# SAMPLE SPACES

## DEFINITION :

The *sample space* is the set of all possible outcomes of an experiment.

**EXAMPLE** : When we *flip a coin* then sample space is

$$\mathcal{S} = \{ H , T \} ,$$

where

$H$ denotes that the coin lands "Heads up"

and

$T$ denotes that the coin lands "Tails up".

For a "*fair coin* " we expect H and T to have the same "*chance* " of occurring, *i.e.*, if we flip the coin many times then about 50 % of the outcomes will be $H$.

We say that the *probability* of H to occur is 0.5 (or 50 %) .

The probability of $T$ to occur is then also 0.5.

## EXAMPLE :

When we *roll a fair die* then the sample space is

$$\mathcal{S} = \{ 1 , 2 , 3 , 4 , 5 , 6 \} .$$

The probability the die lands with $k$ up is $\frac{1}{6}$ , $(k = 1, 2, \cdots, 6)$.

When we roll it 1200 times we expect a 5 up about 200 times.

The probability the die lands with an *even number* up is

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} .$$

**EXAMPLE** :

When we toss a coin 3 times and record the results in the *sequence* that they occur, then the sample space is

$$\mathcal{S} = \{ HHH , HHT , HTH , HTT , THH , THT , TTH , TTT \} .$$

Elements of $\mathcal{S}$ are "*vectors*", "*sequences*", or "*ordered outcomes*".

We may expect each of the 8 outcomes to be equally likely.

Thus the probability of the sequence $HTT$ is $\frac{1}{8}$ .

The probability of a sequence to contain precisely two Heads is

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} .$$

**EXAMPLE** : When we toss a coin 3 times and record the results without paying attention to the order in which they occur, *e.g.*, if we only record the number of Heads, then the sample space is

$$\mathcal{S} = \Big\{ \{H,H,H\} , \{H,H,T\} , \{H,T,T\} , \{T,T,T\} \Big\} .$$

The outcomes in $\mathcal{S}$ are now *sets* ; *i.e.*, order is not important.

Recall that the ordered outcomes are

$$\{ HHH , HHT , HTH , HTT , THH , THT , TTH , TTT \} .$$

Note that

| | | | |
|---|---|---|---|
| $\{H,H,H\}$ | corresponds to | *one* | of the ordered outcomes, |
| $\{H,H,T\}$ | ,, | *three* | ,, |
| $\{H,T,T\}$ | ,, | *three* | ,, |
| $\{T,T,T\}$ | ,, | *one* | ,, |

Thus $\{H,H,H\}$ and $\{T,T,T\}$ each occur with probability $\frac{1}{8}$,

while $\{H,H,T\}$ and $\{H,T,T\}$ each occur with probability $\frac{3}{8}$.

## Events

In Probability Theory subsets of the sample space are called *events*.

**EXAMPLE** : The set of basic outcomes of rolling a die *once* is

$$\mathcal{S} = \{ 1 , 2 , 3 , 4 , 5 , 6 \} ,$$

so the subset $E = \{ 2 , 4 , 6 \}$ is an example of an event.

If a die is rolled *once* and it lands with a 2 *or* a 4 *or* a 6 up then we say that the event $E$ has *occurred*.

We have already seen that the probability that $E$ occurs is

$$P(E) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} .$$

## The Algebra of Events

Since events are *sets*, namely, subsets of the sample space $\mathcal{S}$, we can do the usual *set operations* :

If $E$ and $F$ are events then we can form

| | |
|---|---|
| $E^c$ | the *complement* of $E$ |
| $E \cup F$ | the *union* of $E$ and $F$ |
| $EF$ | the *intersection* of $E$ and $F$ |

We write $E \subset F$ if $E$ is a *subset* of $F$.

**REMARK** : In Probability Theory we use

$E^c$ instead of $\bar{E}$ ,

$EF$ instead of $E \cap F$ ,

$E \subset F$ instead of $E \subseteq F$ .

If the sample space $\mathcal{S}$ is *finite* then we typically allow any subset of $\mathcal{S}$ to be an event.

**EXAMPLE** : If we randomly draw *one character* from a box containing the characters $a$, $b$, and $c$, then the sample space is

$$\mathcal{S} \;=\; \{a \;,\; b \;,\; c\} \;,$$

and there are 8 possible events, namely, those in the set of events

$$\mathcal{E} \;=\; \Big\{ \{\;\} \;,\; \{a\} \;,\; \{b\} \;,\; \{c\} \;,\; \{a,b\} \;,\; \{a,c\} \;,\; \{b,c\} \;,\; \{a,b,c\} \Big\} \;.$$

If the outcomes $a$, $b$, and $c$, are equally likely to occur, then

$$P(\{\;\}) = 0 \;\;,\;\; P(\{a\}) = \frac{1}{3} \;\;,\;\; P(\{b\}) = \frac{1}{3} \;\;,\;\; P(\{c\}) = \frac{1}{3} \;,$$

$$P(\{a,b\}) = \frac{2}{3} \;,\; P(\{a,c\}) = \frac{2}{3} \;,\; P(\{b,c\}) = \frac{2}{3} \;,\; P(\{a,b,c\}) = 1 \;.$$

For example, $P(\{a,b\})$ is the probability the character is an $a$ *or* a $b$.

We always assume that the set $\mathcal{E}$ of allowable events *includes the complements, unions, and intersections* of its events.

**EXAMPLE** : If the sample space is

$$\mathcal{S} = \{a\ ,\ b\ ,\ c\ ,\ d\}\ ,$$

and we start with the events

$$\mathcal{E}_0 = \Big\{ \{a\}\ ,\ \{c,d\} \Big\}\ ,$$

then this set of events needs to be extended to (at least)

$$\mathcal{E} = \Big\{ \{\ \}\ ,\ \{a\}\ ,\ \{c,d\}\ ,\ \{b,c,d\}\ ,\ \{a,b\}\ ,\ \{a,c,d\}\ ,\ \{b\}\ ,\ \{a,b,c,d\} \Big\}\ .$$

**EXERCISE** : Verify $\mathcal{E}$ includes complements, unions, intersections.

# Axioms of Probability

A *probability function* $P$ assigns a real number (the *probability* of $E$) to every event $E$ in a sample space $\mathcal{S}$.

$P(\cdot)$ must satisfy the following basic properties :

- $0 \leq P(E) \leq 1$ ,
- $P(\mathcal{S}) = 1$ ,
- For any *disjoint events* $E_i$ , $i = 1, 2, \cdots, n$, we have

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots P(E_n) .$$

# Further Properties

**PROPERTY 1** :

$$P(E \cup E^c) \;=\; P(E) \;+\; P(E^c) \;=\; 1\,. \qquad (\text{ Why ? })$$

Thus

$$P(E^c) \;=\; 1 \;-\; P(E)\,.$$

**EXAMPLE** :

What is the probability of at least one "H" in *four tosses* of a coin?

**SOLUTION** : The sample space $\mathcal{S}$ will have 16 outcomes. (Which?)

$$P(\text{at least one H}) \;=\; 1 \;-\; P(\text{no H}) \;=\; 1 \;-\; \frac{1}{16} \;=\; \frac{15}{16}\,.$$

**PROPERTY 2** :

$$P(E \cup F) \;=\; P(E) \;+\; P(F) \;-\; P(EF) \;.$$

**PROOF** (using the third axiom) :

$$P(E \cup F) \;=\; P(EF) \;+\; P(EF^c) \;+\; P(E^cF)$$

$$= \; [P(EF) \;+\; P(EF^c)] \;+\; [P(EF) \;+\; P(E^cF)] \;-\; P(EF)$$

$$= \; \mathrm{P(E)} \;+\; \mathrm{P(F)} \;-\; \mathrm{P(EF)} \;.$$ ( Why ? )

**NOTE** :

- Draw a Venn diagram with $E$ and $F$ to see this !
- The formula is similar to the one for the number of elements :

$$n(E \cup F) \;=\; n(E) \;+\; n(F) \;-\; n(EF) \;.$$

So far our sample spaces $\mathcal{S}$ have been *finite*.

$\mathcal{S}$ can also be *countably infinite*, *e.g.*, the set $\mathbb{Z}$ of all integers.

$\mathcal{S}$ can also be *uncountable*, *e.g.*, the set $\mathbb{R}$ of all real numbers.

**EXAMPLE** : Record the low temperature in Montreal on January 8 in each of a large number of years.

We can take $\mathcal{S}$ to be the set of *all real numbers*, *i.e.*, $\mathcal{S} = \mathbb{R}$.

(Are there are other choices of $\mathcal{S}$ ?)

What probability would you expect for the following *events* to have?

(a) $P(\{\pi\})$ (b) $P(\{x \; : \; -\pi < x < \pi\})$

(How does this differ from finite sample spaces?)

We will encounter such infinite sample spaces many times $\cdots$