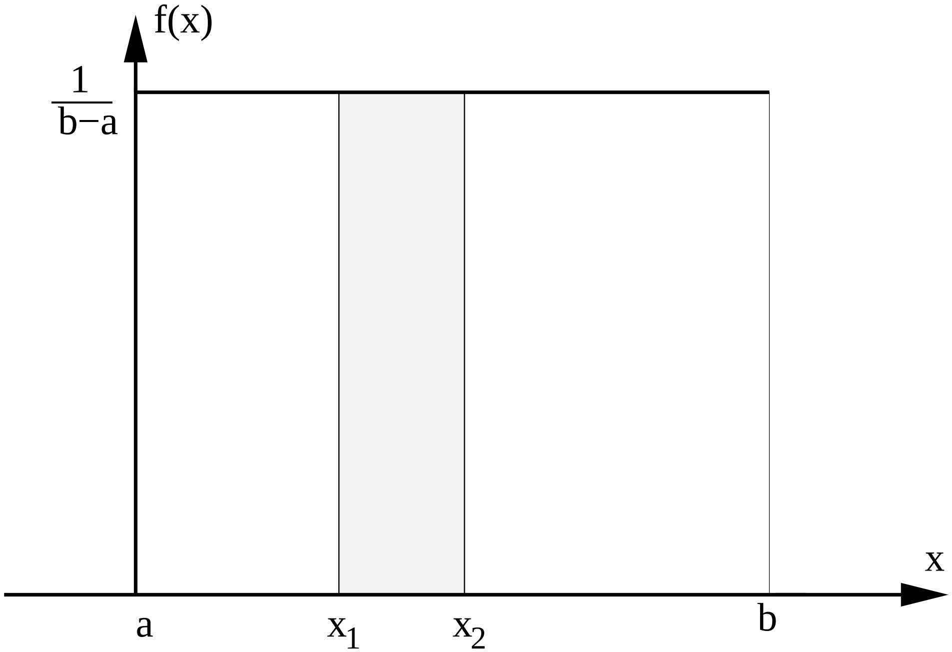
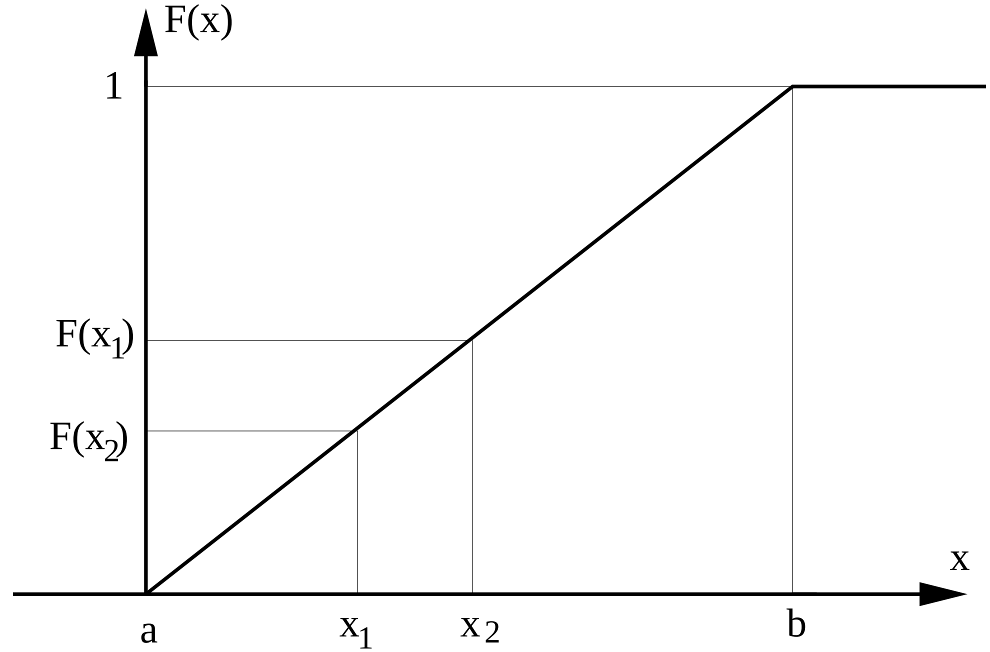
SPECIAL CONTINUOUS RANDOM VARIABLES

The Uniform Random Variable

$$f(x) = \begin{cases} \frac{1}{b-a}\,, & a < x \leq b \\ 0\,, & \text{otherwise} \end{cases} \quad , \quad F(x) = \begin{cases} 0\,, & x \leq a \\ \frac{x-a}{b-a}\,, & a < x \leq b \\ 1\,, & x > b \end{cases}$$





(Already introduced earlier for the special case $a = 0$, $b = 1$.)

EXERCISE :

Show that the *uniform density function*

$$f(x) = \begin{cases} \frac{1}{b-a}\ , & a < x \leq b \\ 0\ , & \text{otherwise} \end{cases}$$

has *mean*

$$\mu = \frac{a+b}{2}\ ,$$

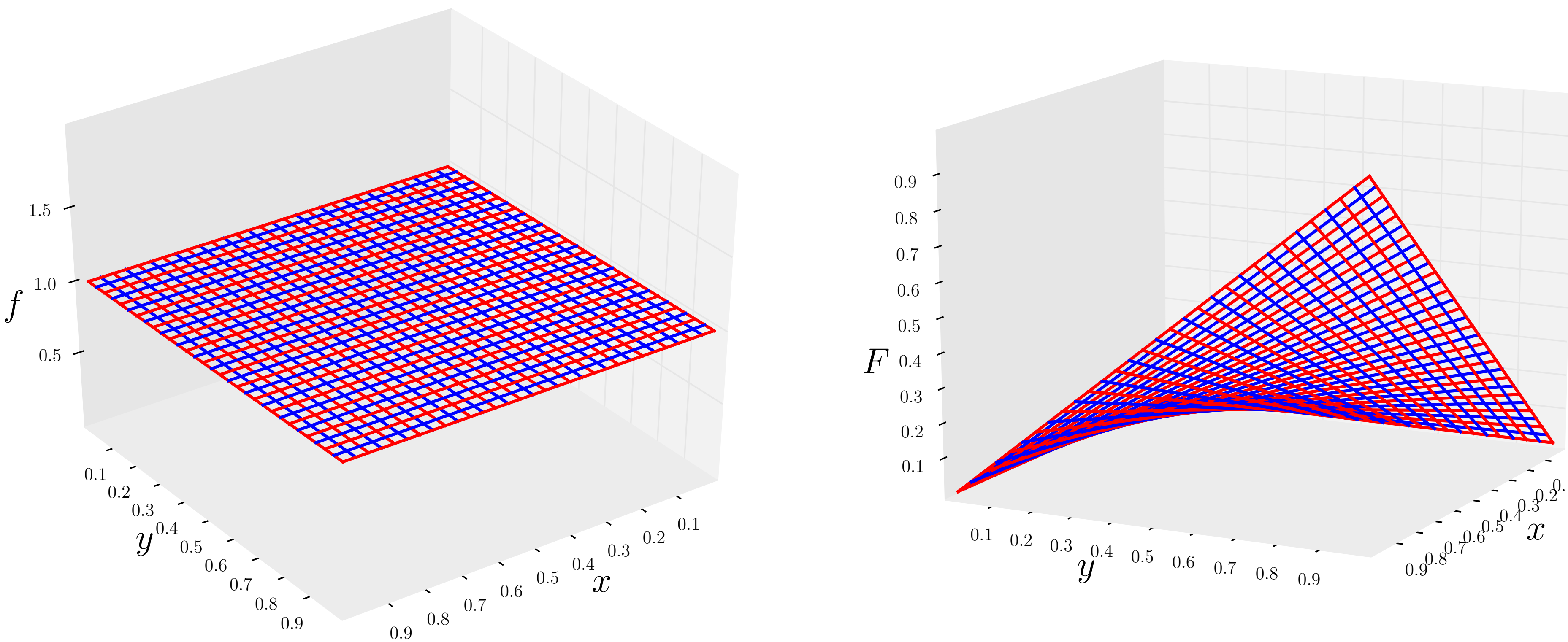
and *standard deviation*

$$\sigma = \frac{b-a}{2\sqrt{3}}\ .$$

A *joint uniform* random variable :

$$f(x,y) = \frac{1}{(b-a)(d-c)} \quad , \quad F(x,y) = \frac{(x-a)(y-c)}{(b-a)(d-c)} \, ,$$

for $x \in (a,b], \; y \in (c,d]$.



Here $x \in [0,1], \; y \in [0,1]$.

EXERCISE :

Consider the *joint uniform density function*

$$f(x,y) = \begin{cases} c & \text{for } x^2 + y^2 \leq 4 , \\ 0 & \text{otherwise} . \end{cases}$$

- What is the value of c ?
- What is $P(X < 0)$?
- What is $P(X < 0 \, , \, Y < 0)$?
- What is $f(\, x \mid y = 1 \,)$?

HINT : No complicated calculations are needed !

The Exponential Random Variable

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}, \quad F(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

with

$$E[X] = \mu = \int_0^\infty x\, \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda} \quad (\text{ Check ! }),$$

$$E[X^2] = \int_0^\infty x^2\, \lambda e^{-\lambda x}\, dx = \frac{2}{\lambda^2} \quad (\text{ Check ! }),$$

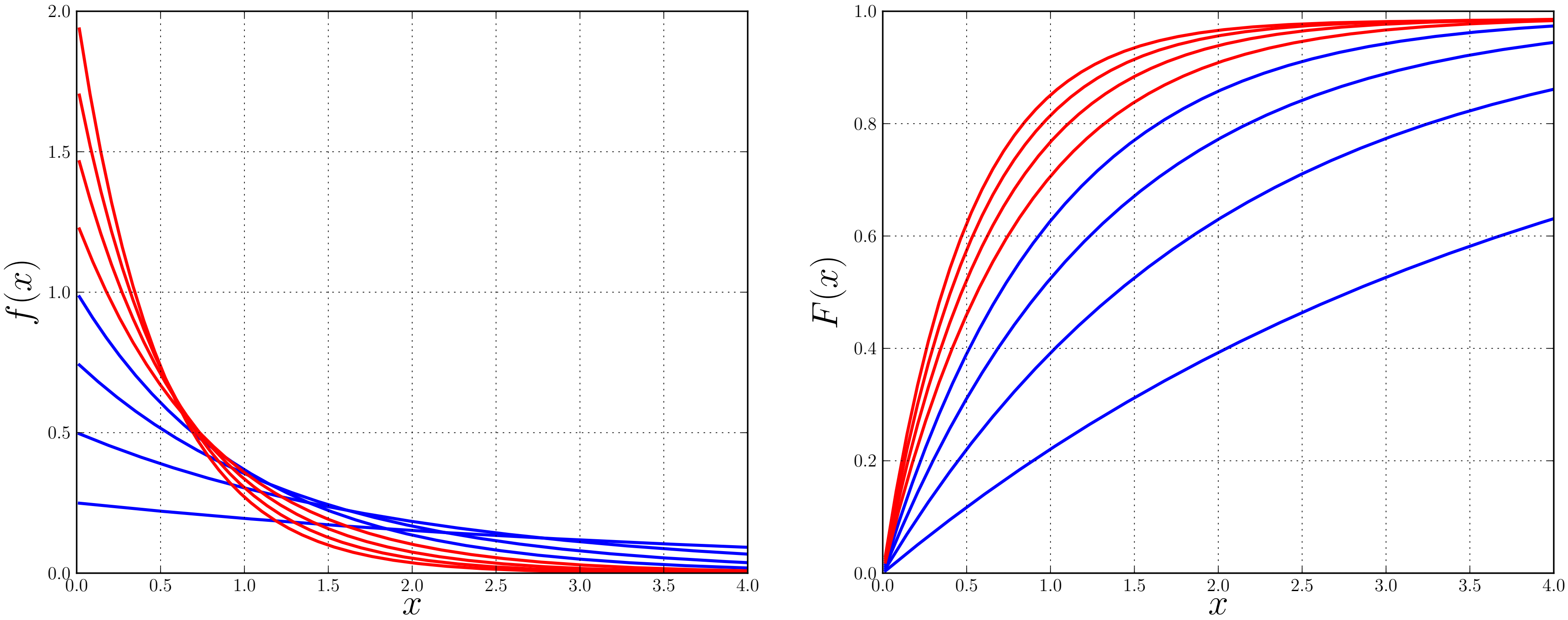
$$Var(X) = E[X^2] - \mu^2 = \frac{1}{\lambda^2},$$

$$\sigma(X) = \sqrt{Var(X)} = \frac{1}{\lambda}.$$

NOTE : The two integrals can be done by "*integration by parts*".

EXERCISE : (Done earlier for $\lambda = 1$) :

Also use the *Method of Moments* to compute $E[X]$ and $E[X^2]$.



The Exponential *density* and *distribution* functions

$$f(x) = \lambda e^{-\lambda x} \quad , \quad F(x) = 1 - e^{-\lambda x} \ ,$$

for $\lambda = 0.25, 0.50, 0.75, 1.00$ (*blue*), $1.25, 1.50, 1.75, 2.00$ (*red*).

PROPERTY : From

$$F(x) \equiv P(X \le x) = 1 - e^{-\lambda x}, \qquad (\text{ for } x > 0),$$

we have $P(X > x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$.

Also, for $\Delta x > 0$,

$$P(X > x + \Delta x \mid X > x) = \frac{P(X > x + \Delta x\ ,\ X > x)}{P(X > x)}$$

$$= \frac{P(X > x + \Delta x)}{P(X > x)} = \frac{e^{-\lambda(x+\Delta x)}}{e^{-\lambda x}} = e^{-\lambda \Delta x} .$$

CONCLUSION : $P(\ X > x + \Delta x \mid X > x\ )$

only depends on Δx (and λ), and *not* on x !

We say that the exponential random variable is "*memoryless*" .

EXAMPLE :

Let the density function $f(t)$ model *failure* of a device,

$$f(t) \quad = \quad e^{-t}\,, \qquad\qquad (\text{taking } \lambda = 1\ )\ ,$$

i.e., the *probability of failure* in the time-interval $(0, t]$ is

$$F(t) \quad = \quad \int_0^t f(t)\, dt \quad = \quad \int_0^t e^{-t}\, dt \quad = \quad 1 \; - \; e^{-t}\,,$$

with

$$F(0) \; = \; 0\,, \qquad\qquad (\text{ the device works at time } 0\ ).$$

and

$$F(\infty) \; = \; 1\,, \qquad\qquad (\text{ the device must fail at some time }).$$

EXAMPLE : (continued $\cdots$) $F(t) = 1 - e^{-t}$.

Let E_t be the *event* that the device still *works* at time t :

$$P(E_t) = 1 - F(t) = e^{-t} .$$

The probability it still works at time $t+1$ is

$$P(E_{t+1}) = 1 - F(t+1) = e^{-(t+1)} .$$

The probability it still works at time $t+1$, given it works at time t is

$$P(E_{t+1}|E_t) = \frac{P(E_{t+1}E_t)}{P(E_t)} = \frac{P(E_{t+1})}{P(E_t)} = \frac{e^{-(t+1)}}{e^{-t}} = \frac{1}{e} ,$$

which is *independent of* t !

QUESTION : Is such an exponential distribution realistic if the "device" is your heart, and time t is measured in decades ?

The Standard Normal Random Variable

The *standard normal* random variable has *density function*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} , \qquad -\infty < x < \infty ,$$

with *mean*

$$\mu = \int_{-\infty}^{\infty} x \, f(x) \, dx = 0 , \qquad (\text{ Check ! })$$

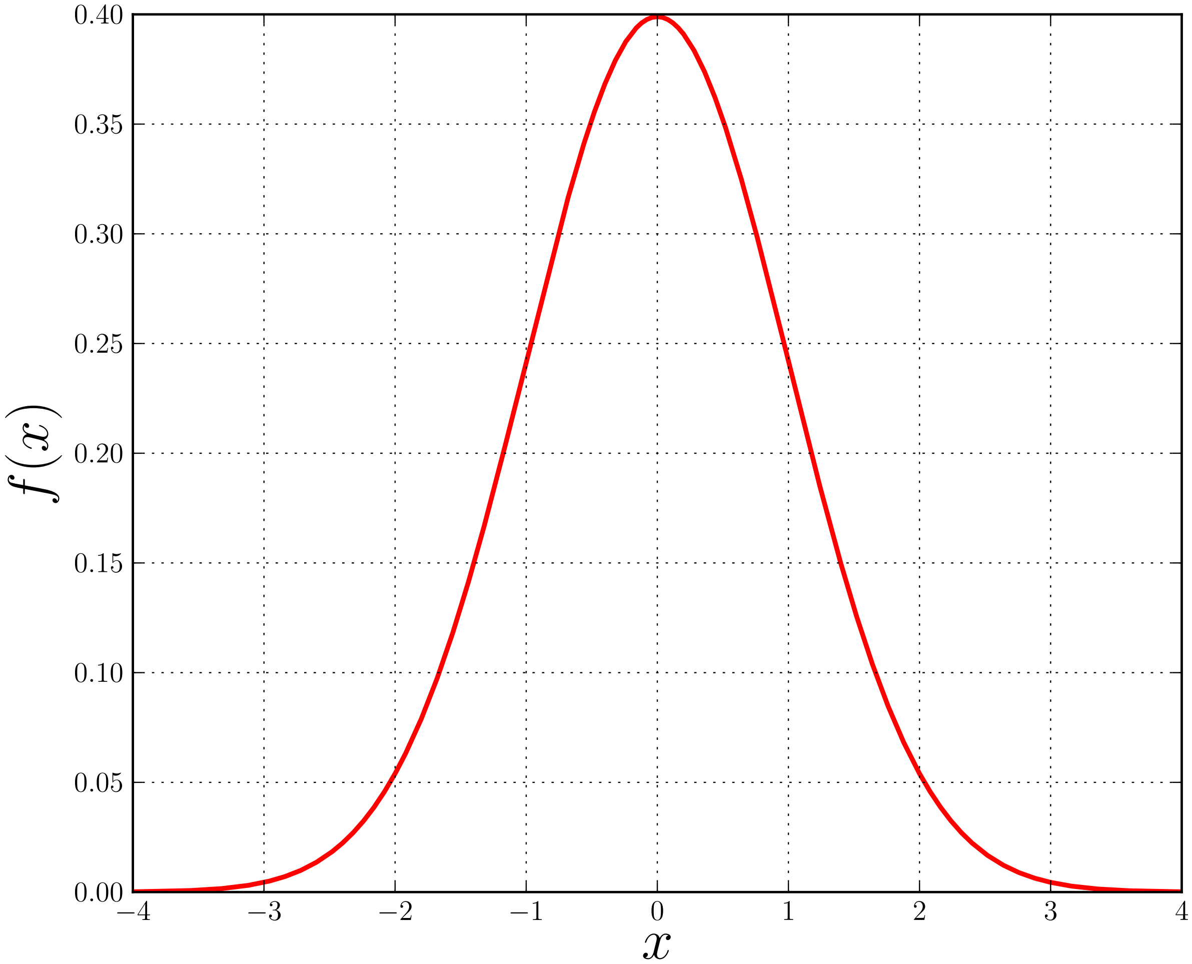
Since

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \, f(x) \, dx = 1 , \qquad (\text{ more difficult } \cdots)$$

we have

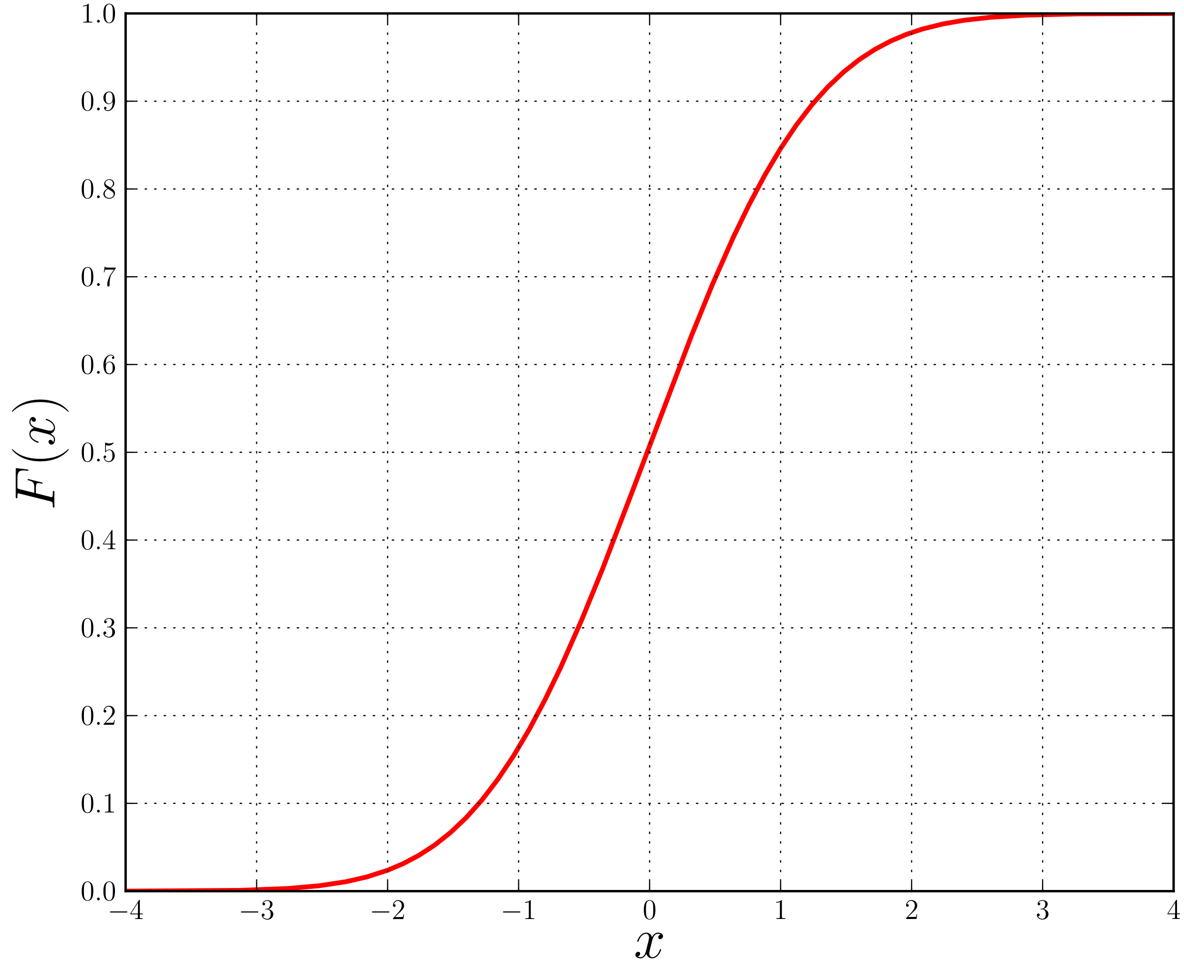
$$Var(X) = E[X^2] - \mu^2 = 1 , \qquad \text{and} \qquad \sigma(X) = 1 .$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$



The *standard normal density function* $f(x)$.

$$\mathbf{\Phi(x)} = F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\, dx$$



The *standard normal distribution function* $F(x)$

(often denoted by $\mathbf{\Phi(x)}$) .

The Standard Normal Distribution $\Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$
0.0	.5000	-1.2	.1151
-0.1	.4602	-1.4	.0808
-0.2	.4207	-1.6	.0548
-0.3	.3821	-1.8	.0359
-0.4	.3446	**-2.0**	**.0228**
-0.5	.3085	-2.2	.0139
-0.6	.2743	-2.4	.0082
-0.7	.2420	-2.6	.0047
-0.8	.2119	-2.8	.0026
-0.9	.1841	-3.0	.0013
-1.0	.1587	-3.2	.0007

(For example, $P(Z \leq -\mathbf{2.0}) = \Phi(-\mathbf{2.0}) = \mathbf{2.28}\%$)

QUESTION : How to get the values of $\Phi(z)$ for *positive* z ?

EXERCISE :

Suppose the random variable X has the *standard normal* distribution.

What are the values of

- $P(\ X\ \leq -0.5\ )$
- $P(\ X\ \leq\ 0.5\ )$
- $P(\ |\,X\,|\ \geq\ 0.5\ )$
- $P(\ |\,X\,|\ \leq\ 0.5\ )$
- $P(\ -1\ \leq\ X\ \leq\ 1\ )$
- $P(\ -1\ \leq\ X\ \leq\ 0.5\ )$

The General Normal Random Variable

The *general normal density function* is

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

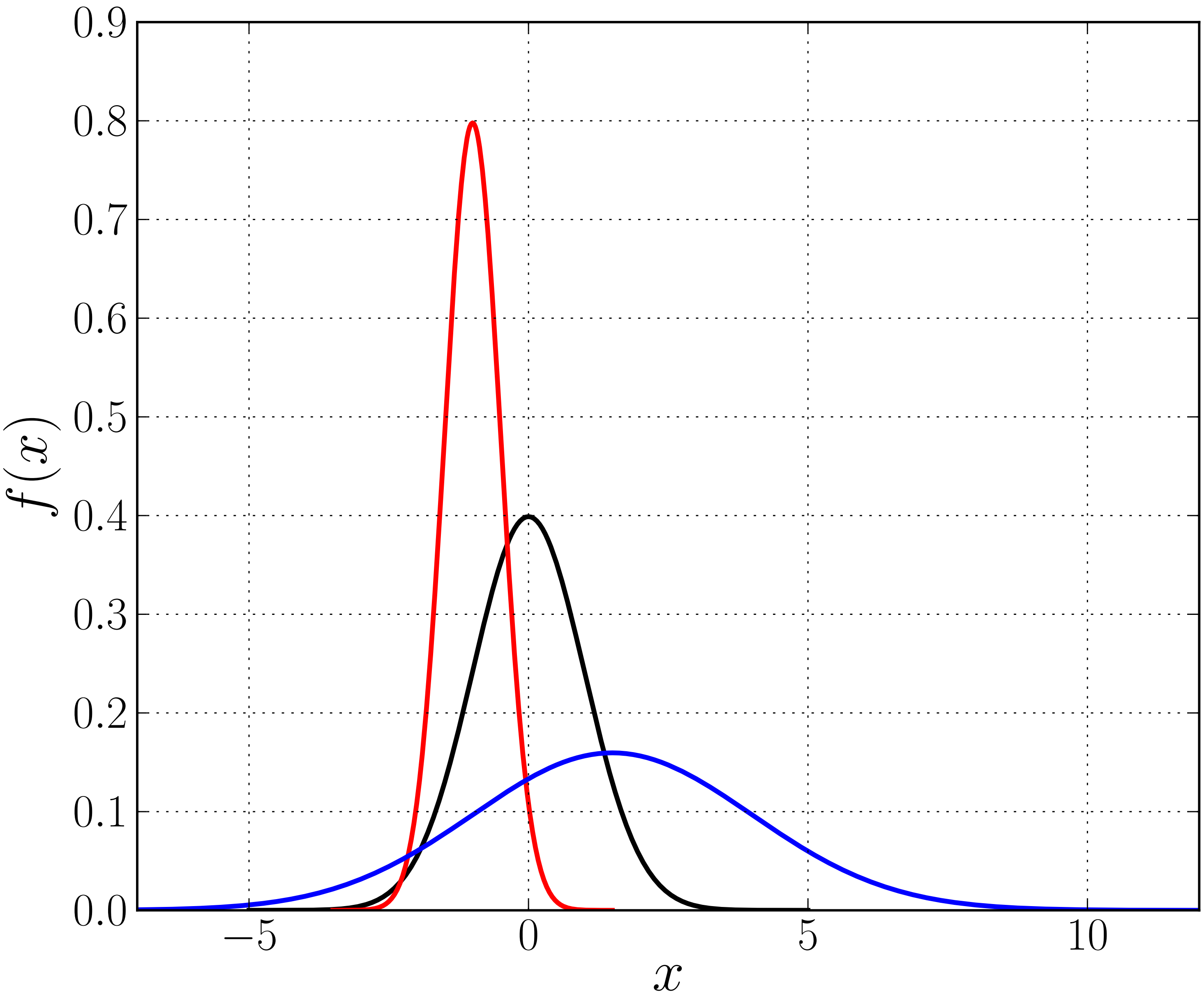
where, not surprisingly,

$$E[X] \quad = \quad \mu \qquad\qquad (\text{ Why ? })$$

One can also show that

$$Var(X) \quad \equiv \quad E[(X-\mu)^2] \quad = \quad \sigma^2 \ ,$$

so that σ is in fact the *standard deviation* .



The standard normal (*black*) and the normal density functions with $\mu = -1$, $\sigma = 0.5$ (*red*) and $\mu = 1.5$, $\sigma = 2.5$ (*blue*).

To compute the *mean* of the *general normal density function*

$$f(x) = \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

consider

$$E[X-\mu] = \int_{-\infty}^{\infty}(x-\mu)\ f(x)\ dx$$

$$= \frac{1}{\sqrt{2\pi}\ \sigma}\int_{-\infty}^{\infty}(x-\mu)\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\ dx$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\ \Big|_{-\infty}^{\infty} = 0\ .$$

Thus the *mean* is indeed

$$E[X] = \mu\ .$$

NOTE : If X is *general normal* we have the *very useful formula* :

$$P\left(\frac{X-\mu}{\sigma} \leq c \right) = \Phi(c) ,$$

i.e., we can use the *Table* of the *standard normal distribution* !

PROOF : For any constant c we have

$$P\left(\frac{X-\mu}{\sigma} \leq c\right) = P(X \leq \mu+c\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\mu+c\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\, dx .$$

Let $y \equiv (x-\mu)/\sigma$, so that $x = \mu + y\sigma$.

Then the new variable y ranges from $-\infty$ to c , and

$$(x-\mu)^2/\sigma^2 = y^2 \quad , \quad dx = \sigma\, dy ,$$

so that

$$P\left(\frac{X-\mu}{\sigma} \leq c\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{1}{2}y^2}\, dy = \Phi(c) .$$

(the *standard normal distribution*)

EXERCISE : Suppose X is normally distributed with

mean $\mu = 1.5$ and *standard deviation* $\sigma = 2.5$.

Use the *standard normal Table* to determine :

- $P(\ X\ \leq -0.5\ )$
- $P(\ X\ \geq\ 0.5\ )$
- $P(\ \mid X - \mu \mid\ \geq\ 0.5\ )$
- $P(\ \mid X - \mu \mid\ \leq\ 0.5\ )$

The Chi-Square Random Variable

Suppose $X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ ,$

are *independent standard normal* random variables.

Then $\chi^2_n \ \equiv \ X_1^2 + X_2^2 + \cdots + X_n^2 \ ,$

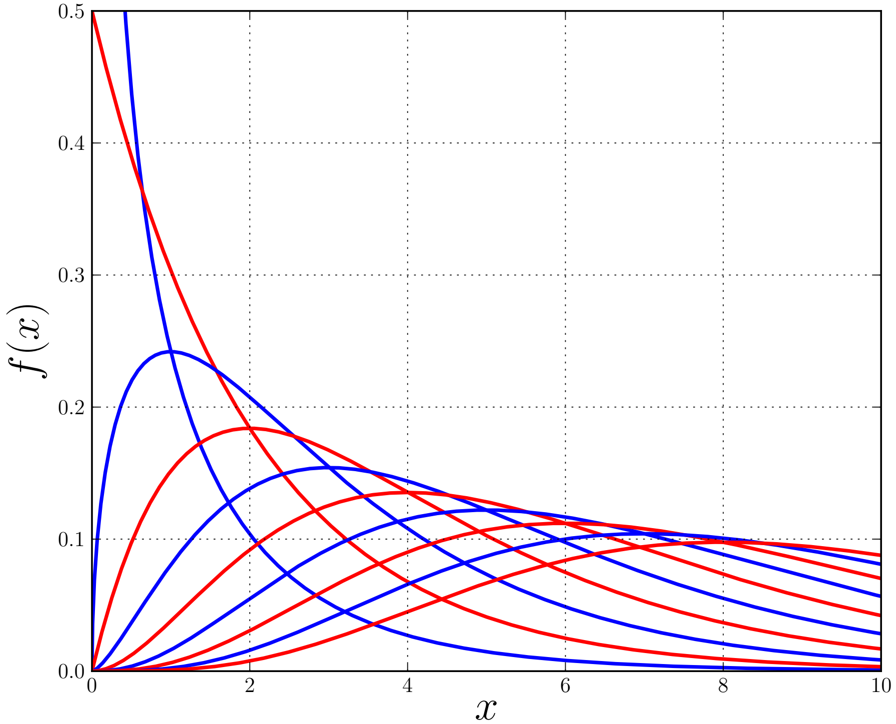
is called the *chi-square random variable* with n *degrees of freedom.*

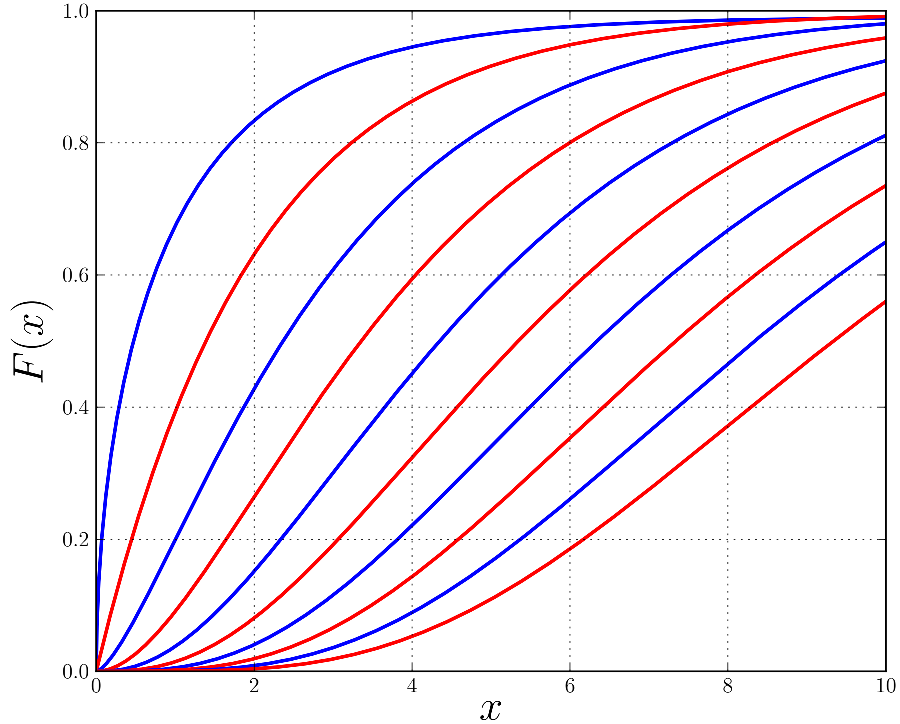
We will show that

$$E[\chi^2_n] = n \quad , \quad Var(\chi^2_n) = 2n \quad , \quad \sigma(\chi^2_n) = \sqrt{2n} \ .$$

NOTE :

The ^{2} in χ^2_n is part of its *name* , while ^{2} in X_1^2, *etc.* is "*power* 2 " !





The Chi-Square *density* and *distribution* functions for $n = 1, 2, \cdots, 10.$

(In the Figure for F, the value of n increases from left to right.)

If $n = 1$ then

$$\chi_1^2 \quad \equiv \quad X_1^2 \ , \quad \text{where} \quad X \equiv X_1 \quad \text{is } \textit{standard normal} \ .$$

We can compute the *moment generating function* of χ_1^2 :

$$E[e^{t\chi_1^2}] \ = \ E[e^{tX^2}] \ = \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{1}{2}x^2} \ dx$$

$$= \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2t)} \ dx$$

Let

$$1 - 2t \ = \ \frac{1}{\hat{\sigma}^2} \ , \quad \text{or equivalently} \ , \quad \hat{\sigma} \ \equiv \ \frac{1}{\sqrt{1-2t}} \ .$$

Then

$$E[e^{t\chi_1^2}] \ = \ \hat{\sigma} \cdot \frac{1}{\sqrt{2\pi}\ \hat{\sigma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2/\hat{\sigma}^2} \ dx \ = \ \hat{\sigma} \ = \ \frac{1}{\sqrt{1-2t}} \ .$$

(integral of a normal density function)

Thus we have found that :

The *moment generating function* of χ_1^2 is

$$\psi(t) \equiv E[e^{t\chi_1^2}] = \frac{1}{\sqrt{1-2t}} ,$$

with which we can compute

$$E[\chi_1^2] = \psi'(0) = 1 , \qquad (\text{ Check ! })$$

$$E[(\chi_1^2)^2] = \psi''(0) = 3 , \qquad (\text{ Check ! })$$

$$Var(\chi_1^2) = E[(\chi_1^2)^2] - E[\chi_1^2]^2 = 2 .$$

We found that

$$E[\chi_1^2] = 1 \quad , \quad Var(\chi_1^2) = 2 .$$

In the *general case* where

$$\chi_n^2 \equiv X_1^2 + X_2^2 + \cdots + X_n^2 ,$$

we have

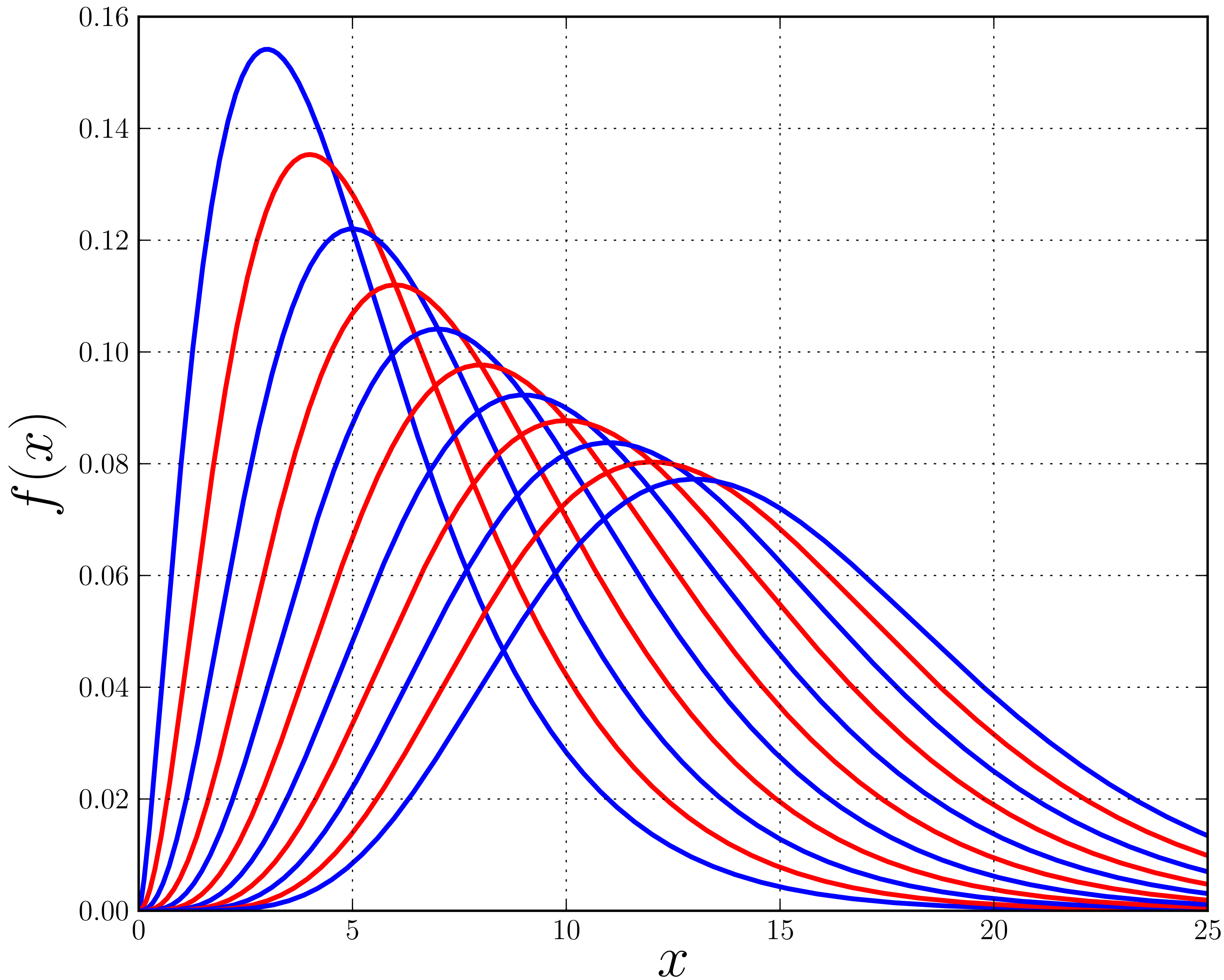
$$E[\chi_n^2] = E[X_1^2] + E[X_2^2] + \cdots + E[X_n^2] = n ,$$

and since the X_i are assumed *independent* ,

$$Var[\chi_n^2] = Var[X_1^2] + Var[X_2^2] + \cdots + Var[X_n^2] = 2n ,$$

and

$$\sigma(\chi_n^2) = \sqrt{2n} .$$



The Chi-Square *density* functions for $n = 5, 6, \cdots, 15$.

(For *large* n they look like *normal* density functions !)

THE CENTRAL LIMIT THEOREM

The density function of the Chi-Square random variable

$$\chi_n^2 \equiv \tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_n ,$$

where

$$\tilde{X}_i = X_i^2 , \quad \text{and} \quad X_i \text{ is standard normal, } i = 1, 2, \cdots, n ,$$

starts looking like a *normal density function* when n gets large.

- This remarkable fact holds much more generally !
- It is known as the *Central Limit Theorem* (CLT).