CONDITIONAL PROBABILITY

Giving more information can change the probability of an event.

EXAMPLE :

If a coin is tossed two times then what is the probability of two Heads?

ANSWER :

$$\frac{1}{4}\ .$$

EXAMPLE :

If a coin is tossed two times then what is the probability of two Heads, *given that the first toss gave Heads* ?

ANSWER :

$$\frac{1}{2}\ .$$

NOTE :

Several examples will be about *playing cards* .

A standard *deck* of *playing cards* consists of 52 cards :

- Four *suits* :

  Hearts , Diamonds (*red*) , and Spades , Clubs (*black*) .

- Each suit has 13 cards, whose *denomination* is

  $2 \ , \ 3 \ , \ \cdots \ , 10$, Jack , Queen , King , Ace .

- The Jack , Queen , and King are called *face cards* .

EXERCISE :

Suppose we draw a card from a shuffled set of 52 playing cards.

- What is the probability of drawing a Queen ?
- What is the probability of drawing a Queen, given that the card drawn is of *suit* Hearts ?
- What is the probability of drawing a Queen, given that the card drawn is a *Face card* ?

What do the answers tell us?

(We'll soon learn the events "Queen" and "Hearts" are *independent* .)

The two preceding questions are examples of *conditional probability* .

Conditional probability is an *important* and *useful* concept.

If E and F are events, *i.e.*, subsets of a sample space $\mathcal{S}$, then

$P(E|F)$ *is the conditional probability of* E *, given* F ,

defined as

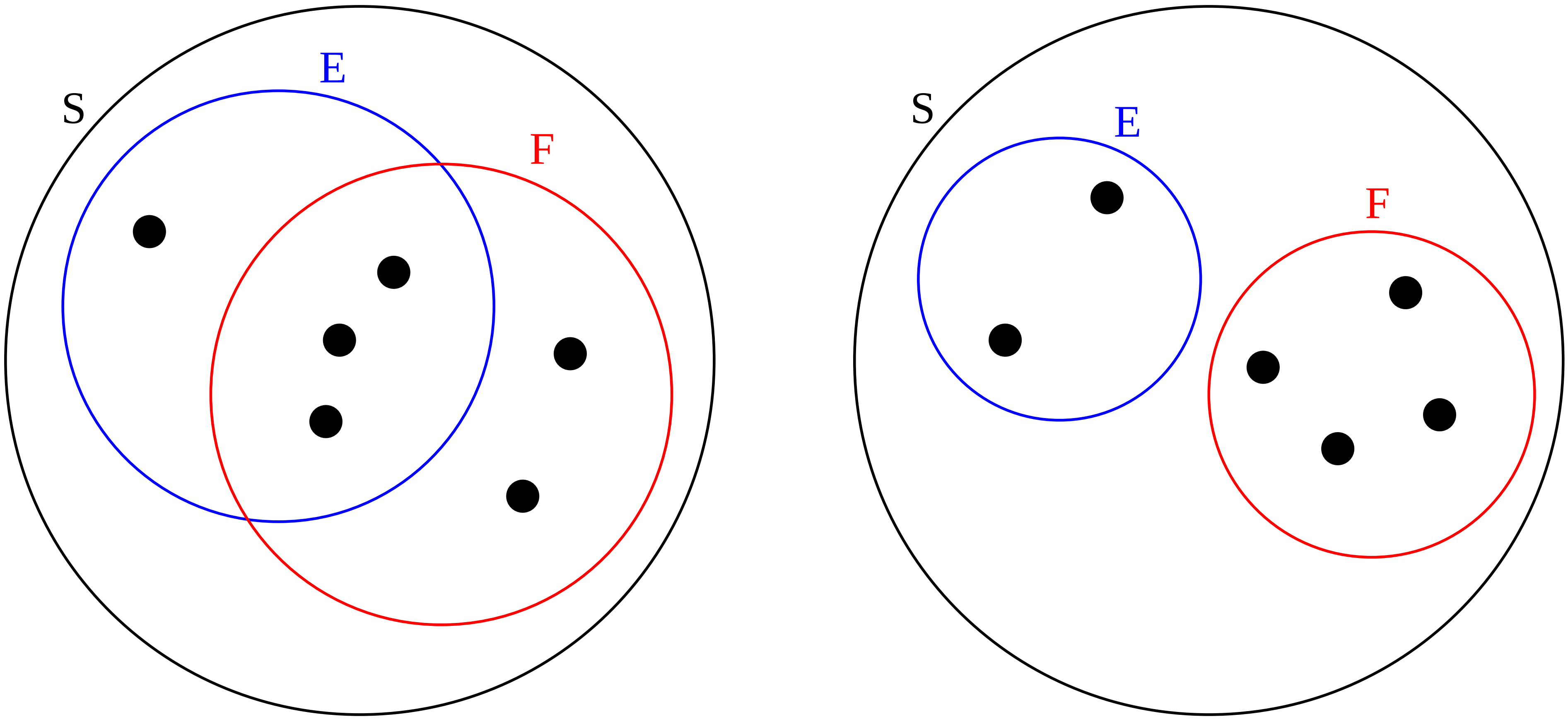
$$P(E|F) \equiv \frac{P(EF)}{P(F)} .$$

or, equivalently

$$P(EF) = P(E|F)\, P(F) ,$$

(assuming that $P(F)$ is not zero).

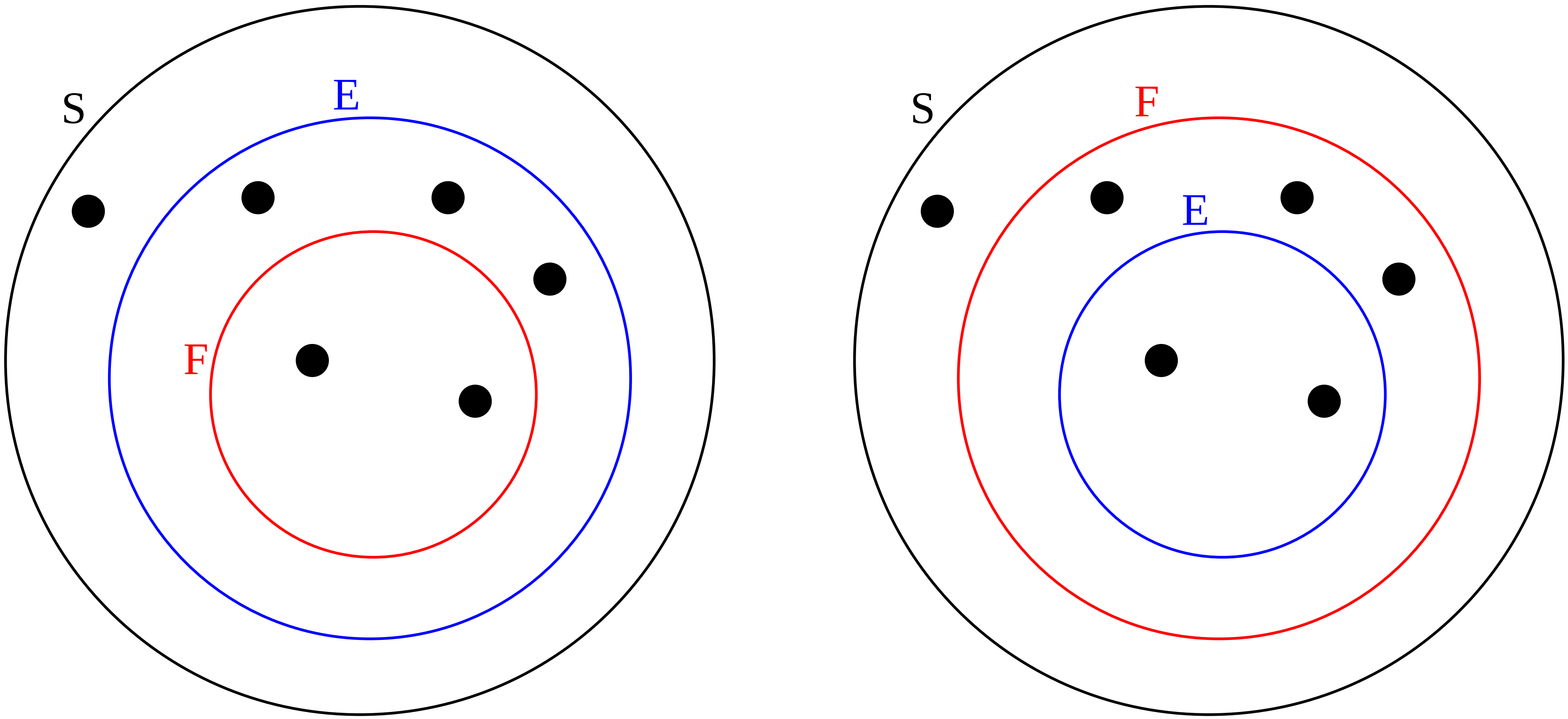
$$P(E|F) \equiv \frac{P(EF)}{P(F)}$$



Suppose that the 6 outcomes in $\mathcal{S}$ are equally likely.

What is $P(E|F)$ in each of these two cases ?

$$P(E|F) \equiv \frac{P(EF)}{P(F)}$$



Suppose that the 6 outcomes in $\mathcal{S}$ are equally likely.

What is $P(E|F)$ in each of these two cases ?

EXAMPLE : Suppose a coin is tossed two times.

The sample space is

$$\mathcal{S} \;=\; \{HH\,,\,HT\,,\,TH\,,\,TT\}\,.$$

Let E be the event "*two Heads*" , *i.e.*,

$$E \;=\; \{HH\}\,.$$

Let F be the event "*the first toss gives Heads*" , *i.e.*,

$$F \;=\; \{HH\,,\,HT\}\,.$$

Then

$$EF \;=\; \{HH\} \;=\; E \qquad (\text{ since } \; E \subset F\;)\,.$$

We have

$$P(E|F) \;=\; \frac{P(EF)}{P(F)} \;=\; \frac{P(E)}{P(F)} \;=\; \frac{\frac{1}{4}}{\frac{2}{4}} \;=\; \frac{1}{2}\,.$$

EXAMPLE :

Suppose we draw a card from a shuffled set of 52 playing cards.

- What is the probability of drawing a Queen, given that the card drawn is of *suit* Hearts ?

ANSWER :

$$P(Q|H) \quad = \quad \frac{P(QH)}{P(H)} \quad = \quad \frac{\frac{1}{52}}{\frac{13}{52}} \quad = \quad \frac{1}{13} \ .$$

- What is the probability of drawing a Queen, given that the card drawn is a *Face card* ?

ANSWER :

$$P(Q|F) \quad = \quad \frac{P(QF)}{P(F)} \quad = \quad \frac{P(Q)}{P(F)} \quad = \quad \frac{\frac{4}{52}}{\frac{12}{52}} \quad = \quad \frac{1}{3} \ .$$

(Here $Q \subset F$, so that $QF = Q$.)

The probability of an event E is sometimes computed more easily

if we condition E *on another event* F *,*

namely, from

$$P(E) = P(\ E(F \cup F^c)\ ) \qquad (\text{ Why ? })$$

$$= P(\ EF \cup EF^c\ ) = P(EF) + P(EF^c) \qquad (\text{ Why ? })$$

and

$$P(EF) = P(E|F)\ P(F) \quad , \quad P(EF^c) = P(E|F^c)\ P(F^c)\ ,$$

we obtain this *basic formula*

$$P(E) = P(E|F) \cdot P(F) + P(E|F^c) \cdot P(F^c)\ .$$

EXAMPLE :

An insurance company has these data :

The probability of an insurance claim in a period of one year is

4 percent for persons under age 30

2 percent for persons over age 30

and it is known that

30 percent of the targeted population is under age 30.

What is the probability of an insurance claim in a period of one year for a randomly chosen person from the targeted population?

SOLUTION :

Let the sample space $\mathcal{S}$ be all persons under consideration.

Let C be the event (subset of $\mathcal{S}$) of persons filing a claim.

Let U be the event (subset of $\mathcal{S}$) of persons under age 30.

Then U^c is the event (subset of $\mathcal{S}$) of persons over age 30.

Thus

$$\begin{aligned} P(C) &= P(C|U)\,P(U) + P(C|U^c)\,P(U^c) \\ &= \frac{4}{100}\,\frac{3}{10} + \frac{2}{100}\,\frac{7}{10} \\ &= \frac{26}{1000} = 2.6\% \,. \end{aligned}$$

EXAMPLE :

Two balls are drawn from a bag with 2 *white* and 3 *black* balls.

There are 20 outcomes (*sequences*) in $\mathcal{S}$. (Why ?)

What is the probability that *the second ball is white* ?

SOLUTION :

Let F be the event that *the first ball is white*.

Let S be the event that *the second second ball is white*.

Then

$$P(S) \;=\; P(S|F)\,P(F) + P(S|F^c)\,P(F^c) \;=\; \frac{1}{4}\cdot\frac{2}{5} \;+\; \frac{2}{4}\cdot\frac{3}{5} \;=\; \frac{2}{5}.$$

QUESTION : Is it surprising that $P(S) \;=\; P(F)$?

EXAMPLE : (continued $\cdots$)

Is it surprising that $P(S) = P(F)$?

ANSWER : Not really, if one considers the sample space $\mathcal{S}$:

$$\begin{Bmatrix} \mathbf{w}_1\mathbf{w}_2 , & \mathbf{w}_1 b_1 , & \mathbf{w}_1 b_2 , & \mathbf{w}_1 b_3 , \\ \mathbf{w}_2\mathbf{w}_1 , & \mathbf{w}_2 b_1 , & \mathbf{w}_2 b_2 , & \mathbf{w}_2 b_3 , \\ b_1\mathbf{w}_1 , & b_1\mathbf{w}_2 , & b_1 b_2 , & b_1 b_3 , \\ b_2\mathbf{w}_1 , & b_2\mathbf{w}_2 , & b_2 b_1 , & b_2 b_3 , \\ b_3\mathbf{w}_1 , & b_3\mathbf{w}_2 , & b_3 b_1 , & b_3 b_2 \end{Bmatrix} ,$$

where outcomes (*sequences*) are assumed equally likely.

EXAMPLE :

Suppose we draw *two cards* from a shuffled set of 52 playing cards.

What is the probability that the second card is a Queen ?

ANSWER :

$$P(2^{\text{nd}} \text{ card } Q) \; =$$

$$P(2^{\text{nd}} \text{ card } Q | 1^{\text{st}} \text{ card } Q) \; \cdot \; P(1^{\text{st}} \text{ card } Q)$$

$$+ \; P(2^{\text{nd}} \text{ card } Q | 1^{\text{st}} \text{ card not } Q) \; \cdot \; P(1^{\text{st}} \text{ card not } Q)$$

$$= \frac{3}{51} \cdot \frac{4}{52} + \frac{4}{51} \cdot \frac{48}{52} = \frac{204}{51 \cdot 52} = \frac{4}{52} = \frac{1}{13} \, .$$

QUESTION : Is it surprising that $P(2^{\text{nd}} \text{ card } Q) = P(1^{\text{st}} \text{ card } Q)$?

A useful formula that "*inverts conditioning*" is derived as follows :

Since we have both

$$P(EF) \;=\; P(E|F)\;P(F)\;,$$

and

$$P(EF) \;=\; P(F|E)\;P(E)\;.$$

If $P(E) \neq 0$ then it follows that

$$P(F|E) \;=\; \frac{P(EF)}{P(E)} \;=\; \frac{P(E|F)\cdot P(F)}{P(E)}\;,$$

and, using the earlier useful formula, we get

$$P(F|E) \;=\; \frac{P(E|F)\cdot P(F)}{P(E|F)\cdot P(F) \;+\; P(E|F^c)\cdot P(F^c)}\;,$$

which is known as *Bayes' formula* .

EXAMPLE : Suppose 1 in 1000 persons has a certain disease.

A test detects the disease in 99 % of diseased persons.

The test also "detects" the disease in 5 % of healthly persons.

With what probability does a positive test diagnose the disease?

SOLUTION : Let

$$D \sim \text{"diseased"} \quad , \quad H \sim \text{"healthy"} \quad , \quad + \sim \text{"positive"}.$$

We are given that

$$P(D) = 0.001 \,, \qquad P(+|D) = 0.99 \,, \qquad P(+|H) = 0.05 \,.$$

By Bayes' formula

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+|D) \cdot P(D) + P(+|H) \cdot P(H)}$$

$$= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 \;+\; 0.05 \cdot 0.999} \;\cong\; 0.0194 \qquad (\,!\,)$$

EXERCISE :

Suppose 1 in 100 products has a certain defect.

A test detects the defect in 95 % of defective products.

The test also "detects" the defect in 10 % of non-defective products.

- With what probability does a positive test diagnose a defect?

EXERCISE :

Suppose 1 in 2000 persons has a certain disease.

A test detects the disease in 90 % of diseased persons.

The test also "detects" the disease in 5 % of healthly persons.

- With what probability does a positive test diagnose the disease?

More generally, if the sample space $\mathcal{S}$ is *the union of disjoint events*

$$\mathcal{S} = F_1 \cup F_2 \cup \cdots \cup F_n ,$$

then for any event E

$$P(F_i|E) = \frac{P(E|F_i) \cdot P(F_i)}{P(E|F_1) \cdot P(F_1) + P(E|F_2) \cdot P(F_2) + \cdots + P(E|F_n) \cdot P(F_n)}$$

EXERCISE :

Machines M_1, M_2, M_3 produce these *proportions* of a article

Production : M_1 : 10 % , M_2 : 30 % , M_3 : 60 % .

The probability the machines produce *defective* articles is

Defects : M_1 : 4 % , M_2 : 3 % , M_3 : 2 % .

What is the probability a random article was made by machine M_1, given that it is defective?

Independent Events

Two events E and F are *independent* if

$$P(EF) = P(E)\,P(F)\ .$$

In this case

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(E)\,P(F)}{P(F)} = P(E)\ ,$$

(assuming $P(F)$ is not zero).

Thus

knowing F occurred doesn't change the probability of E .

EXAMPLE : Draw *one* card from a deck of 52 playing cards.

Counting outcomes we find

$$P(\text{Face Card}) = \frac{12}{52} = \frac{3}{13} ,$$

$$P(\text{Hearts}) = \frac{13}{52} = \frac{1}{4} ,$$

$$P(\text{Face Card and Hearts}) = \frac{3}{52} ,$$

$$P(\text{Face Card}|\text{Hearts}) = \frac{3}{13} .$$

We see that

$$P(\text{Face Card and Hearts}) = P(\text{Face Card}) \cdot P(\text{Hearts}) \quad (= \frac{3}{52}) .$$

Thus the events "*Face Card*" and "*Hearts*" are *independent*.

Therefore we also have

$$P(\text{Face Card}|\text{Hearts}) = P(\text{Face Card}) \quad (= \frac{3}{13}) .$$

EXERCISE :

Which of the following pairs of events are independent?

(1) drawing "Hearts" and drawing "Black" ,

(2) drawing "Black" and drawing "Ace" ,

(3) the event $\{2, 3, \cdots, 9\}$ and drawing "Red" .

EXERCISE : *Two* numbers are drawn at random from the set

$$\{ 1 , 2 , 3 , 4 \} .$$

If *order is not important* then what is the sample space $\mathcal{S}$?

Define the following functions on $\mathcal{S}$:

$$X(\{i,j\}) = i+j , \qquad Y(\{i,j\}) = |i-j| .$$

Which of the following pairs of events are independent?

$$(1) \quad X=5 \quad \text{and} \quad Y=2 ,$$

$$(2) \quad X=5 \quad \text{and} \quad Y=1 .$$

REMARK :

X and Y are examples of *random variables* . (More soon!)

EXAMPLE : If E and F are *independent* then so are E and F^c .

PROOF : $E = E(F \cup F^c) = EF \cup EF^c$, where

EF and EF^c are *disjoint* .

Thus

$$P(E) = P(EF) + P(EF^c) ,$$

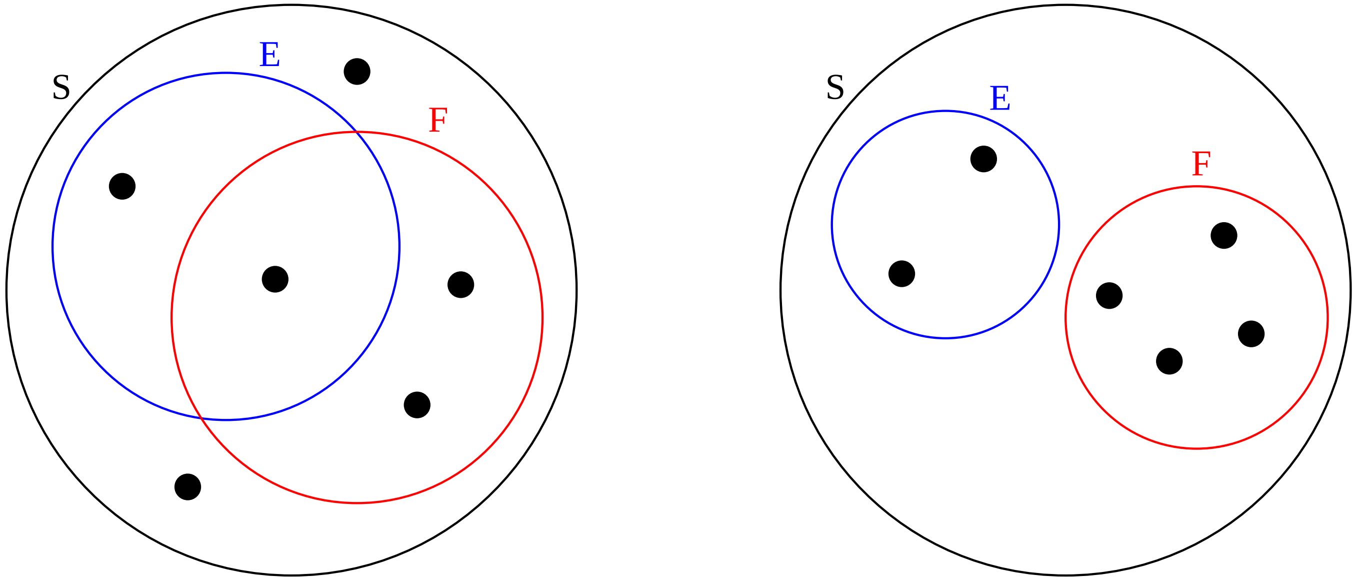
from which

$$\begin{aligned} P(EF^c) &= P(E) - P(EF) \\ &= P(E) - P(E) \cdot P(F) \quad \text{(since } E \text{ and } F \text{ independent)} \\ &= P(E) \cdot (1 - P(F)) \\ &= P(E) \cdot P(F^c) . \end{aligned}$$

EXERCISE :

Prove that if E and F are *independent* then so are E^c and F^c .

NOTE : *Independence* and *disjointness are different things* !



Independent, but not disjoint. Disjoint, but not independent.

(The six outcomes in S are assumed to have equal probability.)

If E and F are *independent* then $P(EF) = P(E)\ P(F)$.

If E and F are *disjoint* then $P(EF) = P(\ \emptyset\ ) = 0$.

If E and F are *independent and disjoint* then one has *zero probability* !

Three events E , F, and G are *independent* if

$$P(EFG) = P(E)\,P(F)\,P(G)\,.$$

and

$$P(EF) = P(E)\,P(F)\,.$$

$$P(EG) = P(E)\,P(G)\,.$$

$$P(FG) = P(F)\,P(G)\,.$$

EXERCISE : Are the three events of drawing

(1) a red card ,

(2) a face card ,

(3) a Heart or Spade ,

independent ?

EXERCISE :

A machine M consists of three *independent parts*, M_1, M_2, and M_3 .

Suppose that

M_1 functions properly with probability $\frac{9}{10}$,

M_2 functions properly with probability $\frac{9}{10}$,

M_3 functions properly with probability $\frac{8}{10}$,

and that

the machine M functions if and only if *its three parts function*.

- What is the probability for the machine M to *function* ?
- What is the probability for the machine M to *malfunction* ?