4.3 Uniform distribution

One of the simplest continuous distributions is the continuous uniform distribution. This distribution is characterized by a density function that is flat and thus the probability is uniform in a finite interval, say $[a, b]$. The density function of the continuous uniform random variable X on the interval $[a, b]$ is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{elsewhere} \end{cases}$$

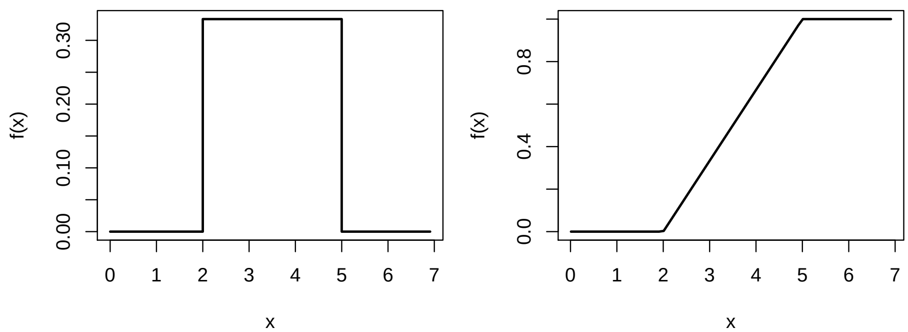

Figure 4.1: Left: uniform density, right: uniform CDF, $a = 2$, $b = 5$

The CDF of a uniformly distributed X is given by

$$F(x) = \int_a^x \frac{1}{b-a}\, dt = \frac{x-a}{b-a}, \quad a \le x \le b$$

The mean and variance of the uniform distribution are

$$\mu = \frac{b+a}{2} \quad \text{and} \quad \sigma^2 = \frac{(b-a)^2}{12}.$$

Example 4.9.
Suppose that a large conference room for a certain company can be reserved for no more than 4 hours. However, the use of the conference room is such that both long and short conferences occur quite often. In fact, it can be assumed that length X of a conference has a uniform distribution on the interval $[0, 4]$.

a) What is the probability density function of X?

b) What is the probability that any given conference lasts at least 3 hours?

Solution. (a) The appropriate density function for the uniformly distributed random variable X in this situation is

$$f(x) = \begin{cases} 1/4 & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

(b)

$$P(X \ge 3) = \int_3^4 \frac{1}{4}\, dx = \frac{1}{4}.$$

□

Example 4.10.
The failure of a circuit board interrupts work by a computing system until a new board is delivered. Delivery time X is uniformly distributed over the interval of at least one but no more than four days. The cost C of this failure and interruption consists of a fixed cost C_0 for the new part and a cost that increases proportionally to X^2, so that

$$C = C_0 + C_1 X^2$$

(a) Find the probability that the delivery time is two or more days.
(b) Find the expected cost of a single failure, in terms of C_0 and C_1.

Solution. a)

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } 1 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

Thus,

$$P(X \ge 2) = \int_2^5 \frac{1}{4}\, dx = \frac{1}{4}(5-2) = \frac{3}{4}$$

b) We know that

$$\mathbb{E}(C) = C_0 + C_1 \mathbb{E}(X^2)$$

so it remains for us to find $\mathbb{E}(X^2)$. This value could be found directly from the definition or by using the variance and the fact that $\mathbb{E}(X^2) = V(X) + \mu^2$. Using the latter approach, we find

$$\mathbb{E}(X^2) = \frac{(b-a)^2}{12} + \left(\frac{a+b}{2}\right)^2 = \frac{(5-1)^2}{12} + \left(\frac{1+5}{2}\right)^2 = \frac{31}{3}$$

Thus, $\mathbb{E}(C) = C_0 + C_1\left(\frac{31}{3}\right)$. □

Exercises

4.15.
For a digital measuring device, rounding errors have Uniform distribution, between -0.05 and 0.05 mm.

a) Find the probability that the rounding error is between -0.01 and 0.03mm
b) Find the expected value and the standard deviation of the rounding error.
c) Calculate and plot the CDF of the rounding errors.

4.16.
The capacitances of "1mF" (microfarad) capacitors are, in fact, Uniform$[0.95, 1.05]$ mF.

a) What proportion of capacitors are 0.98 mF or above?
b) What proportion of capacitors are within 0.03 of the nominal value?

4.17.
For X having a Uniform$[-1, 4]$ distribution, find the mean and variance. Then, use the formula for variance and a little algebra to find $\mathbb{E}(X^2)$.

4.18.
Suppose the radii of spheres R have a uniform distribution on $[2, 3]$. Find the mean volume. $(V = \frac{4}{3}\pi R^3)$. Find the mean surface area. $(A = 4\pi R^2)$.

Exponential distribution

Definition **Exponential distribution**

The continuous random variable X has an exponential distribution, with parameter β, if its density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2.$$

The distribution function for the exponential distribution has the simple form:

$$F(t) = P(X \leq t) = \int_0^t \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = 1 - e^{-\frac{t}{\beta}} \qquad \text{for } t \geq 0$$

The **failure rate function r(t)** is defined as

$$r(t) = \frac{f(t)}{1 - F(t)}, \qquad t > 0 \tag{4.2}$$

Suppose that X, with density f, is a lifetime of an item. Consider the proportion of items currently alive (at the time t) that will fail in the next time interval $(t, t + \Delta t]$, where Δt is small. Thus, by the conditional probability formula,

$$P\{\text{die in the next } (t, t + \Delta t] \mid \text{ currently alive}\} =$$

$$= \frac{P\{X \in (t, t + \Delta t]\}}{P(X > t)} \approx \frac{f(t)\Delta t}{1 - F(t)} = r(t)\Delta t$$

so the rate at which the items fail is $r(t)$.

For the exponential case,

$$r(t) = \frac{f(t)}{1 - F(t)} = \frac{1/\beta \, e^{-t/\beta}}{e^{-t/\beta}} = \frac{1}{\beta}$$

Note that the failure rate $\lambda = \frac{1}{\beta}$ of an item with exponential lifetime does not depend on the item's age. This is known as the *memoryless property* of exponential distribution. The exponential distribution is the only continuous distribution to have a constant failure rate.

In reliability studies, the mean of a positive-valued distribution, is also called *Mean Time To Fail* or MTTF. So, we have exponential MTTF $= \beta$.

Relationship between Poisson and exponential distributions

Suppose that certain events happen at the rate λ, so that the average (expected) number of events on the interval $[0, t]$ is $\mu = \lambda t$. If we assume that the number of events on $[0, t]$ has Poisson distribution, then the probability of no events up to time t is given by

$$pois(0, \lambda t) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}.$$

Thus, if the time of first failure is denoted X, then

$$P(X \le t) = 1 - P(X > t) = 1 - e^{-\lambda t}$$

We see that $P(X \le t) = F(t)$, the CDF for X, has the form of an exponential CDF. Here, $\lambda = \frac{1}{\beta}$ is again the failure rate. Upon differentiating, we see that the density of X is given by

$$f(t) = \frac{dF(t)}{dt} = \frac{d(1 - e^{-\lambda t})}{dt} = \lambda e^{-\lambda t} = \frac{1}{\beta} e^{-t/\beta}$$

and thus X has an exponential distribution.

Some natural phenomena have a constant failure rate (or occurrence rate) property; for example, the arrival rate of cosmic ray alpha particles or Geiger counter clicks. The exponential model works well for interarrival times (while the Poisson distribution describes the total number of events in a given period).

Example
A downtime due to equipment failure is estimated to have Exponential distribution with the mean $\beta = 6$ hours. What is the probability that the next downtime will last between 5 and 10 hours?

Solution. $P(5 < X < 10) =$

$= F(10) - F(5) = 1 - \exp(-10/6) - [1 - \exp(-5/6)] = 0.2457$ □

Example
The number of calls to the call center has Poisson distribution with the rate $\lambda = 4$ calls per minute. What is the probability that we have to wait more than 20 seconds for the next call?

Solution. The waiting time between calls, X, has exponential distribution with parameter $\beta = 1/\lambda = 1/4$. Then, $P(X > \frac{1}{3}) = 1 - F(\frac{1}{3}) = e^{-4/3} = 0.2636$ □

Exercises

1.
Prove another version of the memoryless property of the exponential distribution,

$$P(X > t + s \,|\, X > t) = P(X > s).$$

Thus, an item that is t years old has the same probabilistic properties as a brand-new item. [**Hint:** Use the definition of conditional probability and the expression for exponential CDF.]

2
The service time at the bank teller follows an Exponential distribution with the mean of 1.5 minutes.

a) Find the probability that the next customer will be served within 2 minutes
b) Find the probability that the service takes between 1 and 3 minutes

3.

Big meteorites (10 megaton TNT equivalent and higher) are believed to hit Earth approximately once every 1,000 years. Assuming Poisson distribution,

a) Find the probability that no big meteorites will hit Earth in a given 1,000-year period.

b) Given that no big meteorites hit Earth during the first 500 years, what is the probability that no big meteorites will hit Earth for the entire 1,000-year period?

4

The 1-hour carbon monoxide concentrations in a big city are found to have an exponential distribution with a mean of 3.6 parts per million (ppm).

(a) Find the probability that a concentration will exceed 9 ppm.

(b) A traffic control policy is trying to reduce the average concentration. Find the new target mean β so that the probability in part (a) will equal 0.01

(c) Find the median of the concentrations from part (a).

5

Customers come to a barber shop as a Poisson process with the frequency of 3 per hour. Suppose Y_1 is the time when first customer comes.

a) Find the expected value and the standard deviation of Y_1

b) Find the probability that the store is idle for at least first 30 minutes after opening.

The Gamma distribution

The Gamma distribution derives its name from the well-known gamma function, studied in many areas of mathematics. This distribution plays an important role in both queuing theory and reliability problems. Time between arrivals at service facilities, and time to failure of component parts and electrical systems, often are nicely modeled by the Gamma distribution.

Definition **Gamma function**

The **gamma function**, for $\alpha > 0$, is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\, e^{-x}\, dx$$

$\Gamma(k) = (k-1)!$ for integer k.

Definition **Gamma distribution**

The continuous random variable X has a gamma distribution, with *shape parameter* α and *scale parameter* β, if its density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The mean and variance of the Gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2.$$

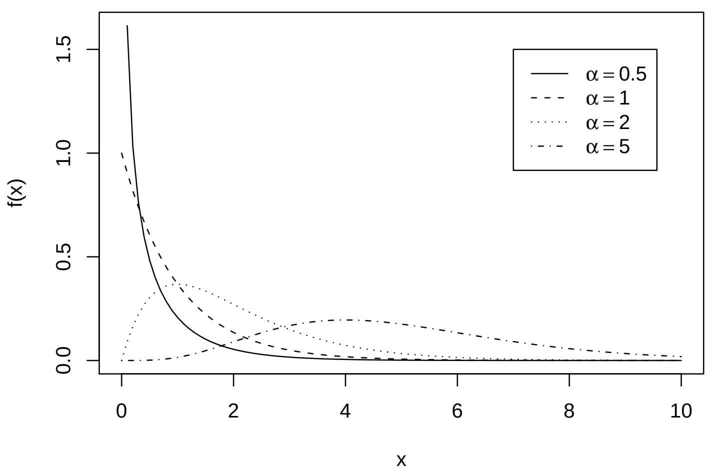

Figure : Gamma densities, all with $\beta = 1$

Note: When $\alpha = 1$, the Gamma reduces to the exponential distribution. Another well-known statistical distribution, chi-square, is also a special case of the Gamma.