Chapter Two

2.1 Fermat's Principle and its Applications

Fermat's principle is a remarkable and important principle in optics, which explains why light propagates along straight lines and the laws of reflection and refraction.

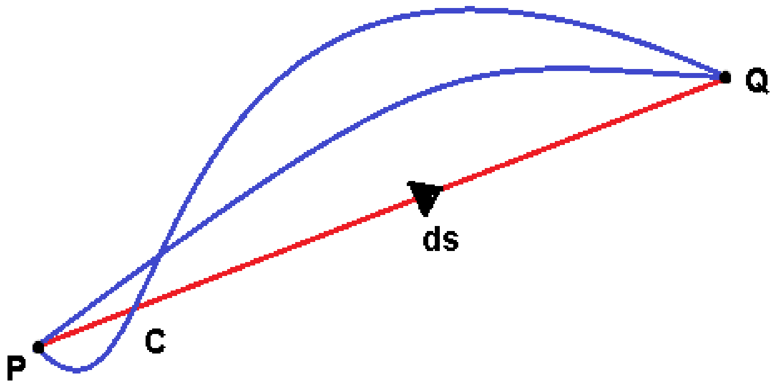
Fermat's Principle of Least Time

Fermat suggested in 1650 that the ***principle of shortest path*** be replaced with the ***principle of least time***. The principle of least time states that:

When a light ray travels between two points P and Q, it follows, out of all possible paths from P to Q, a path, which requires the least time.

Suppose ***P and Q*** is two points in the same medium. Light needs the time *dt* to travel the distance *dL* in the medium.

$$dt = \frac{dL}{v} \quad \text{............(2.1)}$$



The time required to travel the total distance PQ is:

$$t = \int_P^Q \frac{dL}{v} \quad \text{..........(2.2)}$$

As $v = \frac{c}{\mu}$ we can rewrite equation (2.2) as:

$$t = \frac{1}{c}\int_P^Q \mu dL \quad \text{..............(2.3)}$$

The quantity $\int_{P}^{Q} \mu dL$ is the optical path length, Δ.

Fermat's principle state that:

$$\frac{dt}{ds} = 0 \quad \text{……………(2.4)}$$

Where *ds* is a parameter that expresses the difference between any two given paths under comparison.

Equ. (2.3) may be rewritten as: $t = \frac{\Delta}{c}$ …………(2.5)

Therefore restate the Fermat's principle as follows:

Light travels along a path having the minimum optical path length.

Thus, the condition may now be expressed as

$$\frac{d\Delta}{ds} = 0 \quad \text{……………(2.6)}$$

It is found that there are a number of cases in which the real path of light is the one for which the time taken is not a minimum but a maximum. For example in case of a spherical reflector, light prefers maximum time. On the other hand, in elliptical reflectors light ray takes the same time for all paths. In view of these facts the Fermat's principle is to be modified. In its modern form the principle states that

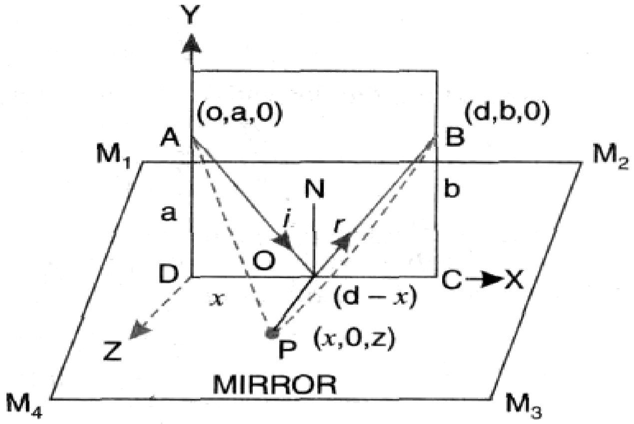
A light ray travelling from one point to another point will traverse a path for which, compared to all neighboring paths, the time required is a minimum or a maximum or stationary.

This is known as ***"Fermat's principle of extremum path"***

2.2 Laws of Reflection

First law of reflection:

Consider a plane mirror. A and B are two points above the mirror plane and are located in a plane ABCD normal to the mirror plane. Light coming from point A is reflected toward B. Suppose the light ray passes through a point P. It means that the light is incident along AP and is reflected along PB.



Thus, APB is a most general conceivable path from A to B.

Draw a plane ABCD normal to the mirror plane M1M2M3M4.

Let DC and DA be the x and y axes

Let DA = a, CB = b and DC = d

The point P has general coordinates (x,o,z). If now AP + PB = L, we get:

$$L = \left[(x-0)^2 + (0-a)^2 + (z-0)^2\right]^{1/2} + [(x-d)^2 + (0-b)^2 + (z-0)^2]^{1/2}$$

$$\text{or} \quad L = \sqrt{x^2 + a^2 + z^2} + \sqrt{(x-d)^2 + b^2 + z^2} \qquad (2.7)$$

We now apply Fermat's principle to get the actual path. The path APB can be varied by varying x and z. We obtain the minimum value of L, which is the shortest optical path by taking the derivative of L with respect to z and setting the derivative equal to zero. Thus,

$$\left(\frac{\partial L}{\partial z}\right)_x = \frac{1}{2}\left[\frac{1}{\sqrt{x^2 + a^2 + z^2}} \cdot 2z\right] + \frac{1}{2}\left[\frac{1}{\sqrt{(x-d)^2 + b^2 + z^2}} \cdot 2z\right] = 0$$

$$\therefore \qquad z\left[\frac{1}{\sqrt{x^2 + a^2 + z^2}} + \frac{1}{\sqrt{(x-d)^2 + b^2 + z^2}}\right] = 0$$

As the factor within the brackets cannot be zero, z must be equal to zero.

$$z = 0$$

The above result means that P must lie in the plane ABCD, which is normal to the mirror. O is such a position for P. If P coincides with O, it is obvious that the incident ray AO, the surface normal ON and the reflected ray OB lie in the same plane. This is the first law of reflection.

Second law of reflection:

Using z = 0 into equ. (2.7), we get

$$L = \sqrt{x^2 + a^2} + \sqrt{(x-d)^2 + b^2} \tag{2.8}$$

Now taking the derivative of L with respect to x, we get

$$\frac{dL}{dx} = \frac{1}{2}\frac{1}{\sqrt{x^2+a^2}}\cdot 2x + \frac{1}{2}\frac{1}{\sqrt{(x-d)^2+b^2}}\cdot 2(x-d)$$

Setting $\frac{dL}{dx} = 0$, we obtain

$$\frac{x}{\sqrt{x^2+a^2}} + \frac{(x-d)}{\sqrt{(x-d)^2+b^2}} = 0$$

$$\therefore \quad \frac{x}{\sqrt{x^2+a^2}} = \frac{-(x-d)}{\sqrt{(x-d)^2+b^2}}$$

or

$$\frac{x}{\sqrt{x^2+a^2}} = \frac{(d-x)}{\sqrt{(x-d)^2+b^2}}$$

Looking at triangles AOD and BOC

$$\sin i = \frac{x}{\sqrt{x^2+a^2}} \text{ and } \sin r = \frac{(d-x)}{\sqrt{(x-d)^2+b^2}}$$

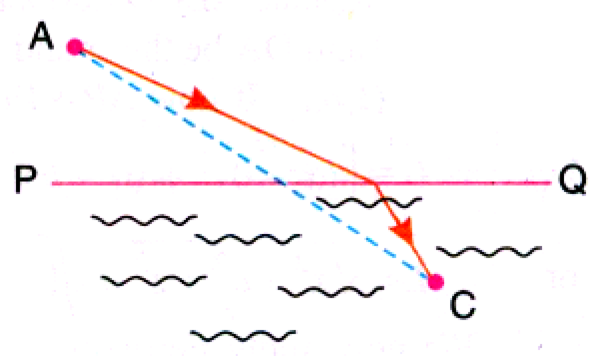
$$\therefore \quad \sin i = \sin r$$

$$\therefore \quad i = r \tag{2.9}$$

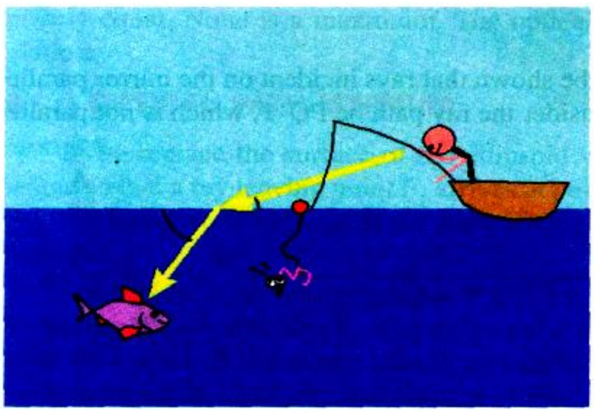
This is the second law of reflection.

2.3 Laws of Refraction

We now apply Fermat's principle for refraction phenomenon. We have a boundary PQ between two media. Light goes from A to C. If the speed of light on both sides of the boundary were the same, the path from A to C would be a straight line. The speed above PQ and that below PQ are different and we assume that the medium above PQ is air and below PQ is water.



Consider a plane surface S separating two media. Let A and C be two points lying in the two different media. We must find the path from A to C. Clearly, the path must consist of two straight lines namely AB in medium 1 and BC in medium 2; the point B in the plane S has to be found. Let K be any general point on the surface S. It has the coordinates (x,0,z). The optical path through the point K (x, 0, z) is given by



It shows how light from the fish is refracted through the air-water interface to the fisherman.

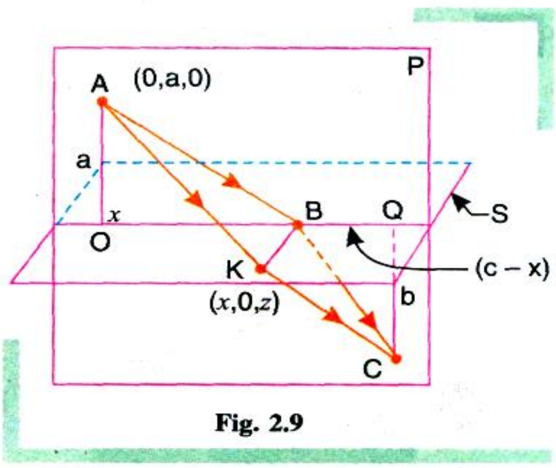


Fig. 2.9

$$\Delta = (AK)\,\mu_1 + (KC)\mu_2$$

$$= \mu_1\left(x^2 + a^2 + z^2\right)^{1/2} + \mu_2\left[(x-c)^2 + b^2 + z^2\right]^{1/2} \qquad (2.10)$$

Now apply Fermat's condition to get the actual path.

$$\left(\frac{\partial \Delta}{\partial z}\right)_x = 0$$

$$\therefore \qquad z = 0$$

It means that the incident ray, the refracted ray and the normal to the plane PQRS lie in the same plane. This is the first law of refraction.

Applying the second condition that $\left(\frac{\partial \Delta}{\partial z}\right)_x = 0$, we get

$$\left(\frac{\partial \Delta}{\partial z}\right)_x = \frac{\mu_1}{2}\left[\frac{2x}{\sqrt{x^2 + a^2 + z^2}}\right] + \frac{\mu_2}{2}\left[\frac{2(x-c)}{\sqrt{(x-c)^2 + b^2 + z^2}}\right] = 0$$

$$\frac{\mu_1 x}{\sqrt{x^2 + a^2 + z^2}} = \frac{\mu_2\,(c-x)}{\sqrt{(x-c)^2 + b^2 + z^2}}$$

$$\mu_1 \sin\, i = \mu_2 \sin\, r$$

$$\therefore \qquad \frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} \qquad (2.11)$$

This is *Snell's law*.

Questions:

1. State and explain Fermat's principle of extremum path and use it to deduce the laws reflection and refraction of light.
2. State and explain Fermat's principle of stationary time. Derive the laws of refraction using this principle. Give an example where the path of light is a relative maximum rather than a minimum.
3. State and explain Fermat's principle of extremum path and analyze a case where the actual path of light may be a maximum. Use Fermat's principle to deduce Snell's law refraction.