SPECIAL DISCRETE RANDOM VARIABLES

The Bernoulli Random Variable

A *Bernoulli trial* has only *two outcomes* , with probability

$$P(X = 1) \quad = \quad p \ ,$$

$$P(X = 0) \quad = \quad 1 - p \ ,$$

e.g., tossing a coin, winning or losing a game, $\cdots$.

We have

$$E[X] \;=\; 1 \cdot p \;+\; 0 \cdot (1 - p) \;=\; p \ ,$$

$$E[X^2] \;=\; 1^2 \cdot p \;+\; 0^2 \cdot (1 - p) \;=\; p \ ,$$

$$Var(X) \;=\; E[X^2] - E[X]^2 \;=\; p - p^2 \;=\; p(1 - p) \ .$$

NOTE : If p *is small* then $Var(X) \cong p$.

EXAMPLES :

- When $p = \frac{1}{2}$ $(e.g.,$ for tossing a coin), we have

$$E[X] = p = \frac{1}{2} \quad , \quad Var(X) = p(1-p) = \frac{1}{4} \,.$$

- When *rolling a die* , with outcome k , $(1 \leq k \leq 6)$, let

$$X(k) = 1 \text{ if the roll resulted in a } six \,,$$

and

$$X(k) = 0 \text{ if the roll did } not \text{ result in a } six \,.$$

Then

$$E[X] = p = \frac{1}{6} \quad , \quad Var(X) = p(1-p) = \frac{5}{36} \,.$$

- When $p = 0.01$, then

$$E[X] = 0.01 \quad , \quad Var(X) = 0.0099 \cong 0.01 \,.$$

The Binomial Random Variable

Perform a Bernoulli trial n times *in sequence* .

Assume the individual trials are *independent* .

An *outcome* could be

$$100011001010 \qquad (n = 12) \ ,$$

with probability

$$P(100011001010) \quad = \quad p^5 \cdot (1-p)^7 \ . \qquad (\text{ Why ? })$$

Let the X be the number of "*successes*" (*i.e.* 1's) .

For example,

$$X(100011001010) \quad = \quad 5 \ .$$

We have

$$P(X = 5) \quad = \quad \binom{12}{5} \cdot p^5 \cdot (1-p)^7 \ . \qquad (\text{ Why ? })$$

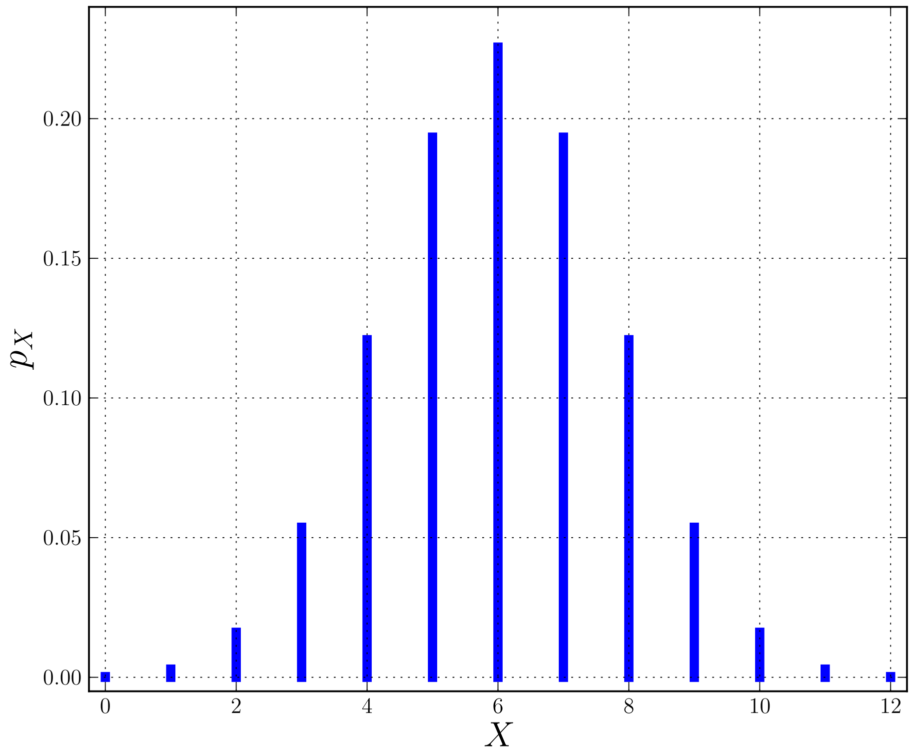
In general, for k successes in a sequence of n trials, we have

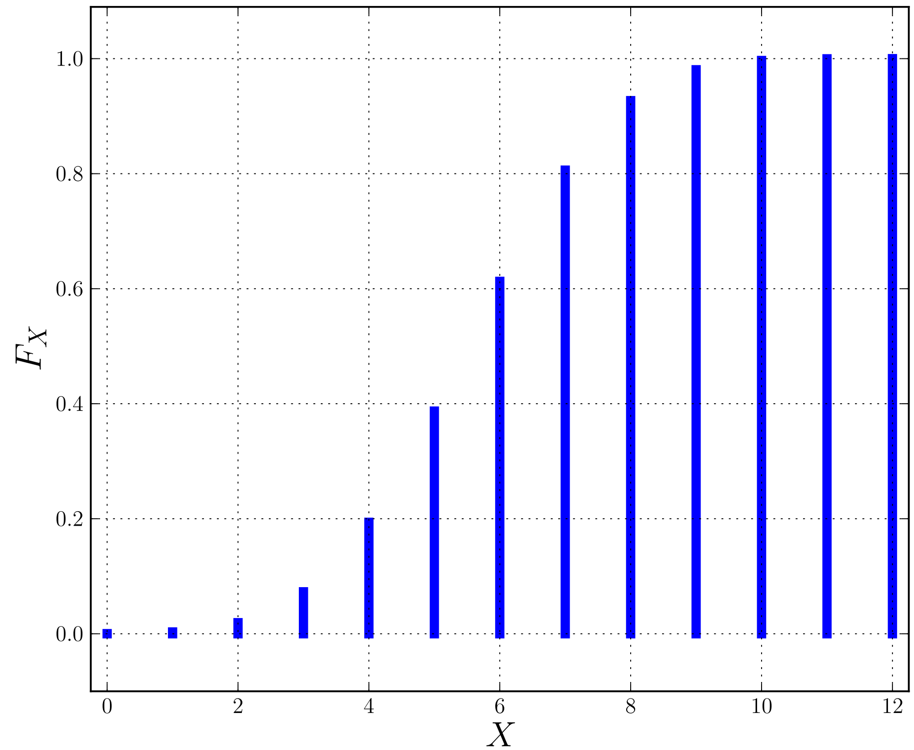
$$P(X = k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \ , \qquad\qquad (0 \le k \le n) \ .$$

EXAMPLE : Tossing a coin 12 times:

$$n = 12 \quad , \quad \mathbf{p} = \mathbf{\tfrac{1}{2}}$$

k	$p_X(k)$	$F_X(k)$
0	1 / 4096	1 / 4096
1	12 / 4096	13 / 4096
2	66 / 4096	79 / 4096
3	220 / 4096	299 / 4096
4	495 / 4096	794 / 4096
5	792 / 4096	1586 / 4096
6	924 / 4096	2510 / 4096
7	792 / 4096	3302 / 4096
8	495 / 4096	3797 / 4096
9	220 / 4096	4017 / 4096
10	66 / 4096	4083 / 4096
11	12 / 4096	4095 / 4096
12	1 / 4096	4096 / 4096





The Binomial *mass* and *distribution* functions for $n = 12$, $p = \frac{1}{2}$

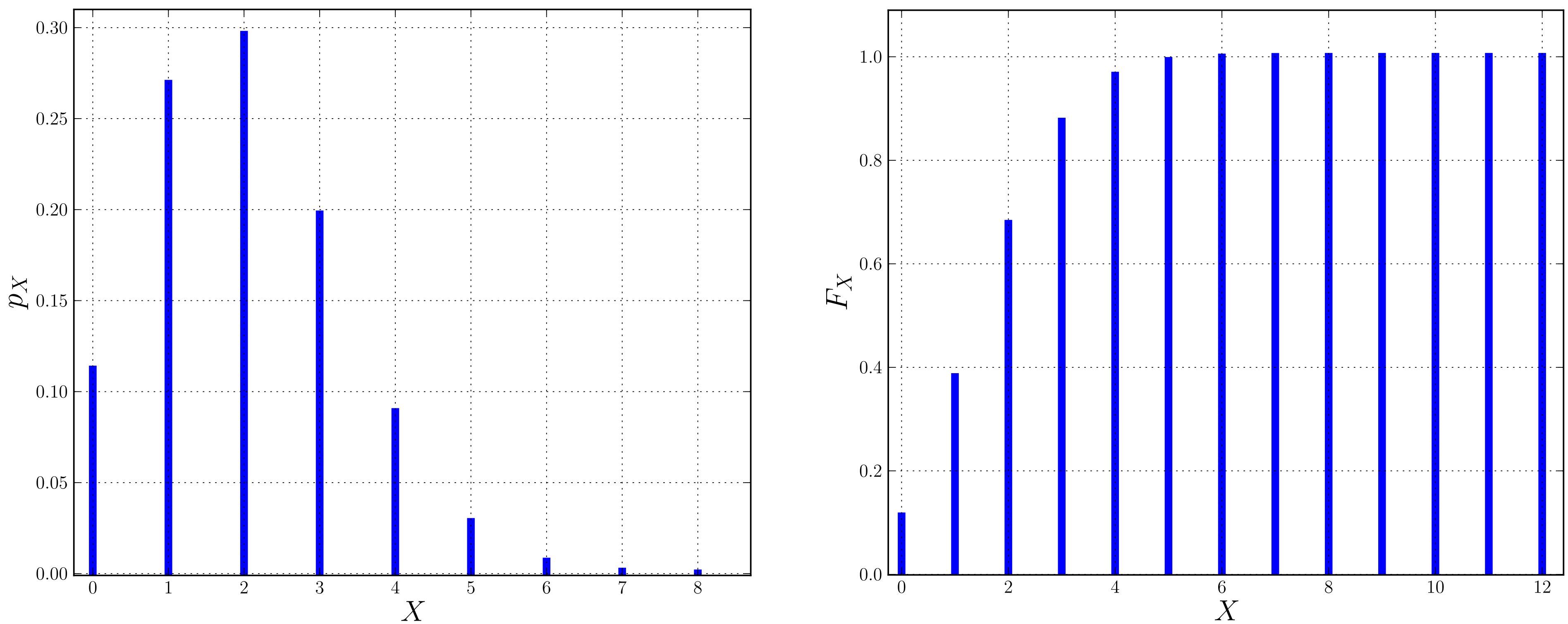
For k successes in a sequence of n trials :

$$P(X=k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \;, \qquad\qquad (0 \leq k \leq n) \;.$$

EXAMPLE : Rolling a die 12 times:

$$n = 12 \quad , \quad \mathbf{p} = \tfrac{\mathbf{1}}{\mathbf{6}}$$

k	$p_X(k)$	$F_X(k)$
0	0.1121566221	0.112156
1	0.2691758871	0.381332
2	0.2960935235	0.677426
3	0.1973956972	0.874821
4	0.0888280571	0.963649
5	0.0284249838	0.992074
6	0.0066324966	0.998707
7	0.0011369995	0.999844
8	0.0001421249	0.999986
9	0.0000126333	0.999998
10	0.0000007580	0.999999
11	0.0000000276	0.999999
12	0.0000000005	1.000000



The Binomial *mass* and *distribution* functions for $n = 12$, $p = \frac{1}{6}$

EXAMPLE :

In 12 *rolls of a die* write the outcome as, for example,

$$100011001010$$

where

1 denotes the roll resulted in a *six* ,

and

0 denotes the roll did *not* result in a *six* .

As before, let X be the number of 1's in the outcome.

Then X represents the *number of sixes* in the 12 rolls.

Then, for example, using the preceding *Table* :

$$P(X = 5) \cong 2.8 \% \quad , \quad P(X \leq 5) \cong 99.2 \% \, .$$

EXERCISE : Show that from

$$P(X=k) \;=\; \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \;,$$

and

$$P(X=k+1) \;=\; \binom{n}{k+1} \cdot p^{k+1} \cdot (1-p)^{n-k-1} \;,$$

it follows that

$$P(X=k+1) \;=\; c_k \cdot P(X=k) \;,$$

where

$$c_k \;=\; \frac{n-k}{k+1} \cdot \frac{p}{1-p} \;.$$

NOTE : This *recurrence formula* is an efficient and stable *algorithm* to compute the binomial probabilities :

$$P(X=0) \;=\; (1-p)^n \;,$$

$$P(X=k+1) \;=\; c_k \cdot P(X=k) \;, \qquad k = 0, 1, \cdots, n-1 \;.$$

Mean and variance of the Binomial random variable :

By definition, the *mean* of a Binomial random variable X is

$$E[X] = \sum_{k=0}^{n} k \cdot P(X = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} ,$$

which can be shown to equal np .

An *easy way* to see this is as follows :

If in a *sequence* of n independent Bernoulli trials we let

X_k = the outcome of the k^{th} Bernoulli trial , $(X_k = 0 \text{ or } 1)$,

then

$$X \equiv X_1 + X_2 + \cdots + X_n ,$$

is the *Binomial random variable* that *counts the successes* ” .

$$X \equiv X_1 + X_2 + \cdots + X_n$$

We know that

$$E[X_k] = p ,$$

so

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = np .$$

We already know that

$$Var(X_k) = E[X_k^2] - (E[X_k])^2 = p - p^2 = p(1-p) ,$$

so, since the X_k are *independent* , we have

$$Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_n) = np(1-p) .$$

NOTE : If p *is small* then $Var(X) \cong np$.

EXAMPLES :

- For 12 tosses of a *coin* , with *Heads* is *success*, we have

  so $$n = 12 \quad , \quad p = \frac{1}{2}$$

  $$E[X] = np = 6 \quad , \quad Var(X) = np(1-p) = 3 .$$

- For 12 rolls of a *die* , with *six* is *success* , we have

  so $$n = 12 \quad , \quad p = \frac{1}{6}$$

  $$E[X] = np = 2 \quad , \quad Var(X) = np(1-p) = 5/3 .$$

- If $n = 500$ and $p = 0.01$, then

$$E[X] = np = 5 \quad , \quad Var(X) = np(1-p) = 4.95 \cong 5 .$$

The Poisson Random Variable

The Poisson variable *approximates* the Binomial random variable :

$$P(X = k) \quad = \quad \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \cong \quad e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad ,$$

when we take

$$\lambda \quad = \quad n\, p \quad (\textit{ the average number of successes }) \; .$$

This approximation is *accurate* if n is *large* and p *small* .

Recall that for the Binomial random variable

$E[X] = n\, p$, and $Var(X) = np(1-p) \;\cong\; np$ when p is small.

Indeed, for the Poisson random variable we will show that

$$E[X] = \lambda \quad \text{and} \quad Var(X) = \lambda \; .$$

A *stable* and *efficient* way to compute the Poisson probability

$$P(X = k) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^k}{k!} , \qquad k = 0, 1, 2, \cdots ,$$

$$P(X = k+1) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^{k+1}}{(k+1)!} ,$$

is to use the *recurrence relation*

$$P(X = 0) \ = \ e^{-\lambda} ,$$

$$P(X = k+1) \quad = \quad \frac{\lambda}{k+1} \cdot P(X = k) , \qquad k = 0, 1, 2, \cdots .$$

NOTE : Unlike the Binomial random variable, the Poisson random variable can have an *arbitrarily large* integer value k.

The Poisson random variable

$$P(X = k) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^k}{k!} \, , \qquad k = 0, 1, 2, \ \cdots \ ,$$

has (as shown later) : $E[X] = \lambda$ and $Var(X) = \lambda$.

The Poisson *distribution function* is

$$F(k) \quad = \quad P(X \leq k) \quad = \quad \sum_{\ell=0}^{k} e^{-\lambda} \frac{\lambda^\ell}{\ell!} \quad = \quad e^{-\lambda} \sum_{\ell=0}^{k} \frac{\lambda^\ell}{\ell!} \, ,$$

with, as should be the case,

$$\lim_{k \to \infty} F(k) \quad = \quad e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \quad = \quad e^{-\lambda} \, e^{\lambda} \quad = \quad 1 \, .$$

(using the *Taylor series* from Calculus for e^λ) .

The Poisson random variable

$$P(X = k) \quad = \quad e^{-\lambda} \cdot \frac{\lambda^k}{k!}\ , \qquad k = 0, 1, 2, \ \cdots \ ,$$

models the probability of k "*successes*" in a given "time" interval, when the *average* number of successes is λ .

EXAMPLE : Suppose customers arrive at the rate of *six* per hour. The probability that k customers arrive in a one-hour period is

$$P(k = 0) \quad = \quad e^{-6} \cdot \frac{6^0}{0!} \quad \cong \quad 0.0024\ ,$$

$$P(k = 1) \quad = \quad e^{-6} \cdot \frac{6^1}{1!} \quad \cong \quad 0.0148\ ,$$

$$P(k = 2) \quad = \quad e^{-6} \cdot \frac{6^2}{2!} \quad \cong \quad 0.0446\ .$$

The probability that more than 2 customers arrive is

$$1 \ - \ (0.0024 \ + \ 0.0148 \ + \ 0.0446) \quad \cong \quad 0.938\ .$$

$$p_{\text{Binomial}}(k) = \binom{n}{k} \, p^k \, (1-p)^{n-k} \quad \cong \quad p_{\text{Poisson}}(k) = e^{-\lambda} \, \frac{\lambda^k}{k!}$$

EXAMPLE : $\lambda = 6$ customers/hour.

For the Binomial take $n = 12$, $p = 0.5$ (0.5 customers/5 minutes) ,

so that indeed $np = \lambda$.

k	p_{Binomial}	p_{Poisson}	F_{Binomial}	F_{Poisson}
0	0.0002	0.0024	0.0002	0.0024
1	0.0029	0.0148	0.0031	0.0173
2	0.0161	0.0446	0.0192	0.0619
3	0.0537	0.0892	0.0729	0.1512
4	0.1208	0.1338	0.1938	0.2850
5	0.1933	0.1606	0.3872	0.4456
6	0.2255	0.1606	0.6127	0.6063
7	0.1933	0.1376	0.8061	0.7439
8	0.1208	0.1032	0.9270	0.8472
9	0.0537	0.0688	0.9807	0.9160
10	0.0161	0.0413	0.9968	0.9573
11	0.0029	0.0225	0.9997	0.9799
12	0.0002	0.0112	1.0000	0.9911 $\star$

Why not 1.0000 ?

Here the approximation is *not so good* $\cdots$

$$p_{\text{Binomial}}(k) = \binom{n}{k} p^k (1-p)^{n-k} \cong p_{\text{Poisson}}(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

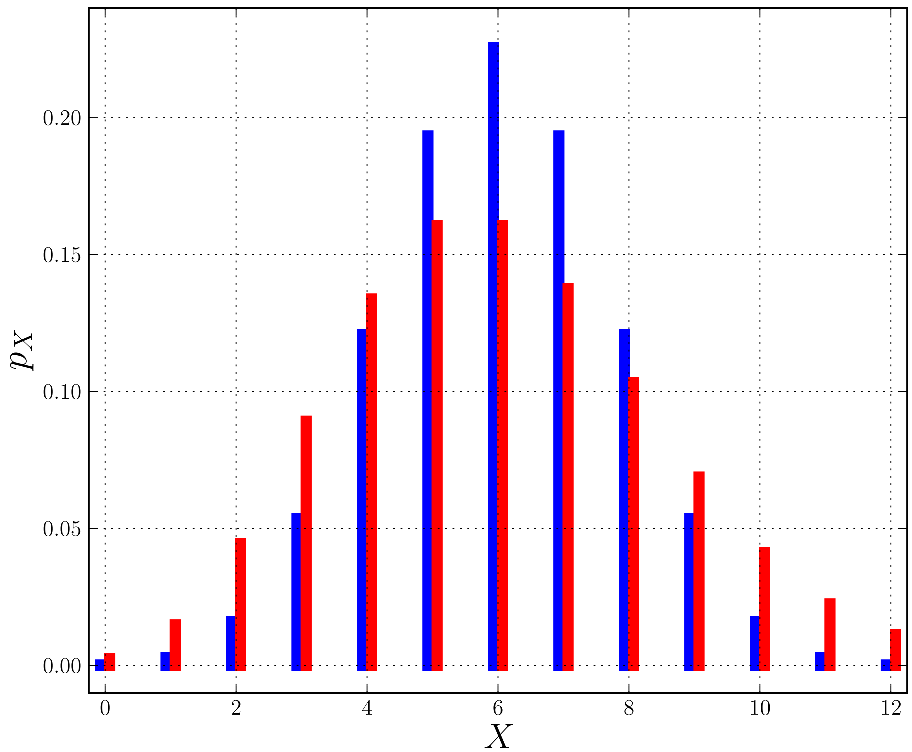
EXAMPLE : $\lambda = 6$ customers/hour.

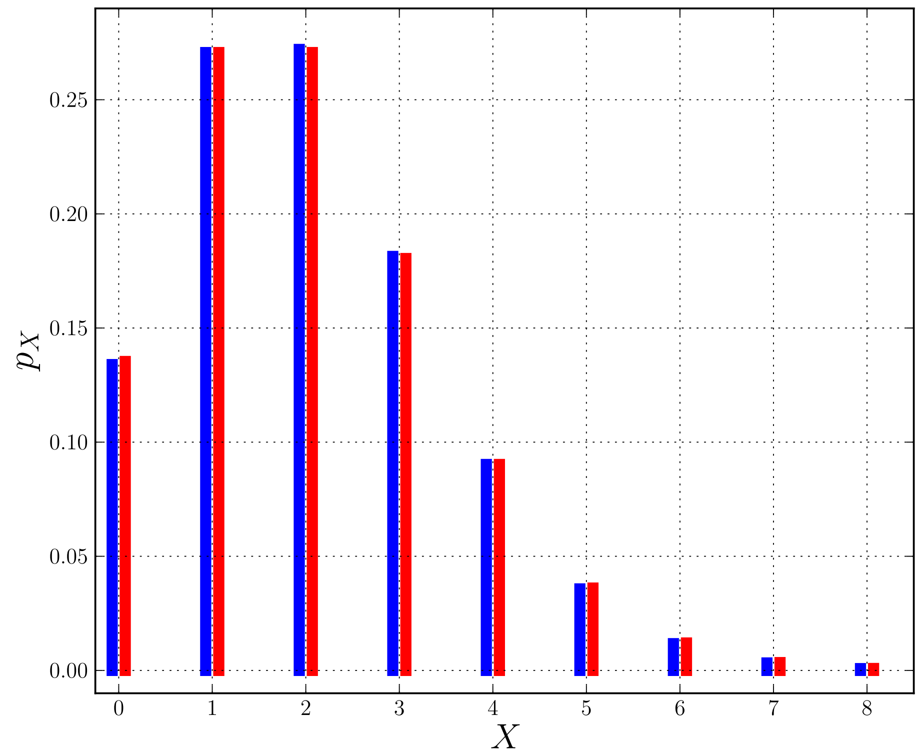
For the Binomial take $n = 60$, $p = 0.1$ (0.1 customers/minute) ,

so that indeed $np = \lambda$.

k	p_{Binomial}	p_{Poisson}	F_{Binomial}	F_{Poisson}
0	0.0017	0.0024	0.0017	0.0024
1	0.0119	0.0148	0.0137	0.0173
2	0.0392	0.0446	0.0530	0.0619
3	0.0843	0.0892	0.1373	0.1512
4	0.1335	0.1338	0.2709	0.2850
5	0.1662	0.1606	0.4371	0.4456
6	0.1692	0.1606	0.6064	0.6063
7	0.1451	0.1376	0.7515	0.7439
8	0.1068	0.1032	0.8583	0.8472
9	0.0685	0.0688	0.9269	0.9160
10	0.0388	0.0413	0.9657	0.9573
11	0.0196	0.0225	0.9854	0.9799
12	0.0089	0.0112	0.9943	0.9911
13	···	···	···	···

Here the approximation is *better* ···





$n = 12 \;\;,\;\; p = \frac{1}{2} \;\;,\;\; \lambda = 6$ $\qquad$ $n = 200 \;\;,\;\; p = 0.01 \;,\;\; \lambda = 2$

The Binomial (*blue*) and Poisson (*red*) probability mass functions.

For the case $n = 200$, $p = 0.01$, the approximation is very good !

For the *Binomial* random variable we found

$$E[X] \;=\; np \qquad \text{and} \qquad Var(X) \;=\; np(1-p)\ ,$$

while for the *Poisson* random variable, with $\lambda = np$ we will show

$$E[X] \;=\; np \qquad \text{and} \qquad Var(X) \;=\; np\ .$$

Note again that

$$np(1-p) \;\cong\; np \quad , \qquad \text{when } p \text{ is } \textit{small}\ .$$

EXAMPLE : In the preceding two *Tables* we have

n=12 , p=0.5

	Binomial	Poisson
$E[X]$	6.0000	6.0000
$Var[X]$	3.0000	6.0000
$\sigma[X]$	1.7321	2.4495

n=60 , p=0.1

	Binomial	Poisson
$E[X]$	6.0000	6.0000
$Var[X]$	5.4000	6.0000
$\sigma[X]$	2.3238	2.4495

FACT : (*The Method of Moments*)

By *Taylor expansion* of e^{tX} about $t = 0$, we have

$$\psi(t) \equiv E[e^{tX}] = E\Big[\, 1 + tX + \frac{t^2 \; X^2}{2!} + \frac{t^3 \; X^3}{3!} + \cdots \Big]$$

$$= 1 + t\, E[X] + \frac{t^2}{2!}\, E[X^2] + \frac{t^3}{3!}\, E[X^3] + \cdots .$$

It follows that

$$\psi'(0) = E[X] \quad , \quad \psi''(0) = E[X^2] \,. \qquad (\text{ Why ? })$$

This sometimes *facilitates computing the mean*

$$\mu = E[X] \,,$$

and the variance

$$Var(X) = E[X^2] - \mu^2 \,.$$

APPLICATION : The *Poisson mean* and *variance* :

$$\psi(t) \equiv E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk}\, P(X=k) = \sum_{k=0}^{\infty} e^{tk}\, e^{-\lambda}\, \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda}\, e^{\lambda\, e^t} = e^{\lambda(e^t-1)} .$$

Here $\psi'(t) = \lambda\, e^t\, e^{\lambda(e^t-1)}$

$$\psi''(t) = \lambda\, \left[\lambda\, (e^t)^2 + e^t\right]\, e^{\lambda(e^t-1)} \qquad (\text{ Check ! })$$

so that $E[X] = \psi'(0) = \lambda$

$$E[X^2] = \psi''(0) = \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$Var(X) = E[X^2] - E[X]^2 = \lambda .$$

EXAMPLE : *Defects* in a wire occur at the rate of *one per* 10 *meter*, with a *Poisson distribution* :

$$P(X=k) \;=\; e^{-\lambda} \cdot \frac{\lambda^k}{k!}\,, \qquad k = 0, 1, 2, \;\cdots \;.$$

What is the probability that :

- A 12-meter roll has at *no* defects?

  ANSWER : Here $\lambda = 1.2$, and $P(X=0) \;=\; e^{-\lambda} \;=\; 0.3012$.

- A 12-meter roll of wire has *one* defect?

  ANSWER : With $\lambda = 1.2$, $P(X=1) \;=\; e^{-\lambda} \cdot \lambda \;=\; 0.3614$.

- Of *five* 12-meter rolls *two* have *one* defect and *three* have *none*?

  ANSWER : $\binom{5}{3} \cdot 0.3012^3 \cdot 0.3614^2 \;=\; 0.0357$. (Why ?)

EXERCISE :

Defects in a certain wire occur at the rate of one per 10 meter.
Assume the defects have a Poisson distribution.
What is the probability that :

- a 20-meter wire has no defects?
- a 20-meter wire has at most 2 defects?

EXERCISE :

Customers arrive at a counter at the rate of 8 per hour.
Assume the arrivals have a Poisson distribution.
What is the probability that :

- no customer arrives in 15 minutes?
- two customers arrive in a period of 30 minutes?