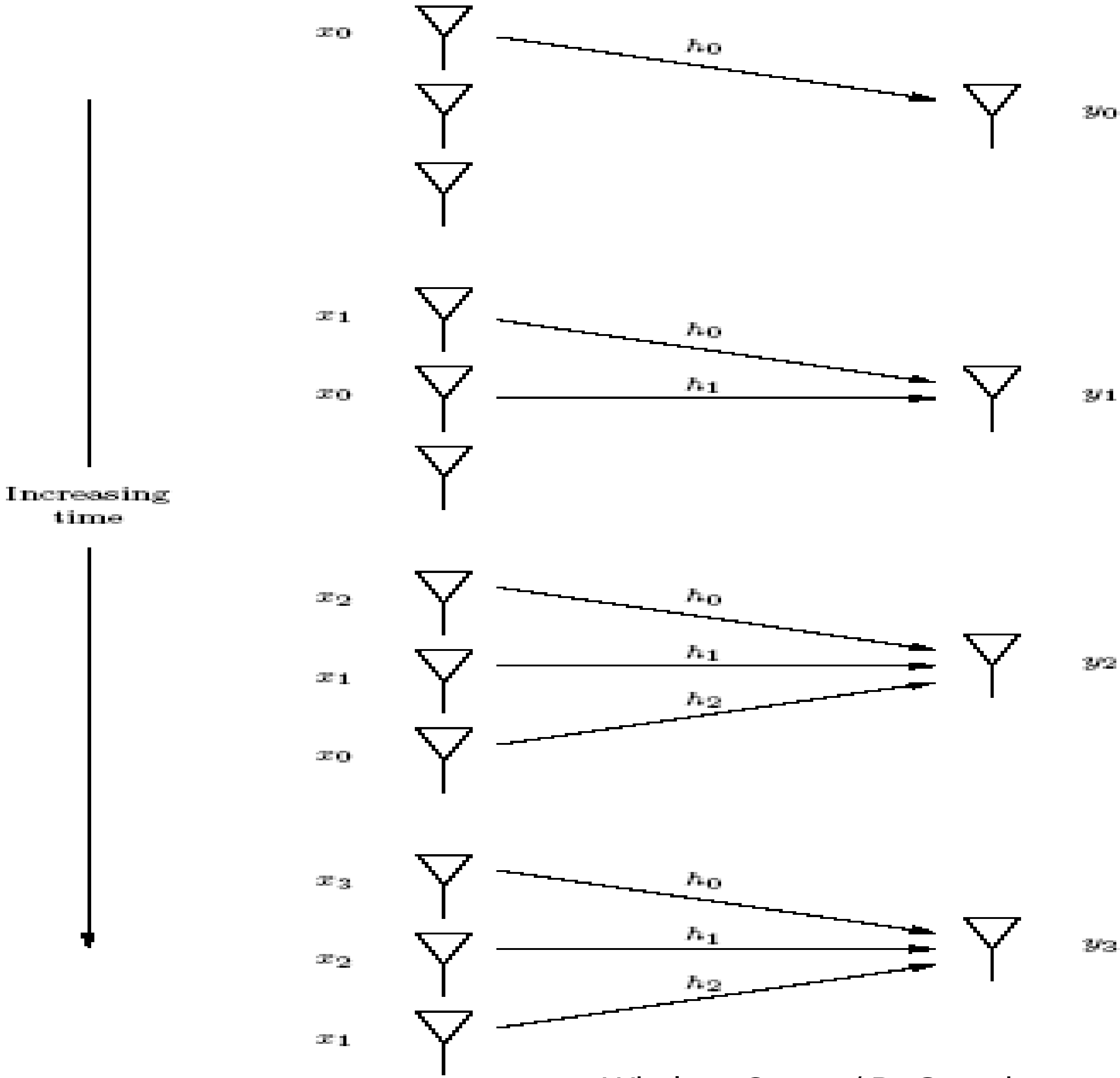
Single Carrier ISI Equalization

- Suppose a sequence of uncoded symbols are transmitted.

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell x[m-\ell] + w[m]$$

- Maximum likelihood sequence detection is performed using the Viterbi algorithm.
- Can full diversity be achieved?
  - transform the frequency-selective channel into a flat fading MISO channel with L transmit antennas and a single receive antenna

Reduction to Transmit Diversity



Space-time code matrix for input sequence $x[0], \ldots, x[N+L-1]$:

$$\mathbf{X} = \begin{bmatrix} x[1] & x[2] & \cdot & \cdot & \cdot & x[N] & \cdot & \cdot & x[N+L-1] \\ 0 & x[1] & x[2] & \cdot & \cdot & \cdot & x[N] & \cdot & x[N+L-2] \\ 0 & 0 & x[1] & x[2] & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & x[1] & x[2] & \cdot & \cdot & x[N] \end{bmatrix}$$

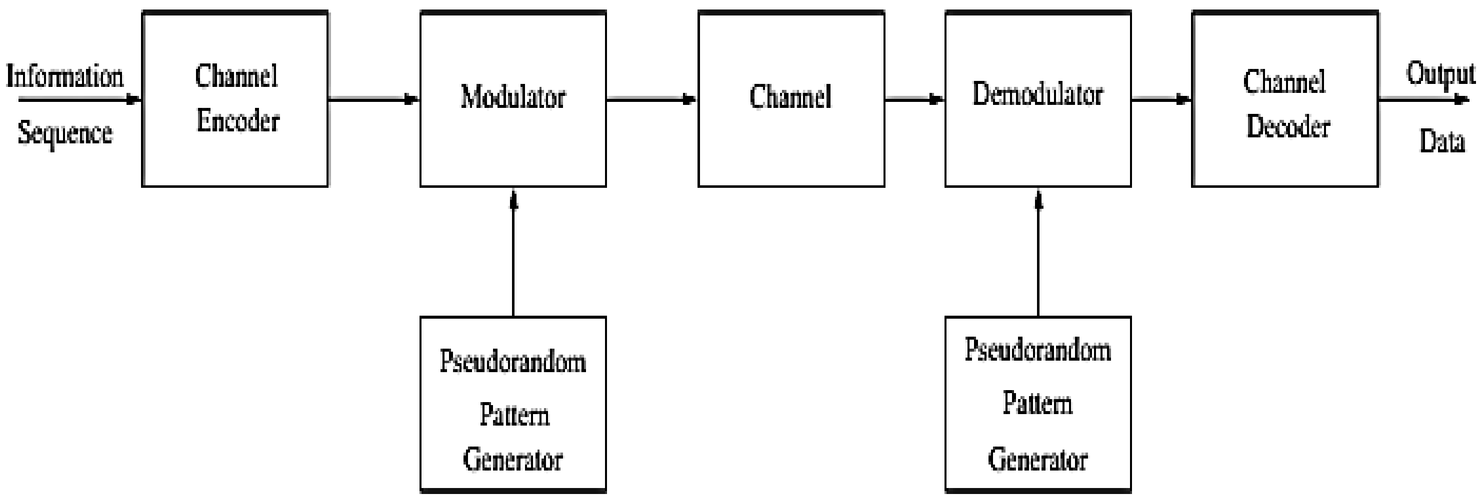
Difference matrix for two sequences first differing at $m^* \le N$:

$$\mathbf{X}_A - \mathbf{X}_B = \begin{bmatrix} 0 & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot \end{bmatrix}$$

is full rank.

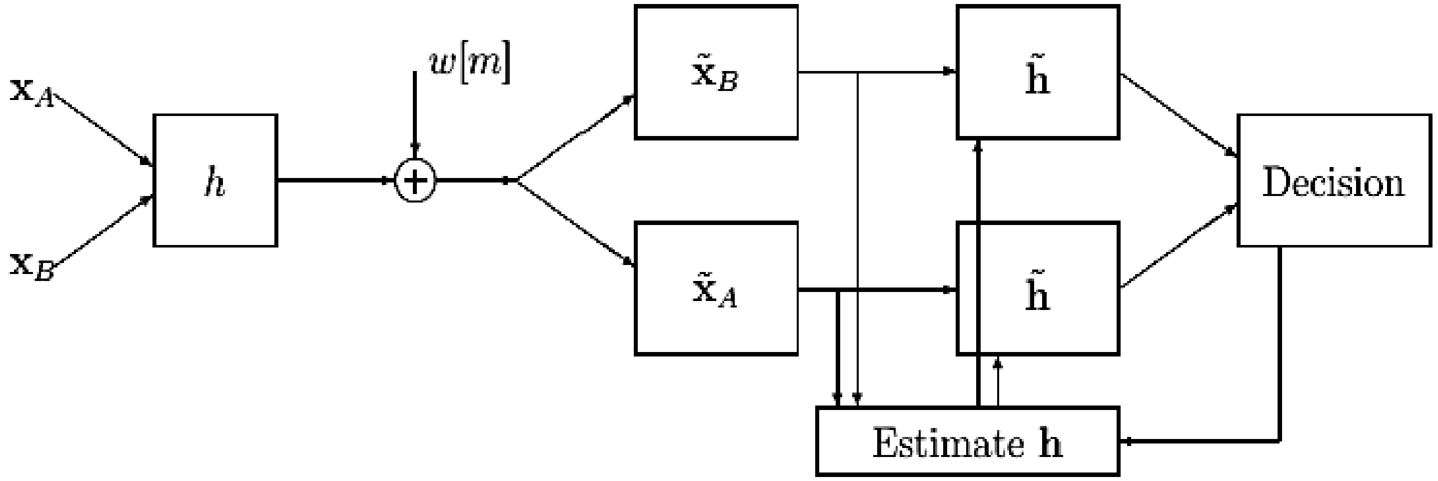
Direct Sequence Spread Spectrum

- Information symbol rate is much lower than chip rate → larger processing gain
- SNR per chip is low
- Few bits are transmitted per d.o.f per user
- ISI is not significant compared to interference from other users and match filtering (Rake) is near optimal

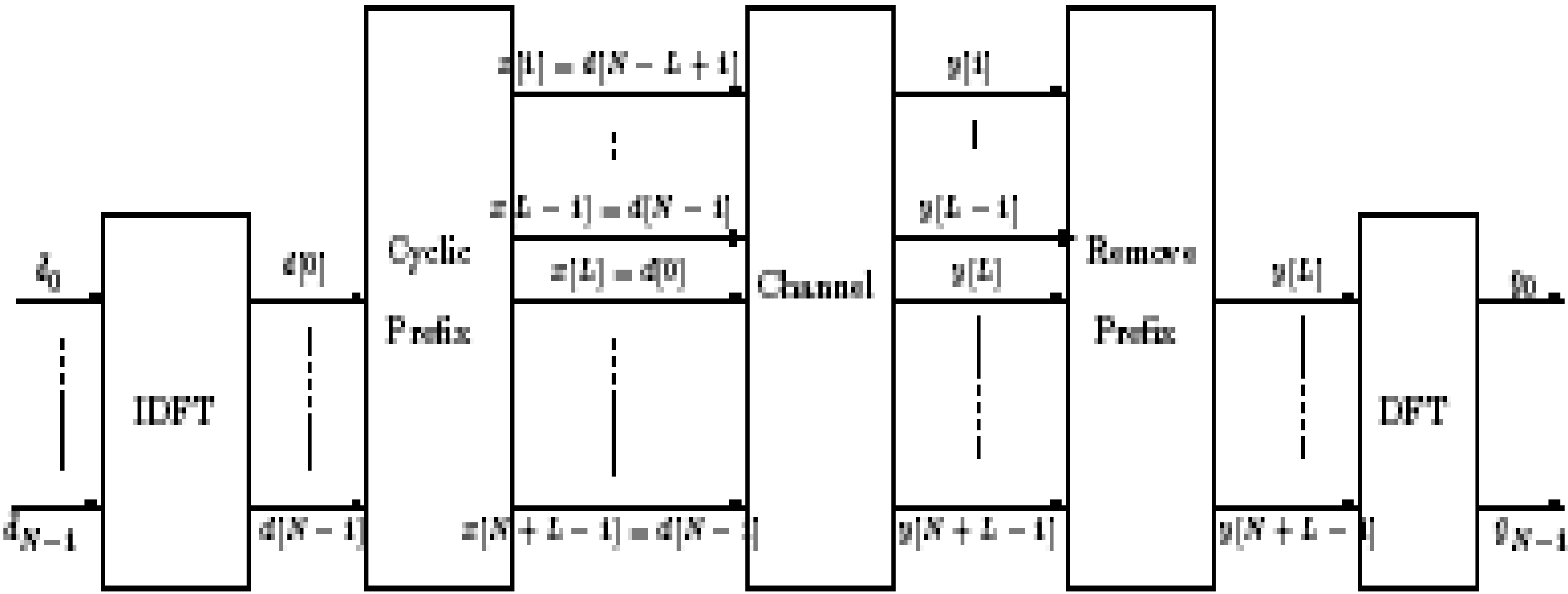


Frequency Diversity via Rake

- Suppose we transmit one of two n-chips long pseudonoise sequences x_A or x_B
- A binary symbol is transmitted over n chips
- Time invariant channel with L taps where $n \ll T_c W$
- Commonly $n \gg T_d W$ (i.e L) → No ISI
- Projecting y = [y[1], . . ., y[n + L]] onto n+L dimensional vectors v_A and v_B, where $v_A = [(h*x_A)[1], \ldots, (h*x_A)[n+L]]$
- Then two matched filters are needed



OFDM



IDFT
Cyclic Prefix
Channel
Remove Prefix
DFT
d[0]
d[N − 1]
x[1] = d[N − L + 1]
x[L − 1] = d[N − 1]
x[L] = d[0]
y[1]
y[L − 1]
y[L]
y[N + L − 1]

OFDM

OFDM transforms the communication problem into the frequency domain:

$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n, \quad n = 0, \ldots, N_c - 1.$$

a bunch of non-interfering sub-channels, one for each sub-carrier.

$$\tilde{h}_n = H_b\left(\frac{nW}{N_c}\right)$$

Can apply time-diversity techniques.

Channel Uncertainty

- In fast varying channels, tap gain measurement errors may have an impact on diversity combining performance
- The impact is particularly significant in channel with many taps each containing a small fraction of the total received energy.