SETS, SUBSETS

2.1. List the elements of the following sets, where $\mathbf{P} = \{1, 2, 3, \ldots\}$:

(a) $A = \{x : x \in \mathbf{P},\ 3 < x < 7\}$,

(b) $B = \{x : x \in \mathbf{P},\ x \text{ is even},\ x < 9\}$,

(c) $C = \{x : x \in \mathbf{P}, x + 4 = 3\}$,

(d) $D = \{x : x \in \mathbf{P},\ x \text{ is a multiple of } 5\}$

(a) A consists of the positive integers between 3 and 7; hence $A = \{4, 5, 6\}$.

(b) B consists of the even positive integers less than 9; hence $B = \{2, 4, 6, 8\}$.

(c) There are no positive integers which satisfy the condition $x + 4 = 3$; hence C contains no elements. In other words $C = \varnothing$, the empty set.

(d) D is infinite, so we cannot list all its elements. However, sometimes we can write $D = \{5, 10, 15, 20, \ldots\}$ assuming everyone understands that we mean the multiples of 5.

2.2. Show that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x : x \in \mathbf{P},\ x \text{ is even}\}$.

It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

2.3. Show that $A = \{2, 3, 4, 5\}$ is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

2.4. Prove Theorem 2.1(iii): If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

We must show that each element in A also belongs to C. Let $x \in A$. Now $A \subseteq B$ implies $x \in B$. But $B \subseteq C$; hence $x \in C$. We have shown that $x \in A$ implies $x \in C$, that is, that $A \subseteq C$.

SET OPERATIONS

2.5. Let $U = \{1, 2, \ldots, 9\}$ be the universal set, and let

$$A = \{1,2,3,4,5\} \qquad C = \{5,6,7,8,9\} \qquad E = \{2,4,6,8\}$$
$$B = \{4,5,6,7\} \qquad D = \{1,3,5,7,9\} \qquad F = \{1,5,9\}$$

Find:
(*a*) $A \cup B$ and $A \cap B$ (*c*) $A \cup C$ and $A \cap C$ (*e*) $E \cup E$ and $E \cap E$
(*b*) $B \cup D$ and $B \cap D$ (*d*) $D \cup E$ and $D \cap E$ (*f*) $D \cup F$ and $D \cap F$

Recall that the union $X \cup Y$ consists of those elements in either X or Y (or both), and that the intersection $X \cap Y$ consists of those elements in both X and Y.

(*a*) $A \cup B = \{1,2,3,4,5,6,7\}$ $A \cap B = \{4,5\}$
(*b*) $B \cup D = \{1,3,4,5,6,7,9\}$ $B \cap D = \{5,7\}$
(*c*) $A \cup C = \{1,2,3,4,5,6,7,8,9\} = U$ $A \cap C = \{5\}$
(*d*) $D \cup E = \{1,2,3,4,5,6,7,8,9\} = U$ $D \cap E = \varnothing$
(*e*) $E \cup E = \{2,4,6,8\} = E$ $E \cap E = \{2,4,6,8\} = E$
(*f*) $D \cup F = \{1,3,5,7,9\} = D$ $D \cap F = \{1,5,9\} = F$

Observe that $F \subseteq D$; so by Theorem 2.2 we must have $D \cup F = D$ and $D \cap F = F$.

2.6. Consider the sets in the preceding Problem 2.5. Find:

(*a*) A^c, B^c, D^c, E^c (*b*) $A\backslash B$, $B\backslash A$, $D\backslash E$, $F\backslash D$ (*c*) $A \oplus B$, $C \oplus D$, $E \oplus F$

(*a*) The complement X^c consists of those elements in the universal set U which do not belong to X. Hence:

$$A^c = \{6,7,8,9\}, \qquad B^c = \{1,2,3,8,9\}, \qquad D^c = \{2,4,6,8\} = E, \qquad E^c = \{1,3,5,7,9\} = D$$

(*b*) The difference $X\backslash Y$ consists of the elements in X which do not belong to Y. Hence:

$$A\backslash B = \{1,2,3\}, \qquad B\backslash A = \{6,7\}, \qquad D\backslash E = \{1,3,5,7,9\} = D, \qquad F\backslash D = \varnothing$$

(*c*) The symmetric difference $X \oplus Y$ consists of the elements in X or Y but not in both X and Y. Hence:

$$A \oplus B = \{1,2,3,6,7\}, \qquad C \oplus D = \{1,3,8,9\}, \qquad E \oplus F = \{2,4,6,8,1,5,9\} = E \cup F$$

2.7. Show that we can have $A \cap B = A \cap C$ without $B = C$.

Let $A = \{1,2\}$, $B = \{2,3\}$, and $C = \{2,4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$. Accordingly,

$$A \cap B = A \cap C \text{ but } B \neq C$$

2.8. Prove: $B\backslash A = B \cap A^c$. Thus the set operation of difference can be written in terms of the operations of intersection and complementation.

$$B\backslash A = \{x : x \in B, \ x \notin A\} = \{x : x \in B, \ x \in A^c\} = B \cap A^c$$

2.9. Prove: $(A \cap B) \subseteq A \subseteq (A \cup B)$ and $(A \cap B) \subseteq B \subseteq (A \cup B)$.

Since every element in $A \cap B$ is in both A and B, it is certainly true that if $x \in (A \cap B)$ then $x \in A$; hence $(A \cap B) \subseteq A$. Furthermore, if $x \in A$, then $x \in (A \cup B)$ (by the definition of $A \cup B$), so $A \subseteq (A \cup B)$. Putting these together gives $(A \cap B) \subseteq A \subseteq (A \cup B)$. Similarly, $(A \cap B) \subseteq B \subseteq (A \cup B)$.

2.10. Prove Theorem 2.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

Suppose $A \subseteq B$ and let $x \in A$. Then $x \in B$, hence $x \in A \cap B$ and $A \subseteq A \cap B$. By Problem 2.9, $(A \cap B) \subseteq A$. Therefore $A \cap B = A$. On the other hand, suppose $A \cap B = A$ and let $x \in A$. Then $x \in (A \cap B)$; hence $x \in A$ and $x \in B$. Therefore, $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cap B = A$.

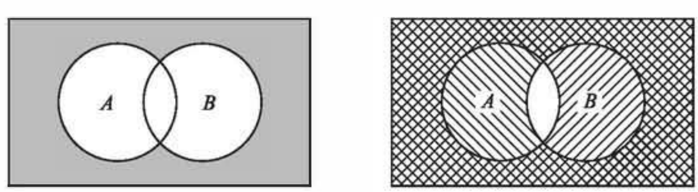
Suppose again that $A \subseteq B$. Let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$ because $A \subseteq B$. In either case, $x \in B$. Therefore $A \cup B \subseteq B$. By Problem 2.9, $B \subseteq A \cup B$. Therefore $A \cup B = B$. Now suppose $A \cup B = B$ and let $x \in A$. Then $x \in A \cup B$ by definition of union of sets. Hence $x \in B = A \cup B$. Therefore $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cup B = B$.

Thus $A \subseteq B$, $A \cup B = A$ and $A \cup B = B$ are equivalent.

VENN DIAGRAMS, ALGEBRA OF SETS, DUALITY

2.11. Illustrate De Morgan's Law $(A \cup B)^c = A^c \cap B^c$ (proved in Section 2.5) using Venn diagrams.

Shade the area outside $A \cup B$ in a Venn diagram of sets A and B. This is shown in Fig. 2-9(a); hence the shaded area represents $(A \cup B)^c$. Now shade the area outside A in a Venn diagram of A and B with strokes in one direction (///), and then shade the area outside B with strokes in another direction (\\\\\\). This is shown in Fig. 2-9(b); hence the cross-hatched area (area where both lines are present) represents the intersection of A^c and B^c, i.e. $A^c \cap B^c$. Both $(A \cup B)^c$ and $A^c \cap B^c$ are represented by the same area; thus the Venn diagrams indicate $(A \cup B)^c = A^c \cap B^c$. (We emphasize that a Venn diagram is not a formal proof, but it can indicate relationships between sets.)



(a) (b)

Fig. 2-9

2.12. Prove the Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Theorem 2.3(4b)).

By the definitions of union and intersection,

$$\begin{aligned} A \cap (B \cup C) &= \{x : x \in A,\ x \in B \cup C\} \\ &= \{x : x \in A,\ x \in B \quad \text{or} \quad x \in A,\ x \in C\} = (A \cap B) \cup (A \cap C) \end{aligned}$$

Here we use the analogous logical law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ where $\wedge$ denotes "and" and $\vee$ denotes "or".

2.13. Prove $(A \cup B)\backslash(A \cap B) = (A\backslash B) \cup (B\backslash A)$. (Thus either one may be used to define the symmetric difference $A \oplus B$.)

Using $X\backslash Y = X \cap Y^c$ and the laws in Table 2-1, including De Morgan's laws, we obtain:

$$\begin{aligned}(A \cup B)\backslash(A \cap B) &= (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cup B^c)\\ &= (A \cup A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c)\\ &= \varnothing \cup (A \cap B^c) \cup (B \cap A^c) \cup \varnothing\\ &= (A \cap B^c) \cup (B \cap A^c) = (A\backslash B) \cup (B\backslash A)\end{aligned}$$

2.14. Write the dual of each set equation:

(*a*) $(U \cap A) \cup (B \cap A) = A$

(*b*) $(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$

(*c*) $(A \cap U) \cap (\varnothing \cup A^c) = \varnothing$

(*d*) $(A \cap U)^c \cap A = \varnothing$

Interchange $\cup$ and $\cap$ and also U and $\varnothing$ in each set equation:

(*a*) $(\varnothing \cup A) \cap (B \cup A) = A$

(*b*) $(A \cap B \cap C)^c = (A \cap C)^c \cup (A \cap B)^c$

(*c*) $(A \cup \varnothing) \cup (U \cap A^c) = U$

(*d*) $(A \cup \varnothing)^c \cup A = U$

FINITE SETS AND THE COUNTING PRINCIPLE

2.15. Determine which of the following sets are finite:

(*a*) $A = \{$seasons in the year$\}$

(*b*) $B = \{$states in the Union$\}$

(*c*) $C = \{$positive integers less than 1$\}$

(*d*) $D = \{$odd integers$\}$

(*e*) $E = \{$positive integral divisors of 12$\}$

(*f*) $F = \{$cats living in the United States$\}$

(*a*) A is finite since there are four seasons in the year, i.e. $n(A) = 4$.

(*b*) B is finite because there are 50 states in the Union, i.e. $n(B) = 50$.

(*c*) There are no positive integers less than 1; hence C is empty. Thus C is finite and $n(C) = 0$.

(*d*) D is infinite.

(*e*) The positive integer divisors of 12 are 1, 2, 3, 4, 6, and 12. Hence E is finite and $n(E) = 6$.

(*f*) Although it may be difficult to find the number of cats living in the United States, there is still a finite number of them at any point in time. Hence F is finite.

2.16. Suppose 50 science students are polled to see whether or not they have studied French (F) or German (G) yielding the following data: 25 studied French, 20 studied German, 5 studied both. Find the number of the students who studied: (*a*) only French, (*b*) French or German, (*c*) neither language.

(*a*) Here 25 studied French, and 5 of them also studied German; hence $25 - 5 = 20$ students only studied French. That is, by Theorem 2.5,

$$n(F\backslash G) = n(F) - n(F \cap G) = 25 - 5 = 20$$

(*b*) By the inclusion–exclusion principle, Theorem 2.6,

$$n(F \cup G) = n(F) + n(G) - n(F \cap G) = 25 + 20 - 5 = 40$$

(*c*) Since 40 studied French or German, $50 - 40 = 10$ studied neither language.

2.17. In a survey of 60 people, it was found that:

25 read *Newsweek* magazine	9 read both *Newsweek* and *Fortune*
26 read *Time*	11 read both *Newsweek* and *Time*
26 read *Fortune*	8 read both *Time* and *Fortune*

3 read all three magazines

(*a*) Find the number of people who read at least one of the three magazines.

(*b*) Fill in the correct number of people in each of the eight regions of the Venn diagram in Fig. 2-10(*a*) where N, T, and F denote the set of people who read *Newsweek*, *Time*, and *Fortune*, respectively.

(*c*) Find the number of people who read exactly one magazine.

(*a*) We want $n(N \cup T \cup F)$, by Corollary 2.7:

$$\begin{aligned} n(N \cup T \cup F) &= n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F) \\ &= 25 + 26 + 26 - 11 - 9 - 8 + 3 = 52 \end{aligned}$$

(*b*) The required Venn diagram in Fig. 2-10(*b*) is obtained as follows:

3 read all three magazines

$11 - 3 = 8$ read *Newsweek* and *Time* but not all three magazines

$9 - 3 = 6$ read *Newsweek* and *Fortune* but not all three magazines

$8 - 3 = 5$ read *Time* and *Fortune* but not all three magazines

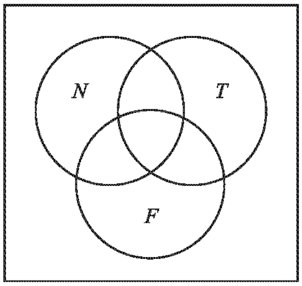
$25 - 8 - 6 - 3 = 8$ read only *Newsweek*

$26 - 8 - 5 - 3 = 10$ read only *Time*

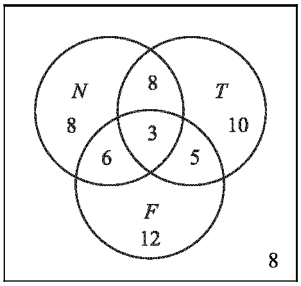
$26 - 6 - 5 - 3 = 12$ read only *Fortune*

$60 - 52 = 8$ read no magazine at all

(*c*) $8 + 10 + 12 = 30$ read only one magazine.



(*a*)



(*b*)

Fig. 2-10

2.18. Prove Theorem 2.6: If A and B are finite sets, then $A \cup B$ and $A \cap B$ are finite and $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

If A and B are finite, then clearly $A \cap B$ and $A \cup B$ are finite.

Suppose we count the element of A and then count the elements of B. Then every element in $A \cap B$ would be counted twice, once in A and once in B. Hence

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Alternatively (Problem 2.66), A is the disjoint union of $A\backslash B$ and $A \cap B$, B is the disjoint union of $B\backslash A$ and $A \cap B$, and $A \cup B$ is the disjoint union of $A\backslash B$, $A \cap B$, and $B\backslash A$. Therefore, by Lemma 2.4,

$$\begin{aligned} n(A \cup B) &= n(A\backslash B) + n(A \cap B) + n(B\backslash A) \\ &= n(A\backslash B) + n(A \cap B) + n(B\backslash A) + n(A \cap B) - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B). \end{aligned}$$

2.19. Show that each set is countable: (*a*) set $\mathbf{Z}$ of integers, (*b*) $\mathbf{P} \times \mathbf{P}$.

A set S is countable if (*a*) S is finite or (*b*) the element of S can be listed in the form of a sequence or, in other words, there is a one-to-one correspondence between the positive integers (counting numbers) $\mathbf{P} = \{1, 2, 3, \ldots\}$ and S. Neither set is finite.

(*a*) The following shows a one-to-one correspondence between $\mathbf{P}$ and $\mathbf{Z}$:

$$\begin{array}{rccccccccc} \text{Counting numbers } \mathbf{P}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{Integers } \mathbf{Z}: & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & \ldots \end{array}$$

That is, $n \in \mathbf{P}$ corresponds to either $n/2$, when n is even, or $(1 - n)/2$, when n is odd. Thus $\mathbf{Z}$ is countable.

(*b*) Figure 2-11 shows that $\mathbf{P} \times \mathbf{P}$ can be written as an infinite sequence as follows:

$$(1, 1), \quad (2, 1), \quad (1, 2), \quad (1, 3), \quad (2, 2), \quad \ldots$$

Specifically, the sequence is determined by "following the arrows" in Fig. 2-11.

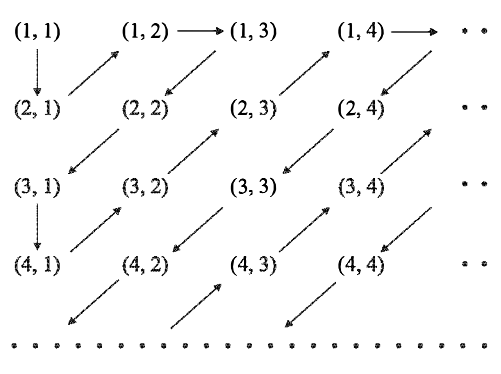


Fig. 2-11

PRODUCT SETS

2.20. Find x and y given that $(3x, x - 2y) = (6, -8)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations $3x = 6$ and $x - 2y = -8$ from which $x = 2$, $y = 5$.

2.21. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find (*a*) $A \times B$, (*b*) $B \times A$.

(*a*) $A \times B$ consists of all ordered pairs with the first component from A and the second component from B. Thus

$$A \times B = \{(1, a),\ (1, b),\ (2, a),\ (2, b),\ (3, a),\ (3, b)\}$$

(*b*) Here the first component is from B and the second component is from A:

$$B \times A = \{(a,1),\ (a,2),\ (a,3),\ (b,1),\ (b,2),\ (b,3)\}$$

2.22. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. Determine the number of elements in (*a*) $A \times B$, (*b*) $B \times A$, (*c*) A^3, (*d*) B^4.

Here $n(A) = 4$ and $n(B) = 3$. To obtain the number of elements in each product set, multiply the number of elements in each set in the product:

(*a*) $n(A \times B) = 4(3) = 12$
(*b*) $n(B \times A) = 3(4) = 12$
(*c*) $n(A^3) = 4(4)(4) = 64$
(*d*) $n(B^4) = 3^4 = 81$

2.23. Each toss of a coin will yield either a head or a tail. Let $C = \{H, T\}$ denote the set of outcomes. Find C^3, $n(C^3)$, and explain what C^3 represents.

Since $n(C) = 2$, we have $n(C^3) = 2^3 = 8$. Omitting certain commas and parentheses for notational convenience,

$$C^3 = \{HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}$$

C^3 represents all possible sequences of outcomes of three tosses of the coin.

2.24. Prove: $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$$\begin{aligned} A \times (B \cap C) &= \{(x, y) : x \in A,\ y \in B \cap C\} \\ &= \{(x, y) : x \in A,\ y \in B, y \in C\} \\ &= \{(x, y) : (x, y) \in A \times B,\ (x, y) \in A \times C\} \\ &= (A \times B) \cap (A \times C) \end{aligned}$$

CLASSES OF SETS, PARTITIONS

2.25. Find the elements of the set $A = [\{1, 2, 3\},\ \{4, 5\},\ \{6, 7, 8\}]$.

A is a class of sets; its elements are the sets $\{1, 2, 3\}$, $\{4, 5\}$, and $\{6, 7, 8\}$.

2.26. Determine the power set $\mathscr{P}(A)$ of $A = \{a, b, c, d\}$.

The elements of $\mathscr{P}(A)$ are the subsets of A. Hence

$$\begin{aligned} \mathscr{P}(A) = [&A,\ \{a,b,c\},\ \{a,b,d\},\ \{a,c,d\},\ \{b,c,d\},\ \{a,b\},\ \{a,c\},\ \{a,d\},\ \{b,c\},\ \{b,d\}, \\ &\{c,d\},\ \{a\},\ \{b\},\ \{c\},\ \{d\},\ \varnothing] \end{aligned}$$

As expected, $\mathscr{P}(A)$ has $2^4 = 16$ elements.

2.27. Let $S = \{a, b, c, d, e, f, g\}$. Determine which of the following are partitions of S:

(*a*) $P_1 = [\{a, c, e\},\ \{b\},\ \{d, g\}]$
(*b*) $P_2 = [\{a, e, g\},\ \{c, d\},\ \{b, e, f\}]$
(*c*) $P_3 = [\{a, b, e, g\},\ \{c\},\ \{d, f\}]$
(*d*) $P_4 = [\{a, b, c, d, e, f, g\}]$

(*a*) P_1 is not a partition of S since $f \in S$ does not belong to any of the cells.

(*b*) P_2 is not a partition of S since $e \in S$ belongs to two of the cells.

(*c*) P_3 is a partition of S since each element in S belongs to exactly one cell.

(*d*) P_4 is a partition of S into one cell, S itself.

2.28. Find all partitions of $S = \{a, b, c, d\}$.

Note first that each partition of S contains either 1, 2, 3, or 4 distinct cells. The partitions are as follows:

(1) $[\{a, b, c, d\}]$

(2) $[\{a\},\ \{b, c, d\}]$, $[\{b\},\ \{a, c, d\}]$, $[\{c\},\ \{a, b, d\}]$, $[\{d\},\ \{a, b, c\}]$,
$[\{a, b\},\ \{c, d\}]$, $[\{a, c\},\ \{b, d\}]$, $[\{a, d\},\ \{b, c\}]$

(3) $[\{a\},\ \{b\},\ \{c, d\}]$, $[\{a\},\ \{c\},\ \{b, d\}]$, $[\{a\},\ \{d\},\ \{b, c\}]$,
$[\{b\},\ \{c\},\ \{a, d\}]$, $[\{b\},\ \{d\},\ \{a, c\}]$, $[\{c\},\ \{d\},\ \{a, b\}]$,

(4) $[\{a\},\ \{b\},\ \{c\},\ \{d\}]$.

There are 15 different partitions of S.

2.29. Let $\mathbf{P} = \{1, 2, 3, \ldots\}$ and, for each $n \in \mathbf{P}$, let

$$A_n = \{x : x \text{ is a multiple of } n\} = \{n,\ 2n,\ 3n, \ldots\}$$

Find (*a*) $A_3 \cap A_5$, (*b*) $A_4 \cap A_6$, (*c*) $\bigcup_{i \in Q} A_i$, where $Q = \{2, 3, 5, 7, 11, \ldots\}$ is the set of prime numbers.

(*a*) Those numbers which are multiples of both 3 and 5 are the multiples of 15; hence $A_3 \cap A_5 = A_{15}$.

(*b*) The multiples of 12 and no other numbers belong to both A_4 and A_6; hence $A_4 \cap A_6 = A_{12}$.

(*c*) Every positive integer except 1 is a multiple of at least one prime number; hence

$$\bigcup_{i \in Q} A_i = \{2, 3, 4, \ldots\} = \mathbf{P} \backslash \{1\}$$

2.30. Prove: Let $\{A_i : i \in I\}$ be an indexed class of sets and let $i_0 \in I$. Then

$$\bigcap_{i \in I} A_i \subseteq A_{i_0} \subseteq \bigcup_{i \in I} A_i$$

Let $x \in \bigcap_{i \in I} A_i$; then $x \in A_i$ for every $i \in I$. In particular, $x \in A_{i_0}$. Hence $\bigcap_{i \in I} A_i \subseteq A_{i_0}$. Now let $y \in A_{i_0}$. Since $i_0 \in I$, $y \in \bigcap_{i \in I} A_i$. Hence $A_{i_0} \subseteq \bigcup_{i \in I} A_i$.

2.31. Prove (De Morgan's law): For any indexed class $\{A_i : i \in I\}$, we have $(\bigcup_i A_i)^c = \bigcap_i A_i^c$.

Using the definitions of union and intersection of indexed classes of sets:

$$\begin{aligned}(\textstyle\bigcup_i A_i)^c &= \{x : x \notin \textstyle\bigcup_i A_i\} = \{x : x \notin A_i \text{ for every } i\} \\ &= \{x : x \in A_i^c \text{ for every } i\} = \textstyle\bigcap_i A_i^c\end{aligned}$$

2.32. Let $\mathscr{A}$ be an algebra (σ-algebra) of subsets of U. Show that: (*a*) U and $\varnothing$ belong to $\mathscr{A}$; and (*b*) $\mathscr{A}$ is closed under finite (countable) intersections.

Recall that $\mathscr{A}$ is closed under complements and finite (countable) unions.

(*a*) Since $\mathscr{A}$ is nonempty, there is a set $A \in \mathscr{A}$. Hence the complement $A^c \in \mathscr{A}$, and the union $U = A \cup A^c \in \mathscr{A}$. Also the complement $\varnothing = U^c \in \mathscr{A}$.

(*b*) Let $\{A_i\}$ be a finite (countable) class of sets belonging to $\mathscr{A}$. By De Morgan's law (Problem 2.31) $(\bigcup_i A_i^c)^c = \bigcap_i A_i^{cc} = \bigcap_i A_i$. Hence $\bigcap_i A_i$ belongs to $\mathscr{A}$, as required.

MATHEMATICAL INDUCTION

2.33. Prove the assertion $A(n)$ that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$A(n)\colon\ 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

The assertion holds for $n = 1$ since

$$A(1)\colon\ 1 = \tfrac{1}{2} \cdot 1(1+1) = 1$$

Assuming $A(n)$ is true, we add $n+1$ to both sides of $A(n)$, obtaining

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \tfrac{1}{2}n(n+1 + (n+1) \\ &= \tfrac{1}{2}[n(n+1) + 2(n+1)] \\ &= \tfrac{1}{2}[(n+1)(n+2)] \end{aligned}$$

which is $A(n+1)$. That is, $A(n+1)$ is true whenever $A(n)$ is true. By the principle of induction, $A(n)$ is true for all n.

2.34. Prove the following assertion (for $n \geq 0$):

$$A(n)\colon\ 1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$$

$A(0)$ is true since $1 = 2^1 - 1$. Assuming $A(n)$ is true, we add 2^{n+1} to both sides of $A(n)$, obtaining

$$\begin{aligned} 1 + 2^1 + 2^2 + \cdots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2(2^{n+1}) - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

which is $A(n+1)$. Thus $A(n+1)$ is true whenever $A(n)$ is true. By the principle of induction, $A(n)$ is true for all $n \geq 0$.

FACTORIAL NOTATION, BINOMIAL COEFFICIENTS

2.35. Compute: (*a*) 4!, 5!, 6!, 7!, 8!, 9!, (*b*) 50!

(*a*) Use $(n+1)! = (n+1)n!$ after calculating 4! and 5!:

$$\begin{aligned} &4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24, && 7! = 7(6!) = 7(720) = 5040 \\ &5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5(24) = 120, && 8! = 8(7!) = 8(5040) = 40{,}320 \\ &6! = 6(5!) = 6(120) = 720, && 9! = 9(8!) = 9(40{,}320) = 362{,}880 \end{aligned}$$

(*b*) Since n is very large, we use Stirling's approximation that $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ (where $e = 2.718$). Thus

$$50! \sim \sqrt{100\pi}\, 50^{50} e^{-50} = N$$

Evaluating N using a calculator, we get $N = 3.04 \times 10^{64}$ (which has 65 digits).

Alternatively, using (base 10) logarithms, we get

$$\begin{aligned}\log N &= \log(\sqrt{100\pi}50^{50}e^{-50})\\ &= \tfrac{1}{2}\log 100 + \tfrac{1}{2}\log\pi + 50\log 50 - 50\log e\\ &= \tfrac{1}{2}(2) + \tfrac{1}{2}(0.4972) + 50(1.6990) - 50(0.4343)\\ &= 64.4836\end{aligned}$$

The antilog yields $N = 3.04 \times 10^{64}$.

2.36. Compute: (*a*) $\dfrac{13!}{11!}$ (*b*) $\dfrac{7!}{10!}$

(*a*) $\dfrac{13!}{11!} = \dfrac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 13\cdot 12 = 156$

Alternatively, this could be solved as follows:

$$\frac{13!}{11!} = \frac{13\cdot 12\cdot 11!}{11!} = 13\cdot 12 = 156$$

(*b*) $\dfrac{7!}{10!} = \dfrac{7!}{10\cdot 9\cdot 8\cdot 7!} = \dfrac{1}{10\cdot 9\cdot 8} = \dfrac{1}{720}$

2.37. Compute: (*a*) $\dbinom{16}{3}$, (*b*) $\dbinom{12}{4}$

Recall that there are as many factors in the numerator as in the denominator.

(*a*) $\dbinom{16}{3} = \dfrac{16\cdot 15\cdot 14}{1\cdot 2\cdot 3} = 560$ (*b*) $\dbinom{12}{4} = \dfrac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4} = 495$

2.38. Compute: (*a*) $\dbinom{8}{5}$, (*b*) $\dbinom{9}{7}$

(*a*) $\dbinom{8}{5} = \dfrac{8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 56$ or, since $8-5=3$, $\dbinom{8}{5} = \dbinom{8}{3} = \dfrac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} = 56$

(*b*) Since $9-7=2$, $\dbinom{9}{7} = \dbinom{9}{2} = \dfrac{9\cdot 8}{1\cdot 2} = 36$

2.39. Prove: $\dbinom{17}{6} = \dbinom{16}{5} + \dbinom{16}{6}$

Now $\dbinom{16}{5} + \dbinom{16}{6} = \dfrac{16!}{5!\,11!} + \dfrac{16!}{6!\,10!}$. Multiply the first fraction by $\dfrac{6}{6}$ and the second by $\dfrac{11}{11}$ to obtain the same denominator in both fractions; and then add:

$$\begin{aligned}\binom{16}{5} + \binom{16}{6} &= \frac{6\cdot 16!}{6\cdot 5!\cdot 11!} + \frac{11\cdot 16!}{6!\cdot 11\cdot 10!} = \frac{6\cdot 16!}{6!\cdot 11!} + \frac{11\cdot 16!}{6!\cdot 11!}\\ &= \frac{6\cdot 16! + 11\cdot 16!}{6!\cdot 11!} = \frac{(6+11)\cdot 16!}{6!\cdot 11!} = \frac{17\cdot 16!}{6!\cdot 11!} = \frac{17!}{6!\cdot 11!} = \binom{17}{6}\end{aligned}$$

2.40. Prove Theorem 2.10: $\dbinom{n+1}{r} = \dbinom{n}{r-1} + \dbinom{n}{r}$

(The technique in this proof is similar to that of the preceding problem.)

Now $\binom{n}{r-1} + \binom{n}{r} = \dfrac{n!}{(r-1)! \cdot (n-r+1)!} + \dfrac{n!}{r! \cdot (n-r)!}$. To obtain the same denominator in both fractions, multiply the first fraction by $\dfrac{r}{r}$ and the second fraction by $\dfrac{n-r+1}{n-r+1}$. Hence

$$\begin{aligned}
\binom{n}{r-1} + \binom{n}{r} &= \frac{r \cdot n!}{r \cdot (r-1)! \cdot (n-r+1)!} + \frac{(n-r+1) \cdot n!}{r! \cdot (n-r+1) \cdot (n-r)!} \\
&= \frac{r \cdot n!}{r!\,(n-r+1)!} + \frac{(n-r+1) \cdot n!}{r!\,(n-r+1)!} \\
&= \frac{r \cdot n! + (n-r+1) \cdot n!}{r!\,(n-r+1)!} = \frac{[r + (n-r+1)] \cdot n!}{r!\,(n-r+1)!} \\
&= \frac{(n+1)n!}{r!\,(n-r+1)!} = \frac{(n+1)!}{r!\,(n-r+1)!} = \binom{n+1}{r}
\end{aligned}$$

COUNTING PRINCIPLES

2.41. Suppose a bookcase shelf has 6 mathematics texts, 3 physics texts, 4 chemistry texts, and 5 computer science texts. Find the number n of ways a student can choose: (*a*) one of the texts, (*b*) one of each type of text.

(*a*) Here the sum rule applies; hence $n = 6 + 3 + 4 + 5 = 18$.

(*b*) Here the product rule applies; hence $n = 6 \cdot 3 \cdot 4 \cdot 5 = 360$.

2.42. A restaurant has a menu with 3 appetizers, 4 entrées, and 2 desserts. Find the number n of ways a customer can order an appetizer, entrée, and dessert.

Here the product rule applies, since the customer orders one of each. Thus $n = 3 \cdot 4 \cdot 2 = 24$.

2.43. A history class contains 7 male students and 5 female students. Find the number n of ways that the class can elect: (*a*) a class representative, (*b*) two class representatives, one male and one female, (*c*) a president and a vice-president.

(*a*) Here the sum rule is used; hence $n = 7 + 5 = 12$.

(*b*) Here the product rule is used; hence $n = 7 \cdot 5 = 35$.

(*c*) There are 12 ways to elect the president and then 11 ways to elect the vice-president. Thus $n = 12 \cdot 11 = 132$.

2.44. There are four bus lines from city A to city B and three bus lines from city B to city C. Find the number n of ways a person can travel by bus: (*a*) from A to C by way of B, (*b*) round-trip from A to C by way of B, (*c*) round-trip from A to C by way of B, without using a bus line more than once.

(*a*) There are 4 ways to go from A to B, and 3 ways from B to C; hence, by the product rule, $n = 4 \cdot 3 = 12$.

(*b*) There are 12 ways to go from A to C by way of B, and 12 ways to return. Thus, by the product rule, $n = 12 \cdot 12 = 144$.

(*c*) The person will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

$$\mathrm{A} \to \mathrm{B} \to \mathrm{C} \to \mathrm{B} \to \mathrm{A}$$

There are 4 ways to go from A to B and 3 ways to go from B to C. Since a bus line is not to be used more than once, there are only 2 ways to go from C back to B and only 3 ways to go from B back to A. Enter these numbers above the corresponding arrows as follows:

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

Thus, by the product rule, $n = 4 \cdot 3 \cdot 2 \cdot 3 = 72$.

PERMUTATIONS, ORDERED SAMPLES

2.45. State the essential difference between permutations and combinations, with examples.

Order counts with permutations, such as words, sitting in a row, or electing a president, vice-president, and treasurer. Order does not count with combinations, such as committees or teams (without counting positions). The product rule is usually used with permutations since the choice for each of the ordered positions may be viewed as a sequence of events.

2.46. A family has 3 boys and 2 girls. (*a*) Find the number of ways they can sit in a row. (*b*) How many ways are there if the boys and girls are each to sit together?

(*a*) The five children can sit in a row in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ ways.

(*b*) There are two ways to distribute them according to sex: BBBGG or GGGBB. In each case, the boys can sit in $3 \cdot 2 \cdot 1 = 3! = 6$ ways, and the girls can sit in $2 \cdot 1 = 2! = 2$ ways. Thus, altogether, there are $2 \cdot 3! \cdot 2! = 2 \cdot 6 \cdot 2 = 24$ ways.

2.47. Suppose repetitions are not allowed. (*a*) Find the number n of three-digit numbers that can be formed from the digits 2, 3, 5, 6, 7, and 9. (*b*) How many of them are even? (*c*) How many of them exceed 400?

There are 6 digits, and the three-digit number may be pictured by __, __, __. In each case, write down the number of ways that one can fill each of the positions.

(*a*) There are 6 ways to fill the first position, 5 ways to fill the second position, and 4 ways to fill the third position. This may be pictured by: $\underline{6}$, $\underline{5}$, $\underline{4}$. Thus $n = 6 \cdot 5 \cdot 4 = 120$.
Alternatively, n is the number of permutations of 6 things taken 3 at a time, so

$$n = P(6, 3) = 6 \cdot 5 \cdot 4 = 120$$

(*b*) Since the numbers must be even, the last digit must be either 2 or 4. Thus the third position is filled first and it can be done in 2 ways. Then there are now 5 ways to fill the middle position and 4 ways to fill the first position. This may be pictured by: $\underline{4}$, $\underline{5}$, $\underline{2}$. Thus $4 \cdot 5 \cdot 2 = 120$ of the numbers are even.

(*c*) Since the numbers must exceed 400, they must begin with 5, 6, 7, or 9. Thus we first fill the first position, which can be done in 4 ways. Then there are 5 ways to fill the second position and 4 ways to fill the third position. This may be pictured by: $\underline{4}$, $\underline{5}$, $\underline{4}$. Thus $4 \cdot 5 \cdot 4 = 80$ of the numbers exceed 400.

2.48. Find the number n of distinct permutations that can be formed from all the letters of each word: (*a*) THEM, (*b*) UNUSUAL, (*c*) SOCIOLOGICAL.

This problem concerns permutations with repetitions.

(*a*) $n = 4! = 24$, since there are 4 letters and no repetitions.

(*b*) $n = \frac{7!}{3!} = 840$, since there are 7 letters of which 3 are U.

(*c*) $n = \frac{12!}{3!\,2!\,2!\,2!}$, since there are 12 letters of which 3 are O, 2 are C, 2 are I, and 2 are L.

2.49. A class contains 8 students. Find the number of ordered samples of size 3: (*a*) with replacement, (*b*) without replacement.

(*a*) Each student in the ordered sample can be chosen in 8 ways; hence there are $8 \cdot 8 \cdot 8 = 8^3 = 512$ samples of size 3 with replacement.

(*b*) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus there are $8 \cdot 7 \cdot 6 = 336$ samples of size 3 without replacement.

2.50. Find n if: (*a*) $P(n, 2) = 72$, (*b*) $2P(n, 2) + 50 = P(2n, 2)$.

(*a*) $P(n, 2) = n(n-1) = n^2 - n$; hence $n^2 - n = 72$ or $n^2 - n - 72 = 0$ or $(n-9)(n+8) = 0$.
Since n must be positive, the only answer is $n = 9$.

(*b*) $P(n, 2) = n(n-1) = n^2 - n$ and $P(2n, 2) = 2n(2n-1) = 4n^2 - 2n$. Hence

$$2(n^2 - n) + 50 = 4n^2 - 2n \quad \text{or} \quad 2n^2 - 2n + 50 = 4n^2 - 2n \quad \text{or} \quad 50 = 2n^2 \quad \text{or} \quad n^2 = 25$$

Since n must be positive, the only answer is $n = 5$.

COMBINATIONS, PARTITIONS

2.51. A class contains 10 students with 6 men and 4 women. Find the number n of ways:

(*a*) a 4-member committee can be selected from the students,
(*b*) a 4-member committee with 2 men and 2 women can be selected,
(*c*) the class can elect a president, vice-president, treasurer, and secretary.

(*a*) This concerns combinations, not permutations, since order does not count. There are "10 choose 4" such committees. That is,

$$n = C(10, 4) = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

(*b*) The 2 men can be chosen from the 6 men in $\binom{6}{2}$ ways, and the 2 women can be chosen from the 4 women in $\binom{4}{2}$ ways. Thus, by the product rule,

$$n = \binom{6}{2}\binom{4}{2} = \frac{6 \cdot 5}{2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} = 15(6) = 90 \text{ ways}$$

(*c*) This concerns permutations, not combinations, since order does count. Thus

$$n = P(6, 4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

2.52. A box contains 7 blue socks and 5 red socks. Find the number n of ways two socks can be drawn from the box if: (*a*) they can be any color, (*b*) they must be the same color.

(*a*) There are "12 choose 2" ways to select 2 of the 12 socks. That is,

$$n = C(12, 2) = \binom{12}{2} = \frac{12 \cdot 11}{2 \cdot 1} = 66$$

(*b*) There are $C(7, 2) = 21$ ways to choose 2 of the 7 blue socks, and $C(5, 2) = 10$ ways to choose 2 of the 5 red socks. By the sum rule, $n = 21 + 10 = 31$.

2.53. Let $A, B, \ldots, J$ be 10 given points in the plane $\mathbf{R}^2$ such that no three of the points lie on the same line. Find the number n of:

(*a*) lines in $\mathbf{R}^2$ where each line contains two of the points,
(*b*) lines in $\mathbf{R}^2$ containing A and one of the other points,
(*c*) triangles whose vertices come from the given points,

(*d*) triangles whose vertices are A and two of the other points.

Since order does not count, this problem involves combinations.

(*a*) Each pair of points determines a line; hence

$$n = \text{“10 choose 2”} = C(10, 2) = \binom{10}{2} = 66$$

(*b*) We need only choose one of the 9 remaining points; hence $n = 9$.

(*c*) Each triple of points determines a triangle; hence

$$n = \text{“10 choose 3”} = C(10, 3) = \binom{10}{3} = 120$$

(*d*) We need only choose two of the 9 remaining points; hence $n = C(9, 2) = 36$.

2.54. There are 12 students in a class. Find the number n of ways that 12 students can take three different tests if four students are to take each test.

There are $C(12, 4) = 495$ ways to choose four students to take the first test; following this, there are $C(8, 4) = 70$ ways to choose four students to take the second test. The remaining students take the third test. Thus $n = 70(495) = 34{,}650$.

2.55. Find the number n of ways 12 students can be partitioned into three teams A_1, A_2, A_3, so that each team contains four students. (Compare with preceding Problem 2.54.)

Let A denote one of the students. There are $C(11, 3) = 165$ ways to choose three other students to be on the same team as A. Now let B be a student who is not on the same team as A. Then there are $C(7, 3) = 35$ ways to choose three from the remaining students to be on the same team as B. The remaining four students form the third team. Thus $n = 35(165) = 5925$.

Alternatively, each partition $[A_1, A_2, A_3]$ can be arranged in $3! = 6$ ways as an ordered partition. By the preceding Problem 2.54, there are 34,650 such ordered partitions. Thus $n = 34\,650/6 = 5925$.

TREE DIAGRAMS

2.56. Construct the tree diagram that gives the permutations of $\{a, b, c\}$.

The tree diagram appears in Fig. 2-12. The six paths from the root of the tree yield the six permutations:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

2.57. Jack has time to play roulette at most five times. At each play he wins or loses \$1. He begins with \$1 and will stop playing before the five times if he loses all his money or if he wins \$3, that is,

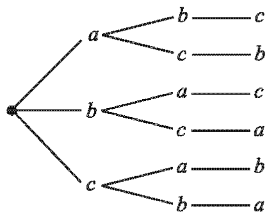


Fig. 2-12

if he has \$4. Find the number of ways the betting can occur, and find the number of times he will stop before betting five times.

Construct the appropriate tree diagram, as shown in Fig. 2-13. Each number in the diagram denotes the number of dollars he has at that moment of time. The betting can occur in 11 ways, and Jack will stop betting before the five times are up in only three of the cases.

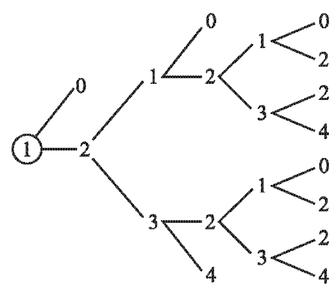


Fig. 2-13

MISCELLANEOUS PROBLEMS

2.58. Prove the binomial theorem 2.9: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$.

The theorem is true for $n = 1$, since

$$\sum_{r=0}^{1} \binom{1}{r} a^{1-r} b^r = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a + b = (a+b)^1$$

We assume the theorem holds for $(a+b)^n$ and prove it is true for $(a+b)^{n+1}$.

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n \\ &= (a+b)\left[a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{r-1} a^{n-r+1} b^{r-1} + \binom{n}{r} a^{n-r} b^r + \cdots + \binom{n}{1} a b^{n-1} + b^n\right]\end{aligned}$$

Now the term in the product which contains b^r is obtained from

$$\begin{aligned} b\left[\binom{n}{r-1} a^{n-r+1} b^{r-1}\right] + a\left[\binom{n}{r} a^{n-r} b^r\right] &= \binom{n}{r-1} a^{n-r+1} b^r + \binom{n}{r} a^{n-r+1} b^r \\ &= \left[\binom{n}{r-1} + \binom{n}{r}\right] a^{n-r+1} b^r \end{aligned}$$

But, by Theorem 2.10 $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. Thus the term containing b^r is $\binom{n+1}{r} a^{n-r+1} b^r$. Note that $(a+b)(a+b)^n$ is a polynomial of degree $n+1$ in b. Consequently,

$$(a+b)^{n+1} = (a+b)(a+b)^n = \sum_{r=0}^{n+1} \binom{n+1}{r} a^{n-r+1} b^r$$

which was to be proved.

2.59. Prove: $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 16$

Note $16 = 2^4 = (1+1)^4$. Expanding $(1+1)^4$, using the binomial theorem, yields:

$$\begin{aligned} 16 = (1+1)^4 &= \binom{4}{0}1^4 + \binom{4}{1}1^3 1^1 + \binom{4}{2}1^2 1^2 + \binom{4}{3}1^1 1^3 + \binom{4}{4}1^4 \\ &= \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \end{aligned}$$

2.60. Let n and $n_1, n_2, \ldots, n_r$ be nonnegative integers such that $n_1 + n_2 + \cdots + n_r = n$. The *multinomial coefficients* are denoted and defined by:

$$\binom{n}{n_1 \quad n_2 \quad \ldots \quad n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Compute the following multinomial coefficients:

(*a*) $\binom{6}{3,2,1}$, (*b*) $\binom{8}{4,2,2,0}$, (*c*) $\binom{10}{5,3,2,2}$

(*a*) $\binom{6}{3,2,1} = \dfrac{6!}{3!\,2!\,1!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1} = 60$

(*b*) $\binom{8}{4,2,2,0} = \dfrac{8!}{4!\,2!\,2!\,0!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1} = 420$

(*c*) The expression $\binom{10}{5,3,2,2}$ has no meaning, since $5 + 3 + 2 + 2 \neq 10$.

Supplementary Problems

SETS AND SUBSETS

2.61. List the elements of the following sets if the universal set is $U = \{a, b, c, \ldots, y, z\}$. Furthermore, identify which of the sets, if any, are equal.

$A = \{x : x \text{ is a vowel}\}$ $\qquad$ $C = \{x : x \text{ precedes } f \text{ in the alphabet}\}$

$B = \{x : x \text{ is a letter in the word "}little\text{"}\}$ $\qquad$ $D = \{x : x \text{ is a letter in the word "}title\text{"}\}$

2.62. Let $A = \{1, 2, \ldots, 8, 9\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 3, 5, 7, 9\}$, $D = \{3, 4, 5\}$, and $E = \{3, 5\}$. Which of the above sets can equal a set X under each of the following conditions?

(*a*) X and B are disjoint $\qquad$ (*c*) $X \subseteq A$ but $X \not\subseteq C$

(*b*) $X \subseteq D$ but $X \not\subseteq B$ $\qquad$ (*d*) $X \subseteq C$ but $X \not\subseteq A$

SET OPERATIONS

Problems 2.63–2.66 refer to the universal set $U = \{1, 2, 3, \ldots, 8, 9\}$ and the sets:

$$A = \{1, 2, 5, 6\}, \qquad B = \{2, 5, 7\}, \qquad C = \{1, 3, 5, 7, 9\}$$

2.63. Find: (*a*) $A \cap B$ and $A \cap C$, (*b*) $A \cup B$ and $B \cup C$, (*c*) A^c and C^c.

2.64. Find: (*a*) $A \backslash B$ and $A \backslash C$, (*b*) $A \oplus B$ and $A \oplus C$.

2.65. Find: (*a*) $(A \cup C)\backslash B$, (*b*) $(A \cup B)^c$, (*c*) $(B \oplus C)\backslash A$.

2.66. Let A and B be any sets. Prove:

(*a*) A is the disjoint union of $A\backslash B$ and $A \cap B$.

(*b*) $A \cup B$ is the disjoint union of $A\backslash B$, $A \cap B$, and $B\backslash A$.

2.67. Prove the following:

(*a*) $A \subseteq B$ if and only if $A \cap B^c = \varnothing$.

(*b*) $A \subseteq B$ if and only if $A^c \cup B = U$.

(*c*) $A \subseteq B$ if and only if $B^c \subseteq A^c$.

(*d*) $A \subseteq B$ if and only if $A\backslash B = \varnothing$.

(Compare results with Theorem 2.2.)

2.68. Prove the absorption laws: (*a*) $A \cup (A \cap B) = A$, (*b*) $A \cap (A \cup B) = A$.

2.69. The formula $A\backslash B = A \cup B^c$ defines the difference operation in terms of the operations of intersection and complement. Find a formula that defines the union $A \cup B$ in terms of the operations of intersection and complement.

VENN DIAGRAMS, ALGEBRA OF SETS, DUALITY

2.70. The Venn diagram in Fig. 2-14 shows sets A, B, C. Shade the following sets:

(*a*) $A\backslash(B \cup C)$, (*b*) $A^c \cap (B \cap C)$, (*c*) $(A \cup C) \cap (B \cup C)$.

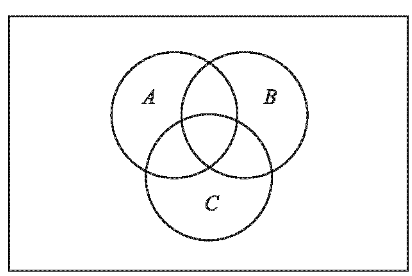


Fig. 2-14

2.71. Write the dual of each equation:

(*a*) $A \cup (A \cap B) = A$, (*b*) $(A \cap B) \cup (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c) = U$

2.72. Use the laws in Table 2-1 to prove $(A \cap B) \cup (A \cap B^c) = A$.

FINITE SETS AND THE COUNTING PRINCIPLE

2.73. Determine which of the following sets are finite:

(*a*) lines parallel to the x-axis,

(*b*) letters in the English alphabet,

(*c*) animals living on the earth,

(*d*) circles through the origin (0, 0).

2.74. Given $n(U) = 20$, $n(A) = 12$, $n(B) = 9$, $n(A \cap B) = 4$, find:

(*a*) $n(A \cup B)$, (*b*) $n(A^c)$, (*c*) $n(B^c)$, (*d*) $n(A\backslash B)$, (*e*) $n(\varnothing)$.

2.75. Among 120 freshmen at a college, 40 take mathematics, 50 take English, and 15 take both mathematics and English. Find the number of freshmen who:

(*a*) do not take mathematics,
(*b*) take mathematics or English,
(*c*) take mathematics, but not English,
(*d*) take English, but not mathematics,
(*e*) take exactly one of the two subjects,
(*f*) take neither mathematics nor English.

2.76. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found:

15 had air-conditioning	5 had air-conditioning and power windows
12 had radio	9 had air-conditioning and radio
11 had power windows	4 had radio and power windows

3 had all three options

Find the number of cars that had: (*a*) only power windows, (*b*) only air-conditioning, (*c*) only radio, (*d*) radio and power windows but not air-conditioning, (*e*) air-conditioning and radio, but not power windows, (*f*) only one of the options, (*g*) none of the options.

2.77. Use Theorem 2.6 to prove Corollary 2.7: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

PRODUCT SETS

2.78. Find x and y if: (*a*) $(x+2, 3) = (7, 2x+y)$, (*b*) $(y-3, 2x+1) = (x+2, y+4)$.

2.79. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Find: (*a*) $A \times B$, (*b*) $B \times A$.

2.80. Let $C = \{H, T\}$, the set of possible outcomes if a coin is tossed. Find: (*a*) $C^2 = C \times C$, (*b*) $C^4 = C \times C \times C \times C$.

2.81. Suppose $n(A) = 3$, and $n(B) = 5$. Find the number of elements in: (*a*) $A \times B$, $B \times A$, (*b*) A^2, B^2, A^3, B^3; (*c*) $A \times A \times B \times A$.

CLASSES OF SETS, PARTITIONS

2.82. Find the power set $\mathcal{P}(A)$ of $A = \{a, b, c, d, e\}$.

2.83. Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether each of the following is a partition of S:

(*a*) $[\{1, 3, 5\}, \{2, 4\}, \{3, 6\}]$
(*b*) $[\{1, 5\}, \{2\}, \{3, 6\}]$
(*c*) $[\{1, 5\}, \{2\}, \{4\}, \{3, 6\}]$
(*d*) $[\{1\}, \{3, 6\}, \{2, 4, 5\}, \{3, 6\}]$
(*e*) $[\{1, 2, 3, 4, 5, 6\}]$
(*f*) $[\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}]$

2.84. Find all partitions of $S = \{1, 2, 3\}$.

2.85. For each positive integer $n \in \mathbf{P}$, let $A_n = \{n, 2n, 3n, \ldots\}$, the multiples of n. Find: (*a*) $A_2 \cap A_7$, (*b*) $A_6 \cap A_8$, (*c*) $A_5 \cup A_{20}$, (*d*) $A_5 \cap A_{20}$, (*e*) $A_s \cup A_{st}$, where $s, t \in \mathbf{P}$, (*f*) $A_s \cap A_{st}$, where $s, t \in \mathbf{P}$.

2.86. Prove: If $J \subseteq \mathbf{P}$ is infinite, then $\cap(A_i\colon i \in J) = \varnothing$. (Here the A_i are the sets in Problem 2.85.)

2.87. Let $[A_1, A_2, \ldots, A_m]$ and $[B_1, B_2, \ldots, B_n]$ be partitions of S. Show that the collection of sets

$$[A_i \cap B_j; \qquad i = 1, \ldots, m, \qquad j = 1, \ldots, n] \backslash \varnothing$$

(where the empty set $\varnothing$ is deleted), is also a partition of S, called the *cross partition*.

2.88. Prove: For any indexed class of sets $\{A_i: i \in I\}$ and any set B: (*a*) $B \cup (\bigcap_i A_i) = \bigcap_i (B \cup A_i)$, (*b*) $B \cap (\bigcup_i A_i) = \bigcup_i (B \cap A_i)$.

2.89. Prove (De Morgan's law): $(\bigcap_i A_i)^c = \bigcup_i A_i^c$.

2.90. Show that each of the following is an algebra of subsets of U: (*a*) $\mathscr{A} = \{\varnothing, U\}$, (*b*) $\mathscr{B} = \{\varnothing, A, A^c, U\}$, (*c*) $\mathscr{P}(U)$, the power set of U.

2.91. Let $\mathscr{A}$ and $\mathscr{B}$ be algebras (σ-algebras) of subsets of U. Prove that the intersection $\mathscr{A} \cap \mathscr{B}$ is also an algebra (σ-algebra) of subsets of U.

MATHEMATICAL INDUCTION

2.92. Prove: $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

2.93. Prove: $1 + 4 + 7 + \cdots + (3n-2) = 2n(3n-1)$.

2.94. Prove: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

2.95. Prove that, for $n \geq 4$: (*a*) $n! \geq 2^n$; (*b*) $2^n \geq n^2$; (*c*) $n^2 \geq 2n + 5$.

FACTORIAL NOTATION AND BINOMIAL COEFFICIENTS

2.96. Find: (*a*) 10!, 11!, 12!, (*b*) 60! (Hint: Use Stirling's approximation to $n!$.)

2.97. Simplify: (*a*) $\dfrac{(n+1)!}{n!}$, (*b*) $\dfrac{n!}{(n-2)!}$, (*c*) $\dfrac{(n-1)!}{(n+2)!}$, (*d*) $\dfrac{(n-r+1)!}{(n-r-1)!}$

2.98. Evaluate: (*a*) $\dbinom{5}{2}$, (*b*) $\dbinom{7}{3}$, (*c*) $\dbinom{14}{2}$, (*d*) $\dbinom{6}{4}$, (*e*) $\dbinom{20}{17}$, (*f*) $\dbinom{18}{15}$

2.99. Show that:

(*a*) $\dbinom{n}{0} + \dbinom{n}{1} + \dbinom{n}{2} + \dbinom{n}{3} + \cdots + \dbinom{n}{n} = 2^n$

(*b*) $\dbinom{n}{0} - \dbinom{n}{1} + \dbinom{n}{2} - \dbinom{n}{3} + \cdots + \dbinom{n}{n} = 0$

2.100. Evaluate the following multinomial coefficients (defined in Problem 2.58):

(*a*) $\dbinom{6}{2, 3, 1}$, (*b*) $\dbinom{8}{4, 3, 1, 0}$, (*c*) $\dbinom{8}{3, 3, 2}$, (*d*) $\dbinom{9}{4, 3, 2, 1}$

COUNTING PRINCIPLES, SUM AND PRODUCT RULES

2.101. A store sells clothes for men. It has 3 different kinds of jackets, 6 different kinds of shirts, and 4 different kinds of pants. Find the number of ways a person can buy: (*a*) one of the items for a present, (*b*) one of each of the items for a present.

2.102. A restaurant has, on its dessert menu, 4 kinds of cakes, 3 kinds of cookies, and 5 kinds of ice cream. Find the number of ways a person can select: (*a*) one of the desserts, (*b*) one of each kind of dessert.

2.103. A class contains 8 male students, and 6 female students. Find the number of ways that the class can elect: (*a*) a class representative, (*b*) two class representatives, one male and one female, (*c*) a president and a vice-president.

2.104. There are 6 roads between A and B and 4 roads between B and C. Find the number of ways a person can drive: (*a*) from A to C by way of B, (*b*) round-trip from A to C by way of B, (*c*) round-trip from A to C by way of B without using the same road more than once.

2.105. Suppose a code consists of two letters followed by a digit. Find the number of: (*a*) codes, (*b*) codes with distinct letters, (*c*) codes with the same letters.

PERMUTATIONS, ORDERED SAMPLES

2.106. Find the number n of ways a judge can award first, second, and third places in a contest with 18 contestants.

2.107. Find the number n of ways 6 people can ride a toboggan where: (*a*) anyone can drive, (*b*) one of three must drive.

2.108. Find the number n of permutations that can be formed from all the letters of each word: (*a*) QUEUE, (*b*) COMMITTEE, (*c*) PROPOSITION, (*d*) BASEBALL.

2.109. A box contains 10 light bulbs. Find the number n of ordered samples of size 3: (*a*) with replacement, (*b*) without replacement.

2.110. A class contains 6 students. Find the number n of ordered samples of size 4: (*a*) with replacement, (*b*) without replacement.

COMBINATIONS, PARTITIONS

2.111. A class contains 9 boys and 3 girls. Find the number of ways a teacher can select a committee of 4.

2.112. Repeat Problem 2.111, but where: (*a*) there are to be 2 boys and 2 girls, (*b*) there is to be exactly one girl, (*c*) there is to be at least one girl.

2.113. A box contains 6 blue socks and 4 white socks. Find the number of ways two socks can be drawn from the box where: (*a*) there are no restrictions, (*b*) they are different colors, (*c*) they are to be the same color.

2.114. A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner.

2.115. Repeat Problem 2.114, but where: (*a*) two of the friends are married and will not attend separately, (*b*) two of the friends are not on speaking terms and will not attend together.

2.116. A student is to answer 8 out of 10 questions on an exam. Find the number of choices.

2.117. Repeat Problem 2.116, but where: (*a*) the first three questions must be answered, (*b*) at least 4 of the first 5 questions must be answered.

2.118. There are 9 students in a class. Find the number of ways the students can take three tests if 3 students are to take each test.

2.119. There are 9 students in a class. Find the number of ways the students can be partitioned into three teams containing 3 students each. (Compare with Problem 2.118.)

2.120. Find the number of ways 9 toys may be divided among four children if the youngest is to receive 3 toys and each of the others 2 toys.

TREE DIAGRAMS

2.121. Teams A and B play in the World Series of baseball, where the team that first wins four games wins the series. Find the number n of ways the series can occur given that A wins the first game and that the team that wins the second game also wins the fourth game, and list the n ways the series can occur.

2.122. Suppose $A, B, \ldots, F$ in Fig. 2-15 denote islands, and the lines connecting them bridges. A man begins at A and walks from island to island. He stops for lunch when he cannot continue to walk without crossing the same bridge twice. (*a*) Construct the appropriate tree diagram, and find the number of ways he can take his walk before eating lunch. (*b*) At which islands can he eat his lunch?

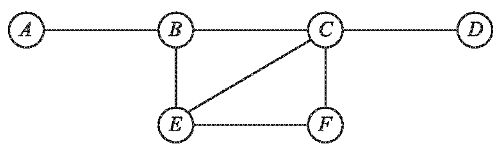


Fig. 2-15

MISCELLANEOUS PROBLEMS

2.123. Suppose n objects are partitioned into r ordered cells with $n_1, n_2, \ldots, n_r$ elements. Show that the number of such ordered partitions is

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_r!}$$

2.124. There are n married couples at a party. (*a*) Find the number of (unordered) pairs at the party. (*b*) Find the number of handshakes if each person shakes hands with every other person other than his or her spouse.

Answers to Supplementary Problems

2.61. $A = \{\text{a, e, i, o, u}\}$, $B = D = \{\text{l, i, t, e}\}$, $C = \{\text{a, b, c, d, e}\}$

2.62. (*a*) C, E; (*b*) D, E; (*c*) B; (*d*) none

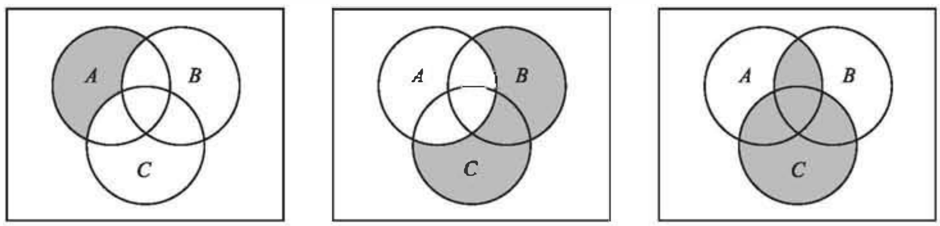
2.63. (*a*) $A \cap B = \{2, 5\}$, $A \cap C = \{5\}$; (*b*) $A \cup B = \{1, 2, 5, 6, 7\}$, $B \cup C = \{1, 2, 3, 5, 7, 9\}$; (*c*) $A^c = \{3, 4, 7, 8, 9\}$, $C^c = \{2, 4, 6, 8\}$

2.64. (*a*) $A \backslash B = \{1, 6\}$, $A \backslash C = \{2, 6\}$; (*b*) $A \oplus B = \{1, 6, 7\}$, $A \oplus C = \{2, 6, 7, 9\}$

2.65. (*a*) $\{1, 3, 6, 9\}$, (*b*) $\{3, 4, 8, 9\}$, (*c*) $\{3, 9\}$

2.69. $A \cup B = (A^c \cap B^c)^c$

2.70. See Fig. 2-16.



(a) (b) (c)

Fig. 2-16

2.71. (a) $A \cap (A \cup B) = A$, (b) $(A \cup B) \cap (A^c \cup B) \cap (A \cup B^c) \cap (A^c \cup B^c) = \varnothing$

2.73. (b), (d), (e)

2.74. (a) 17, (b) 8, (c) 11, (d) 8, (e) 0

2.75. (a) 80, (b) 75, (c) 25, (d) 35, (e) 60, (f) 45

2.76. (a) 5, (b) 4, (c) 2, (d) 1, (e) 6, (f) 11, (g) 2

2.78. (a) $x = 5$, $y = 7$; (b) $x = 8$, $y = 13$

2.79. $A \times B = \{a1, a2, a3, b1, b2, b3\}$, $B \times A = \{1a, 1b, 2a, 2b, 3a, 3b\}$

2.80. $C^2 = \{HH, HT, TH, TT\}$,
$C^4 = \{HHHH,\ HHHT,\ HHTH,\ HHTT,\ HTHH,\ HTHT,\ HTTH,\ HTTT,\ THHH,\ THHT,\ THTH,\ THTT,\ TTHH,\ TTHT,\ TTTH,\ TTTT\}$

2.81. (a) 15, 15, 9, 25; (b) 45, 27

2.82. $P(A) = [\varnothing, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, de, abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde, abcd, abce, abde, acde, bcde, A]$. Note $n(P(a)) = 2^5 = 32$.

2.83. (a) and (b): no. Others: yes.

2.84. [S], [{1, 2}, {3}], [{1, 3}, {2}], [{2, 3}, {1}], [{1}, {2}, {3}]

2.85. (a) A_{14}, (b) A_{24}, (c) A_5, (d) A_{20}, (e) A_s, (f) A_{st}

2.96. (a) 3,628,800; 39,916,800; 479,001,600, (b) $\log(60!) = 81.92$ so $60! = 6.59 \times 10^{81}$

2.97. (a) $n + 1$, (b) $n(n-1) = n^2 - n$, (c) $1/[n(n+1)(n+2)]$, (d) $(n-r)(n-r+1)$

2.98. (a) 10, (b) 35, (c) 91, (d) 15, (e) 1140, (f) 816

2.99. Hint: (a) expand $(1+1)^n$, (b) expand $(1-1)^n$

2.100. (a) 60, (b) 280, (c) 560, (d) not defined

2.101. (*a*) 13, (*b*) 72

2.102. (*a*) 12, (*b*) 60

2.103. (*a*) 14, (*b*) 48, (*c*) 182

2.104. (*a*) 24, (*b*) 576, (*c*) 360

2.105. (*a*) 6760, (*b*) 6500, (*c*) 260

2.106. $n = 18 \cdot 17 \cdot 16 = 4896$

2.107. (*a*) $6! = 720$, (*b*) $3 \cdot 5! = 360$

2.108. (*a*) 30, (*b*) $\dfrac{9!}{2!\,2!\,2!} = 45{,}360$, (*c*) $\dfrac{11!}{2!\,3!\,2!} = 1{,}663{,}200$, (*d*) $\dfrac{8!}{2!\,2!\,2!} = 5040$

2.109. (*a*) $10^3 = 1000$, (*b*) $10 \cdot 9 \cdot 8 = 720$

2.110. (*a*) $6^4 = 1296$, (*b*) $6 \cdot 5 \cdot 4 \cdot 3 = 360$

2.111. $C(12, 4) = 495$

2.112. (*a*) $C(9, 2) \cdot C(3, 2) = 108$, (*b*) $C(9, 3) \cdot C(3, 1) = 252$,
(*c*) $9 + 108 + 252 = 369$ or $C(12, 4) - C(9, 4) = 495 - 126 = 369$

2.113. (*a*) $C(10, 2) = 45$, (*b*) $6 \cdot 4 = 24$, (*c*) $C(6, 2) + C(4, 2) = 21$ or $45 - 24 = 21$

2.114. $C(11, 5) = 462$

2.115. (*a*) 210, (*b*) 252

2.116. $C(10, 8) = C(10, 2) = 45$

2.117. (*a*) $C(7, 5) = C(7, 2) = 21$, (*b*) $25 + 10 = 35$

2.118. 1680

2.119. 280

2.120. 7560

2.121. Construct the appropriate tree diagram as in Fig. 2-17. Note that the tree begins at A, the winner of the first game, and that there is only one choice in the fourth game, the winner of the second game. The diagram shows that $n = 15$ and that the series can occur in the following 15 ways:

AAAA, AABAA, AABABA, AABABBA, AABABBB, ABABAA, ABABABA, ABABABB,
ABABBAA, ABABBAB, ABABBB, ABBBAAA, ABBBAAB, ABBBAB, ABBBB

2.122. (*a*) See Fig. 2-18. There are 11 ways to take his walk. (*b*) B, D, or E

2.124. (*a*) $C(2n, 2) = 2n(2n - 1)/2$, (*b*) $C(2n, 2) - n = 2n(2n - 1)/2 - n$

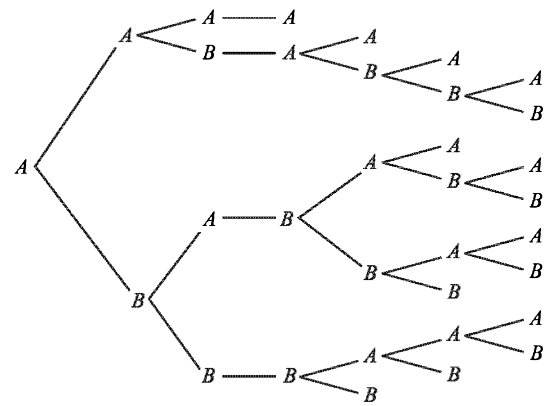


Fig. 2-17

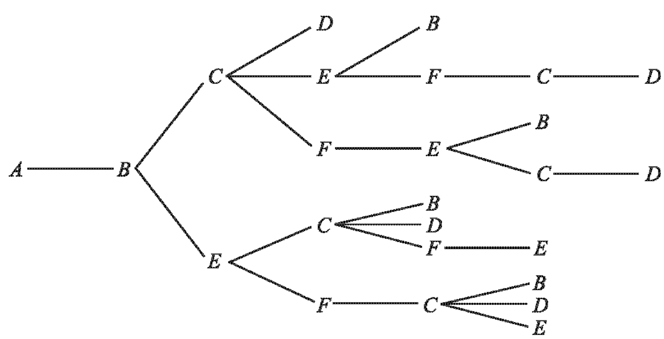


Fig. 2-18